Landau Levels

Particle in Magnetic Field

With an external magnetic field the free Hamiltonian \( H_o = \frac{p^2}{2m} \) becomes

\[
H = \frac{\pi^2}{2m} = \frac{(\vec{p} - q\vec{A})^2}{2m}
\]

where \( \vec{A} \) is the vector potential that defines the magnetic field \( \vec{B} = \nabla \times \vec{A} \).

Choosing the Landau gauge \( \vec{A} = B_0 x \hat{y} \) for \( \vec{B} = B_0 \hat{z} \), we have

\[
H = \frac{p^2}{2m} - \frac{qB_0 p_y}{m} x + \frac{q^2 B_o^2}{2m} x^2
\]

If the particles are constrained to move in the \( x - y \) plane, the ansatz

\[
\psi_{p_y} = e^{ip_y y} \phi_{p_y}(x), \ p_y = \hbar k_y
\]

is valid since \( H \) is translationally invariant in the \( y \) direction. Thus by

\[
H\psi_{p_y} = E(p_y)\psi_{p_y} \Rightarrow
\]

\[
\left( \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{qB_0 p_y x}{m} + \frac{q^2 B_o^2}{2m} x^2 \right) \phi_{p_y} = E(p_y)\phi_{p_y} \Rightarrow
\]

\[
\left( \frac{p_x^2}{2m} + \frac{(qB_0)^2}{2m} \left( \frac{\hbar^2 k_y^2}{(qB_o)^2} - 2 \frac{\hbar k_y}{qB_o} x + x^2 \right) \right) \phi_{p_y} = E(p_y)\phi_{p_y}
\]

Define \( \ell_B^2 = \frac{n}{qB}, \ \omega_c \equiv \frac{|qB_o|}{m} \) and complete the square

\[
\frac{1}{2m} \left( p_x^2 + m^2 \omega_c^2 (x - k_y \ell_B^2)^2 \right) \phi_{p_y} = E(p_y)\phi_{p_y}
\]

This is a harmonic oscillator at \( x = k_y \ell_B^2 \) with energy levels

\[
E_n = \hbar \omega_c (n + \frac{1}{2})
\]

And the final wave function

\[
\psi_{n,p_y} = e^{ik_y y} H_n(x - k_y \ell_B^2) e^{-\frac{(x-k_y \ell_B^2)^2}{4\ell_B^2}}
\]

where \( H_n \) are the Hermite polynomials. The energy levels (6) are called Landau levels. There are many quantum states for every Landau level i.e. for a given \( n \), every \( p_y \) corresponds to a state with the same energy \( E_n \).
Number of States

Suppose the system is of size $L_x \times L_y$, then the separation between harmonic oscillators

$$\Delta x = \Delta k_y \ell_B^2 = \left(\frac{2\pi}{L_y}\right) \ell_B^2$$

Thus the number of oscillators we can fit into the system

$$N = \frac{L_x}{\Delta x} = \frac{L_x L_y}{2\pi \ell_B^2}$$

Had we chosen a different gauge, say $\vec{A} = -B_0 y \hat{x}$, then we would have had

$$N = \frac{L_y}{\Delta y} = \frac{L_x L_y}{2\pi \ell_B^2}$$

Plugging in $\ell_B^2 = \frac{\hbar}{qB}$ we see that for electrons

$$N = \frac{q}{\hbar} BL_x L_y = \frac{B L_x L_y}{\hbar/e} = \frac{\phi}{\phi_0}$$

$N$ is the number of flux quanta, which is a measurable quantity. It is comforting that $N$ is gauge invariance, since physical quantities should obey the same physics regardless of description. The choice of gauge is a choice of description of the physics (much like coordinates), but this choice of description should not affect the result of the underlying physics.