Mathematics of Gambling

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1 Course Description

Take the simplest gambling game, i.e. flipping a fair coin, and I claim that instead of betting on your luck, you should use mathematics to make decisions.

The goal of this mini-course is trying to understand the mathematics hidden behind these games of chances, e.g. flipping a coin, rolling a roulette, or investing in the Chinese stock market (depending on your beliefs, you may disagree with me on the last one, but it is OK).

There are many ways to formulate a theory of "mathematical gambling", and the tool we choose here is probability theory. This may be the most natural choice, since probability theory originates from the study of gambling-related problems.

The style of the course is to avoid proofs as much as possible, and instead emphasize intuitions and direct calculations. We will have to introduce only a few concepts from measure theory, but all of them should not cause any technical problem.

We will try to cover at least the following two famous problems:

Problem 1.1 (The St. Petersburg paradox)**.**

You and I are gambling with each other. The rule is simple. In each play, one of us will flip a coin. To make it even simpler, let this coin be fair, meaning that it lands on either side of Head (H) or Tail (T) with probability 1/2. The game stops immediately once the coin lands on H, and you will win 2^n dollars only if the game stops at the nth flip. For example, suppose the outcome of coin flips is: T, T, T, H, then I will pay you $2^4 = 16$ dollars.

Of course, to make this game "fair", you have to pay me some "entrance fee" otherwise I would never join the game (think about you are buying some sort of options from me, if you know some finance or your friends do). The paradox here is that, if you compute the expectation of your winning, let us say it is denoted by X, you will find that $EX = \sum 2^n 2^{-n} = \infty$, but you clearly wouldn't pay me an infinite amount to start this game (of course, no one ever has an infinite amount of wealth).

Trying to solve this paradox, economists invented concepts like expected utility and risk aversion, which do not always make sense in real life. However, as it turns out, by interpreting "fair" in another way and using probability theory, namely the Weak Law of Large Numbers, one can compute a price for each game if the game is played n times.

Problem 1.2 (Gambler's ruin)**.**

A gambler enters a game with initial wealth being one dollar. He bets one dollar at each play on a flipping coin with probability p landing on H (p need not be 1/2 this time), and he wins one dollar on an H while losing one dollar on a T.

Assume that the gambler's goal is to win at least A dollars, and he will be ruined (or be kicked out) once his wealth drops to $-B$ dollars. Now, what is the probability that he will achieve his goal, or in another word, his wealth reaches A before drops to $-B$? And what is the average ruin time, meaning the expectation of the number of plays after which his wealth drops to $-B$? To answer these questions, we will develop concepts of conditional expectations, stopping times, and martingales, and use martingale theory to analyze the ruin problem.

More generally, replace the gambler by an insurance company, and suppose that the company collects one dollar per day, but independently pays a claim, eacah day with probability p, of two dollars. We can thus make prediction of the ruin time of the insurance company based on martingale theory.

2 Prerequisites

Since this is actually a math course, not a course about real gambling, you need some mathematical preparation to get something out of this course.

The ideal background would consist of a course on probability theory, and some familiarity with basic mathematical analysis by which I mean you are comfortable with the $\epsilon - \delta$ type of arguments. But in practice, all you really need to follow the course is just calculus, so you at least need to know differentiation and integration. That being said, what is more important than any particular math course taken before is curiosity and willingness of studying new mathematics, and/or their applications.

3 Miscellaneous Things

The actual time length of this course will be determined as it goes on. If time permits, we may include a training session on academic writing of mathematical papers using Latex.

There will be no homeworks and exams. Also no credits will be given (for SURE).

The course materials will be presented mainly in English, and we encourage you to take this chance and familiarize yourself with writing mathematics in English if you plan to participate in some contests like the MCM (The Mathematical Contest in Modeling).