Efficient Algorithms for Dynamic Pricing Problem with Reference Price Effect

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We analyze a finite horizon dynamic pricing model in which demand at each period depends on not only the current price but also past prices through reference prices. A unique feature but also a significant challenge in this model is the asymmetry in reference price effects which implies the underlying optimization problem is non-smooth and no standard optimization methods can be applied. We identify a few key structural properties on the problems, which enable us to develop strongly polynomial time algorithms to compute the optimal prices for several plausible scenarios. We complement our exact algorithms by proposing an approximation heuristic which also provides bounds on the optimal objective value. Finally, we demonstrate the efficiency and robustness of our algorithms by applying them to a practical problem with real data.

Key words: dynamic pricing, reference price effects, dynamic programming, piece-wise quadratic functions

1. Introduction

Dynamic pricing, in which a firm varies the prices of its product so as to maximize the profit collected over a time horizon, has been widely studied in academia in the past few decades. In the meantime, there is also a considerable growth in industries that adopt the practice of dynamic pricing. Despite of the numerous different problem settings for dynamic pricing, the efforts have been mainly spent on two issues both in academia and industry. One issue is to capture the relationship between demand and price accurately. A recent progress in this direction is the incorporation of consumers' behavior into consideration, such as strategic or bounded rational behavior of consumers. The other issue is solving the optimal prices efficiently. The latter issue becomes pertinent if one further wants to put the dynamic pricing problem in a practical decision support system which incorporates other operations management decisions, e.g., inventory decisions, and manages prices for thousands of products.

One class of well-studied consumer behavioral models in the marketing and economics literature is memory-based reference price models. Such models argue that consumers develop price expectations from historical prices (referred to as reference prices) and use them to judge the current selling price of a product. That is, reference price is used as an internal anchor formed in consumers' minds as a result of experience based on information such as prices in observed periods (Kalyanaram and Little 1994). Although in most cases it can not be physically observed, researchers notice that "comparison of the market price to . . . reference price (Raman and Bass 2002)" influences consumers' evaluation on potential purchases before making their decision, especially "in a market with repeated interactions (Popescu and Wu 2007)." There is a vast amount of literature that supports memory-based reference price models at both individual/household level as well as aggregate level (Briesch et al. 1997, Hardie et al. 1993, Greenleaf 1995, etc.). Indeed, reference price models are now accepted as an empirical generalization in the marketing literature (see the review paper Mazumdar et al. 2005).

Driven by these evidences on the significant impact of reference price in shaping consumer demand, a stream of literature has studied how firm should set its price when facing such reference price dependent demand and obtained many useful managerial insights. However, to our best knowledge, how to compute the optimal prices accurately and efficiently remains largely unexplored. As Fibich et al. (2003) point out, "calculations of optimal strategies were limited to numerical simulations using dynamic programming". In practice, the revenue of many firms in retail industry, where a large portion of empirical evidences on reference price effects are found, can exceed $30 billion per year (see Talluri and Van Ryzin 2006). This fact highlights the importance of even small incremental gains from revenue management. Thus, computing the optimal prices accurately becomes a critical question. On the other hand, a firm not only needs to coordinate pricing decisions with other operations management decisions such as inventory decisions but also makes all those decisions for over thousands of products. When the more realistic reference price models are incorporated in such decision support systems, the efficiency of computing the simplest basic model becomes crucial.

In this paper, we look into the computational issue of the dynamic pricing problem in a memory-based reference price model. In this model, a firm sells a product over a finite time horizon and faces possibly time-varying demands, which depend on not only the price the firm sets in that period but also all the historical prices the firm set through reference price effects. Like most of the previous literature, the demand in each period is assumed to be deterministic. This assumption is plausible in many practical settings, in which a firm can predict future demands quite accurately from historical sales data. A unique feature and also a significant challenge in this model is the asymmetry in reference price effects. More specifically, consumers' perception of gains (when the shelf price is less than the reference price) and losses (when the shelf price is greater than the reference price) could be different. This leads to a non-smooth optimization problem, for which no standard optimization methods can be applied.

Facing with such challenges, our paper develops algorithms that balance between accuracy in the optimal solutions and efficiency in computations. When consumers are loss-averse (they respond more to losses than gains), we provide a strongly polynomial time algorithm to compute the optimal prices under a certain mild technical condition on the input parameters. Although our model assumes piecewise linear demand function, the core part of the algorithm may also be used to other forms of demand function. Also, a few properties we found along with the algorithm are potentially applicable to other non-smooth optimization problems as well. When consumers are gain-seeking (they respond more to gains than losses), we provide a different strongly polynomial time algorithm to solve the optimal prices for a special case. For all other inputs of parameters (when the technical condition fails for loss-averse consumers or other scenarios for gain-seeking consumers), an approximation heuristic is proposed. Lower and upper bounds on the optimal objective value are obtained from the heuristic. Finally, we apply our algorithms to a practical problem from industry with real data and demonstrate the efficiency and robustness of the algorithms.

The stream of works on dynamic pricing problems incorporating reference price effects is most closely related to our paper. Greenleaf (1995) empirically validates reference price model with linear demand function and examines numerically the profitability of price promotions. Kopalle et al. (1996) observe through numerical studies that the optimal price path converges to a constant price when consumers are loss-averse and tends to cycle when consumers are gain-seeking. Fibich et al. (2003) explicitly solve the optimal pricing strategy when consumers are loss-averse under a continuous time model. Although with explicit solutions they obtain valuable insights such as a constant price is optimal in the long run, deriving the explicit solutions relies heavily on the stationary assumption of the demand and the assumption that prices are unconstrained. Popescu and Wu (2007) examine the discrete time counterpart and prove for general demand function and loss-averse consumers that a constant price is optimal in the long run. Nasiry and Popescu (2011) consider the dynamic pricing problem with a peak-end based reference price model and loss-averse consumers; they also conclude the observation by Kopalle et al. (1996). Hu et al. (2015), on the other hand, show that when consumers are gain-seeking, even myopic pricing strategy can lead to complicated dynamics and they prove for a special case that a cyclic skimming pricing strategy is optimal. It is worth mentioning that all of the above papers assume stationary demand. Although this is a convenient assumption for studying the long run behavior as well as the dynamics of the optimal prices, it is not always satisfied in practice. In fact, in many retail industries, seasonality is a major factor. All of our algorithms, however, can be implemented in the situation of a changing market environment.

Recently, there are a few attempts that incorporate reference price effects into integrated inventory and pricing models. Gimpl-Heersink (2008) analyzes a stochastic periodic review finite horizon model in which demand is a function of both the current price and the reference price with additive random noise. Chen et al. (2015) study a similar model and introduce a novel transformation technique to convert a non-concave single-period revenue function to a modified revenue function that is concave. They characterize various structures of optimal solutions. Based on a different mechanism rather than the reference price models, Ahn et al. (2007) develop algorithms for a periodic review finite horizon deterministic model in which demand depends on past prices. Even though the concentration of our paper is on developing algorithms for the pure pricing problem, our algorithms can serve as a building block in a deterministic joint inventory and pricing problem.

The remainder of this paper is organized as follows. In Section 2 the mathematical formulation of our model is presented. In Section 3, we analyze the model with loss-averse consumers and develop a strongly polynomial time algorithm to solve the optimal prices exactly. When consumers are gain-seeking, a strongly polynomial time algorithm is provided for a special case in Section 4. Section 5 proposes an approximation heuristic to deal with all the other cases. Our algorithms are tested on real data in Section 6 to solve a practical industry problem with analysis on the efficiency as well as the robustness of the algorithms. Finally, we conclude the paper in the last section with some suggestions for future research. To maintain a clear presentation, all technical proofs are presented in the appendix.

2. Model

We assume that the firm sells a product over a finite horizon of T periods. In our model, demand at each period depends on not only the current selling price but also past observed prices through reference price effects. Specifically, reference price is generated by exponentially weighting past prices. That is, starting with a given initial reference price r_1, the reference price at period t, denoted by r_t, evolves as

$$r_{t+1} = \alpha r_t + (1-\alpha) p_t, \quad t = 1, 2, \cdots, T. \tag{1}$$

In the above evolution equations, $\alpha \in [0,1)$ is called the memory factor or carryover constant (Kalyanaram and Little 1994). Observe that as α increases, consumers are less sensitive to the new price information and in the extreme case when $\alpha = 1$, reference prices remain a constant r_1 over the whole planning horizon. Thus, we restrict $\alpha < 1$ to avoid the case that past prices have no impact on demand.

Following Greenleaf (1995), Kopalle and Winer (1996), Fibich et al. (2003) and Nasiry and Popescu (2011), the demand at period t, with a given price p and a reference price r, is modeled as

$$D_t(p, r) = b_t - a_t p + \eta(r - p),$$

where $D_t(p,p) = b_t - a_t p$ is the base demand independent of reference prices, $\eta(r-p)$ is the additional demand or demand loss induced by the reference price effect. The potential market size b_t and the price sensitivity a_t are non-negative and are allowed to be time-varying to reflect a dynamic market conditions. The difference between reference price and selling price, i.e., $r-p$, in the above demand model is usually referred to as a perceived surcharge/discount (Popescu and Wu 2007). If $r < p$, consumers perceive this as a loss, while if $r > p$, they perceive it as a gain. In this paper, we assume that η is a kinked function consisting of two linear pieces as

$$\eta(z) = \eta^+ \max\{z, 0\} + \eta^- \min\{z, 0\},$$

where non-negative coefficients η^+ and η^- represent the marginal reference price effect associated with gains and losses, respectively. The kink of the function η indicates that consumers respond to losses and gains differently. Consumers are classified as loss-averse, loss/gain neutral and gain-seeking depending on whether $\eta^+ < \eta^-$, $\eta^+ = \eta^-$ or $\eta^+ > \eta^-$. It is common in the marketing literature to assume that consumers are more sensitive to losses than gains. Indeed, the loss-averse assumption is consistent with the prospect theory (Tversky and Kahneman 1991) and has also been validated by several empirical studies (see, for example, Putler 1992, Hardie et al. 1993). However, there are evidences that indicate consumers may be more sensitive to gains than losses in some situations (e.g., Greenleaf 1995, Krishnamurthi et al. 1992, Hu et al. 2015). In this paper we consider the general case and make no assumption on the relative magnitudes of the two coefficients. Since these two coefficients, along with the memory factor α reflect consumers' internal perceptions of reference prices, losses and gains, they are assumed to be time-invariant even though our algorithms can be easily extended to the case when they vary with time.

The linear form of the demand function seems somewhat restrictive. It is nevertheless a very practical assumption. In practice, all the parameters in the demand function can be conveniently estimated from real data on historical prices and demands using linear regression (see, for example, Greenleaf 1995, Hu et al. 2015). Furthermore, in the loss/gain neutral case, the unconstrained dynamic pricing problem can be easily solved using the standard linear quadratic control techniques (see the formulation below).

Facing the reference price dependent demands and an initial reference price r_1, the firm then maximizes its total profit over the planning horizon by determining the optimal price in each period. That is,

$$\begin{aligned} \max_{p_t:1\le t\le T} \ & \pi_1(r_1,p_1)+\pi_2(r_2,p_2)+\cdots+\pi_T(r_T,p_T) \\ \text{s.t.} \quad & r_{t+1}=\alpha r_t+(1-\alpha)p_t, \quad p_t\in[L_t,U_t], \quad t=1,\cdots,T. \end{aligned} \tag{2}$$

In the problem formulation (2), $\pi_t(r_t,p_t)=p_tD_t(r_t,p_t)$ is the profit collected in period t, $1\le t\le T$, where we have implicitly assumed that the marginal cost is zero for simplicity and all our results can be extended to the case when there is a non-zero marginal cost. The lower bounds L_t and upper bounds U_t on prices are non-negative and are also allowed to be time-varying. One reason for this lies in our formulation of time-varying demands. Since demands can not be negative, this naturally generates a time-varying upper-bounds on prices. Also, in some scenarios, firm has other objectives such as minimum sales or maximum allowable discount on prices which could vary season by season and result in a time-varying constraints on prices.

Since the effect of past prices on period t's demand is summarized by the reference price r_t, it will be sometimes convenient to express profit in terms of reference prices. In particular, given the reference prices r_t,r_{t+1} at periods $t,t+1$, respectively, the price p_t and the profit, denoted by $\Pi_t(r_t,r_{t+1})$, at period t can be expressed as

$$p_t=\frac{r_{t+1}-\alpha r_t}{1-\alpha}, \quad \Pi_t(r_t,r_{t+1})=\pi_t\left(r_t,\frac{r_{t+1}-\alpha r_t}{1-\alpha}\right).$$

3. Loss-averse Consumers

In this section we focus on the case when consumers are loss-averse, i.e., $\eta^->\eta^+$. We remark here that in the loss-neutral case ($\eta^-=\eta^+$), if there are no constraints on prices, then explicit solutions can be computed via standard linear-quadratic control techniques with computational complexity of $O(T)$ (see, for instance, Anderson and Moore 2007). On the other hand, with price constraints, the loss-neutral case can be reduced to a special case analyzed in this section and consequently our algorithms can be applied.

In the following, we formulate problem (2) as a dynamic programming problem and discuss the potential challenges in solving exactly the dynamic programming problem. In the first subsection, we lift our discussion to a more general problem setting and explore the essential properties in the problem that help us to overcome the challenges. In the second subsection, we then show that under a technical assumption, these properties hold in our dynamic programming problem and we develop a strongly polynomial time algorithm to solve the problem exactly.

We first formulate problem (2) as a dynamic programming problem. Let $G_{t+1}(r_{t+1})$, $t \le T$, be the maximal accumulated profit up to period t when reference price r_{t+1} is specified at period $t+1$. That is,

$$\begin{aligned} G_{t+1}(r_{t+1}) = \max\ & \Pi_1(r_1,r_2) + \Pi_2(r_2,r_3) + \cdots + \Pi_t(r_t,r_{t+1}), \\ \text{s.t.}\quad & \alpha r_s + (1-\alpha)p_s = r_{s+1}, \quad p_s \in [L_s, U_s], \quad s \le t. \end{aligned}$$

Apparently solving problem (2) amounts to maximizing $G_{T+1}(r)$. Thus, it suffices to determine the expression of $G_{T+1}(r)$, which can be iteratively derived for $t = 2, \cdots, T$ through solving the problem

$$\begin{aligned} G_{t+1}(q) &= \max_r \{\Pi_t(r,q) + G_t(r) : \frac{q-\alpha r}{1-\alpha} \in [L_t, U_t]\}, \\ r_t(q) &= \arg\max_r \{\Pi_t(r,q) + G_t(r) : \frac{q-\alpha r}{1-\alpha} \in [L_t, U_t]\}, \end{aligned} \tag{3}$$

where $G_2(q) = \Pi_1(r_1, q)$ for $q \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1]$. Note here that due to the price constraints $p \in [L_t, U_t]$, $G_{t+1}(q)$ is only defined for those states that lead to feasible solutions. For convenience we specify $G_{t+1}(q) = -\infty$ if q leads to an empty feasible set in the above problem and the effective domain of $G_{t+1}(q)$ is then defined as $\{q : G_{t+1}(q) > -\infty\}$. In particular, the first profit-to-go function $G_2(q)$ is of the following form

$$G_2(q) = \begin{cases} g_2(q), & \text{if} \quad q \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1], \\ -\infty, & \text{otherwise}, \end{cases}$$

where $g_2(q) = \Pi_1(r_1, q)$ consists of two quadratic pieces and is concave and continuously differentiable except at r_1.

The main challenge in solving problem (3) efficiently is associated with non-differentiability in the objective function. For instance, even for $t = 2$, both $G_t(r)$ and $\Pi_t(r,q)$ are non-differentiable functions and consist of two different quadratic pieces. Thus, we need to answer the question that in general for $t > 2$, how "simple" can $G_t(r)$ be? In other words, at how many points will $G_t(r)$ be non-differentiable and how many quadratic pieces will $G_t(r)$ be consisted of?

Here, we give a brief sketch of our approach in dealing with the above challenge. Suppose we already have the analytical expressions for $G_t(r)$. As we will later show that there exist $\underline{r}_t, \overline{r}_t$ such that $G_t(r)$ follows the form

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty), \end{cases}$$

where $g_t(r)$ is a continuous function defined on the whole real line. We consider the following problem first:

$$\begin{aligned} f(q) &= \max_r \{\Pi_t(r,q) + g_t(r)\}, \\ r^*(q) &= \arg\max_r \{\Pi_t(r,q) + g_t(r)\}. \end{aligned} \tag{4}$$

Note that in problem (4), we ignore the price constraints and extend the effective domain of $G_t(r)$ to the whole real line by replacing $G_t(r)$ with $g_t(r)$. This allows us to concentrate on the issue of non-differentiability, which is addressed in the following subsection in a more general problem setting. Specifically, we develop an efficient algorithm to solve problem (4) and show that the structure of $f(q)$ is "as simple as" $g_t(r)$.

We then consider the problem

$$\begin{aligned} f_c(q) &= \max_r \{\Pi_t(r,q) + g_t(r) : \frac{q - \alpha r}{1 - \alpha} \in [L_t, U_t]\}, \\ r_c^*(q) &= \arg\max_r \{\Pi_t(r,q) + g_t(r) : \frac{q - \alpha r}{1 - \alpha} \in [L_t, U_t]\}, \end{aligned} \tag{5}$$

where we use subscript "c" in the solution and optimal value function to emphasize the presence of price constraint in problem (5). The second subsection deals with the issue of computing $r_c^*(q), f_c(q)$ from $r^*(q), f(q)$ and computing $r_t(q), G_{t+1}(q)$ from $r_c^*(q), f_c(q)$.

3.1. A General Problem

The purpose of this subsection is two fold. First, we believe the presentation of the algorithm can be made clean and organized by highlighting only those properties that are essential for our algorithm. Second, the treatment in a more general setting naturally makes our algorithm robust to other forms of demand function. Furthermore, some of the properties we identified are potentially applicable to other problems as well. The following problem, which is a generalization of problem (4), is considered in this subsection.

$$
\begin{aligned}
f(q) &= \max_r \{\Pi(r,q)+g(r)\}, \\
r^*(q) &= \arg\max_r \{\Pi(r,q)+g(r)\}.
\end{aligned} \tag{6}
$$

We impose the following two assumptions on both the input functions $\Pi(r,q)$ and $g(r)$ as well as the output functions $f(q)$ and $r^*(q)$ throughout this subsection.

Assumption 1. *(a) $\Pi(r,q):\mathbb{R}^2\to\mathbb{R}$ is continuous, strictly supermodular in (r,q) and strictly concave in r. Furthermore, $\Pi(r,q)$ has the form*

$$
\Pi(r,q)=\begin{cases} \Pi^+(r,q), & q\le r, \\ \Pi^-(r,q), & q\ge r, \end{cases}
$$

where $\Pi^+(r,q)$ and $\Pi^-(r,q)$ are continuously differentiable functions defined on $\mathbb{R}^2$. We denote $\pi(q)=\Pi(q,q)=\Pi^+(q,q)=\Pi^-(q,q)$.

(b) $g(r):\mathbb{R}\to\mathbb{R}$ is concave and continuously differentiable except at finite points $r_1,...,r_m$, with $-\infty=r_0<r_1<...<r_m<r_{m+1}=+\infty$. More specifically,

$$
g(r)=\begin{cases} g_1(r), & r_0\le r\le r_1, \\ \dots & \\ g_{m+1}(r), & r_m\le r\le r_{m+1}, \end{cases}
$$

where $g_j(\cdot)$, $1\le j\le m+1$ are all continuously differentiable functions defined on $\mathbb{R}$. We call $r_1,...,r_m$ the kink points of $g(r)$.

An immediate consequence from the strict concavity and continuity assumptions in Assumption 1 is that $r^*(q)$ is single valued and continuous (see, for example, Ok 2007). We further impose the following assumptions on $f(q)$ and $r^*(q)$.

ASSUMPTION 2. *(a) $f(q)$ is concave.*

(b) $q - r^*(q)$ satisfies the single crossing property. That is, for $q' > q''$, $q'' - r^*(q'') > 0$ implies $q' - r^*(q') > 0$, and $q'' - r^*(q'') \geq 0$ implies $q' - r^*(q') \geq 0$.

Finding general conditions on $\Pi(r,q)$ and $g(r)$ such that Assumption 2 holds is an interesting research topic itself (for instance, Assumption 2 (a) will hold if we further assume $\Pi(r,q)$ to be jointly concave) and is beyond the scope of this paper. However, by using a transformation technique developed in Chen et al. (2015) we will prove in the next subsection that under some mild conditions Assumption 2 holds for our dynamic pricing problem.

The rest of this subsection is divided into three parts. (a) We first prove that $r^*(q)$ can be decomposed into the solutions of two simpler problems and $r^*(q)$ has certain monotonic structures. (b) Then we show that even though $\Pi(r,q)$ may not be differentiable along the line $r = q$, $f(q)$ is, surprisingly, continuously differentiable except at at most m points, whose candidates are exactly $r_1, ..., r_m$, the kink points of $g(r)$. In the extreme case, when $g(r)$ is continuously differentiable, then $f(r)$ is also continuously differentiable. That is, the kink points of $g(r)$ are, in some sense, "preserved" under the maximization of problem (6). This observation connects the algorithm developed in this subsection with the dynamic programming algorithm in the next subsection. (c) Finally, we consider the computational issue. We show how the structures of $r^*(q)$ allow us to develop an efficient algorithm to compute explicitly $r^*(q)$ and $f(q)$ once we know the functional form of the input functions $g(r)$ and $\Pi(r,q)$.

3.1.1. Structures. We introduce the following two problems

$$\begin{aligned} f^+(q) &= \max_r \{\Pi^+(r,q) + g(r)\}, \\ r^+(q) &= \arg\max_r \{\Pi^+(r,q) + g(r)\}, \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
f^-(q) &= \max_r \{\Pi^-(r,q) + g(r)\}, \\
r^-(q) &= \arg\max_r \{\Pi^-(r,q) + g(r)\}.
\end{aligned}
\tag{8}
$$

Our first result states that $r^*(q)$ can be decomposed into $r^+(q)$, $r^-(q)$ in the following way.

Proposition 1. *There exist $\underline{Q} \le \overline{Q}$, such that*

$$
r^*(q) = \begin{cases} r^+(q), & q < \underline{Q}, \\ q, & \underline{Q} \le q \le \overline{Q}, \\ r^-(q), & q > \overline{Q}, \end{cases} \quad \textit{and} \quad f(q) = \begin{cases} f^+(q), & q < \underline{Q}, \\ \pi(q) + g(q), & \underline{Q} \le q \le \overline{Q}, \\ f^-(q), & q > \overline{Q}. \end{cases}
$$

Next, we closely examine the structure of $r^+(q)$ and $r^-(q)$.

Lemma 1. *There exist $-\infty = \overline{q}_0 < \underline{q}_1 \le \overline{q}_1 < \underline{q}_2 \le \overline{q}_2 < \ldots < \underline{q}_m \le \overline{q}_m < \underline{q}_{m+1} = +\infty$ such that for $1 \le j \le m$, $r^+(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^+(q)$ is strictly increasing elsewhere.*

Lemma 1 also holds for $r^-(q)$. That is, for $1 \le j \le m$, $r^-(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^-(q)$ is strictly increasing elsewhere. However, the threshold values $\underline{q}_j$ and $\overline{q}_j$ may be different from that for $r^+(q)$. Now combining this observation and Proposition 1 we arrive at the following result. With a slight abuse of notations, we still use $\underline{q}_j$ and $\overline{q}_j$ to denote the new threshold values.

Proposition 2. *There exist $-\infty = \overline{q}_0 < \underline{q}_1 \le \overline{q}_1 < \underline{q}_2 \le \overline{q}_2 < \ldots < \underline{q}_m \le \overline{q}_m < \underline{q}_{m+1} = +\infty$ such that for $1 \le j \le m$, $r^*(q) = r_j$ on $[\underline{q}_j, \overline{q}_j]$ and $r^*(q)$ is strictly increasing elsewhere.*

Graphically, as illustrated in Figure 1, Proposition 2 suggests a ladder shape of $r^*(q)$. That is, $r^*(q)$ alternates between a strictly increasing piece and a constant piece with the constant piece corresponding to a kink point of $g(\cdot)$. Clearly, if we can compute the threshold values and determine the expressions of the strictly increasing pieces of $r^*(q)$, then the whole expressions of $r^*(q)$ can be obtained easily. We will address how to determine $\overline{q}_{j-1}$ and $\underline{q}_j$ as well as compute $r^*(q)$ on $(\overline{q}_{j-1}, \underline{q}_j)$ in the third part of this subsection.

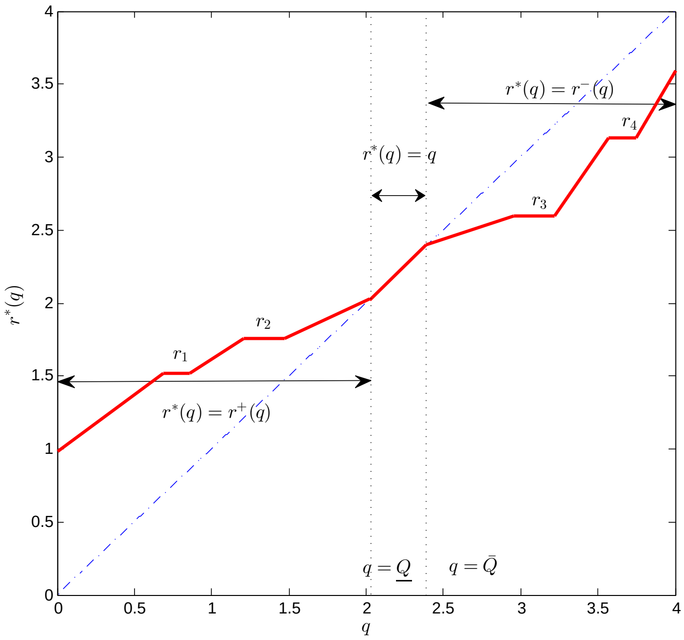


Figure 1 **Illustration of** $r^*(q)$

3.1.2. Preservation. Now we turn our attention to the optimal value function $f(q)$. Despite the fact that $\Pi(r,q)$ may be neither differentiable in r nor in q, we show that the number of kink points of $f(q)$ are less than or equal to $g(r)$ and the candidate kink points of $f(q)$ are exactly $r_1, \dots, r_m$. That is, $f(q)$ has certain similar properties as $g(r)$, which is desirable in a dynamic optimization setting.

The preservation of kink points relies critically on the following observation.

LEMMA 2. *$f^+(q)$ is continuously differentiable on $(-\infty, \underline{Q})$ and $f^-(q)$ is continuously differentiable on $(\overline{Q}, +\infty)$.*

We would like to point out that the only assumptions required for Lemma 2 to hold (or even extend to the whole domain of q) are the concavity of $f^+(q)$ and $f^-(q)$ and the continuous differentiability of $\Pi^{\pm}(r,q)$ with respect to q.

An immediate consequence of Lemma 2 is that $f(q)$ is also continuously differentiable on $(-\infty, \underline{Q})$ and $(\overline{Q}, +\infty)$. On the other hand, $f(q) = \pi(q) + g(q)$, when $q \in (\underline{Q}, \overline{Q})$ and the kink points of $f(q)$ are then solely determined by the kink points of $g(q)$. The following result further claims that $f(q)$ is differentiable at $\underline{Q}$ and $\overline{Q}$ unless $\underline{Q}, \overline{Q} \in \{r_1, \dots, r_m\}$.

PROPOSITION 3. *$f(q)$ has at most m kink points. The possible kink points are $r_1, ..., r_m$.*

In the extreme case when $m = 0$, then $f(q)$ is continuously differentiable even if the objective function is neither differentiable in the decision variable r nor in the parameter q. In contrast, many envelope theorems that study the differentiability properties of the value function of a parameterized optimization problem (see, for example, Milgrom and Segal 2002, Clausen and Strub 2012) assume the objective function to be differentiable in the parameter.

3.1.3. Computation. To address the issue of computation, we need to impose the following assumption in the remaining of this subsection on the functional forms of each continuously differentiable piece of $g(r)$.

ASSUMPTION 3. *For $1 \le j \le m+1$, $g_j(r)$ has the following functional form on $[r_{j-1}, r_j]$. There exist $n_j \ge 1$ and $r_{j-1} = r_j^{(0)} < r_j^{(1)} < ... < r_j^{(n_j-1)} < r_j^{(n_j)} = r_j$ such that*

$$g_j(r) = \begin{cases} g_j^{(1)}(r), & r_j^{(0)} \le r \le r_j^{(1)}, \\ ... & \\ g_j^{(n_j)}(r), & r_j^{(n_j-1)} \le r \le r_j^{(n_j)}, \end{cases}$$

where $g_j^{(i)}(r)$, $1 \le i \le n_j$, are functions defined on $\mathbb{R}$ and have analytical forms such that the following optimization problems

$$\max_r \{\Pi^{\pm}(r, q) + g_j^{(i)}(r)\},$$

can be solved in $O(1)$ time.

One example satisfying Assumption 3 is the case when $\Pi^{\pm}(r, q)$ and $g_j^{(i)}(r)$ are all quadratic functions. For convenience, for $1 \le j \le m+1$, we call $r_j^{(i)}$, $1 \le i \le n_j$ (except $r_{m+1}^{(n_{m+1})} = r_{m+1} = +\infty$) the *breakpoints* of $g(r)$ and denote $n = \sum_{j=1}^{m+1} n_j - 1$ to be the total number of breakpoints of $g(\cdot)$. Note that the breakpoints include the kink points we defined in Assumption 1. The key difference here is that at a breakpoint, the analytical forms of the function on the two sides of the breakpoint are different. However, different analytical forms may still result in the left derivative at the breakpoint having the same value as the right derivative. When the left derivative does not equal to the right derivative at the breakpoint, we then call this breakpoint as a kink point.

By Proposition 2, on $(\overline{q}_{j-1}, \underline{q}_j)$, $r^*(q)$ is strictly increasing and $r_{j-1} < r^*(q) < r_j$. As a result, there exist $\overline{q}_{j-1} = q_j^{(0)} < q_j^{(1)} < ... < q_j^{(n_j-1)} < q_j^{(n_j)} = \underline{q}_j$, such that for $1 \le i \le m$, on $(q_j^{(i-1)}, q_j^{(i)})$, $r_j^{(i-1)} < r^*(q) < r_j^{(i)}$. This suggests that on $(q_j^{(i-1)}, q_j^{(i)})$,

$$f(q) = \max_r \{\Pi(r,q) + g_j^{(i)}(r)\},$$

$$r^*(q) = \arg\max_r \{\Pi(r,q) + g_j^{(i)}(r)\}.$$

Note that this observation also applies to $r^+(q)$ and $r^-(q)$ on the regions where they are strictly increasing. Applying Proposition 1, by comparing q with the threshold values $\underline{Q}, \overline{Q}$, whether $\Pi(r,q) = \Pi^+(r,q)$, $\Pi(r,q) = \Pi^-(r,q)$ or $r^*(q) = q$ can be determined unambiguously in the above problem. This leads to the following result to bound the breakpoints of $f(q)$.

PROPOSITION 4. *$f(q)$ has at most $n+m+2$ breakpoints. The candidates for the breakpoints are $q_j^{(i)}$, $1 \le j \le m+1$, $0 \le i \le n_j$ (except $q_1^{(0)} = \overline{q}_0 = -\infty$ and $q_{m+1}^{(n_{m+1})} = \underline{q}_{m+1} = +\infty$) and $\underline{Q}, \overline{Q}$.*

In actual computation, however, the breakpoints are not known before hand. Algorithm 1 provides a way to both compute the breakpoints and determine the analytical form of the function between the breakpoints.

To see the computational complexity, first note that by Assumption 3, the optimization problem inside the loops only needs $O(1)$ time to solve. Consequently, the expressions for $r^{\pm}(q)$ can be determined in $O(\sum_{j=1}^{m+1} n_j) + O(m) = O(n+m)$ time. Finding $\underline{Q}$, $\overline{Q}$ clearly depends on the number of analytical pieces of $r^{\pm}(q)$ and takes $O(n+m)$ time. Finally, by Proposition 4 there are at most $n+m+2$ breakpoints, $r^*(q)$ and $f(q)$ can be computed in $O(n+m)$ time. In summary, the overall computational complexity is $O(n+m)$.

3.2. Dynamic Programming Algorithm

In this subsection, we verify that Assumptions 1-3 hold in our dynamic pricing problem and we show how to utilize Algorithm 1 as a subroutine in our dynamic programming algorithm.

To verify Assumption 2, we need the following technical condition on the problem parameters.

Algorithm 1

for $j = 1, ..., m+1$ **do**

for $i = 1, ..., n_j$ **do**

solve

$$r_{j,\pm}^{(i)}(q) = \arg\max_r \{\Pi^{\pm}(r,q) + g_j^{(i)}(r)\}$$

set

$$q_{j,\pm}^{(i-1)} = \{q : r_{j,\pm}^{(i)}(q) = r_j^{(i-1)}\}$$

$$q_{j,\pm}^{(i)} = \{q : r_{j,\pm}^{(i)}(q) = r_j^{(i)}\}$$

end for

end for

let

$$r^{\pm}(q) = \begin{cases} r_{j,\pm}^{(i)}(q), & q \in [q_{j,\pm}^{(i-1)}, q_{j,\pm}^{(i)}], \quad j = 1, ..., m+1, \quad i = 1, ..., n_j \\ r_j, & q \in [q_{j,\pm}^{(n_j)}, q_{j+1,\pm}^{(0)}], \quad j = 1, ..., m \end{cases}$$

set

$$\underline{Q} = \sup\{q : q - r^{+}(q) < 0\}$$

$$\overline{Q} = \inf\{q : q - r^{-}(q) > 0\}$$

let

$$r^*(q) = \begin{cases} r^{+}(q), & q < \underline{Q}, \\ q, & \underline{Q} \le q \le \overline{Q}, \\ r^{-}(q), & q > \overline{Q}. \end{cases}$$

ASSUMPTION 4.

$$\eta^- - \eta^+ \leq 2a_t - 2\alpha a_{t+1}, \quad \forall \quad 1 \leq t \leq T-1.$$

Assumption 4 holds under the plausible setting when consumers have short memories (α is small) and the direct price effect dominates the reference price effect ($\eta^+ < \eta^- \leq a_t$). We will impose Assumption 4 throughout this subsection. However, one should keep in mind that Assumption 4 is merely a sufficient condition to guarantee that Assumption 2 holds. As we will show in Section 6 that Assumption 2 can be verified in an on-line fashion as the algorithm implements and may still hold even if Assumption 4 fails.

Even though the per-period profit function $\Pi_t(r,q)$ is not jointly concave in r and q, the following proposition shows iteratively that under Assumption 4, the function G_t is strongly concave.

PROPOSITION 5. *Suppose Assumption 4 holds. Then for $2 \leq t \leq T+1$, G_t is strongly concave with concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. That is, $\hat{G}_t(r) = G_t(r) + A_t r^2$ is also a concave function. Furthermore, there exist $\underline{r}_t, \overline{r}_t$, such that $G_t(r)$ follows the form*

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty), \end{cases}$$

where $g_t(r)$ a continuous function defined on the whole real line.

Before verifying Assumptions 1-3, we show how one can solve problem (3) from the solutions to problem (4). The following lemma helps us in establishing the connections among problems (3), (4) and (5). Let $p_c^*(q) = \frac{q - \alpha r_c^*(q)}{1-\alpha}$.

LEMMA 3. *The solution to problem (5): $r_c^*(q)$ is single-valued and continuous in q. Furthermore, both $r_c^*(q)$ and $p_c^*(q)$ are monotonically increasing in q.*

Now we construct the solutions to problem (3) from the solutions to problem (4) in the following two steps.

From $r^*(q)$ to $r_c^*(q)$: By monotonicity of $p_c^*(q)$, when $\alpha > 0$, we know there exist q_L, q_U, $q_L < q_U$ such that for $q < q_L$, $p_c^*(q) = L_t$ and $r_c^*(q) = \frac{q-(1-\alpha)L_t}{\alpha}$, for $q_U < q$, $p_c^*(q) = U_t$ and $r_c^*(q) = \frac{q-(1-\alpha)U_t}{\alpha}$, while for $q_L < q < q_U$, $L_t < p_c^*(q) < U_t$. When $\alpha = 0$, we can set $q_L = -\infty$ and $q_U = +\infty$. It follows that

$$r_c^*(q) = \begin{cases} \dfrac{q-(1-\alpha)L_t}{\alpha}, & q \in (-\infty, q_L), \\ r^*(q), & q \in [q_L, q_U], \\ \dfrac{q-(1-\alpha)U_t}{\alpha}, & q \in (q_U, +\infty), \end{cases} \tag{9}$$

and

$$f_c(q) = \begin{cases} \Pi_t(\dfrac{q-(1-\alpha)L_t}{\alpha}, q) + g_t(\dfrac{q-(1-\alpha)L_t}{\alpha}), & q \in (-\infty, q_L), \\ f(q), & q \in [q_L, q_U], \\ \Pi_t(\dfrac{q-(1-\alpha)U_t}{\alpha}, q) + g_t(\dfrac{q-(1-\alpha)U_t}{\alpha}), & q \in (q_U, +\infty). \end{cases} \tag{10}$$

Finally, when $\alpha > 0$, q_L and q_U can be computed through $q_L = \sup\{q : r^*(q) > \frac{q-(1-\alpha)L_t}{\alpha}\}$ and $q_U = \inf\{q : r^*(q) < \frac{q-(1-\alpha)U_t}{\alpha}\}$.

From $r_c^*(q)$ to $r_t(q)$: First, note that when $\alpha > 0$, the price constraint is equivalent to $r \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$. If $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}] \cap [\underline{r}_t, \overline{r}_t] = \emptyset$, then $G_{t+1}(q) = -\infty$. That is, the effective domain for $G_{t+1}(q)$ is $[\alpha\underline{r}_t + (1-\alpha)L_t, \alpha\overline{r}_t + (1-\alpha)U_t]$. When $\alpha = 0$, clearly, the effective domain for $G_{t+1}(q)$ is simply $[L_t, U_t]$.

Let us restrict our attention to the effective domain of $G_{t+1}(q)$, i.e., $q \in [\alpha\underline{r}_t + (1-\alpha)L_t, \alpha\overline{r}_t + (1-\alpha)U_t]$.

If $\underline{r}_t \le r_c^*(q) \le \overline{r}_t$, then clearly $r_t(q) = r_c^*(q)$.

If $r_c^*(q) < \underline{r}_t$, since $r_c^*(q) \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ (or $(-\infty, +\infty)$ if $\alpha = 0$) and q is in the effective domain, we must have $\underline{r}_t \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ (or $(-\infty, +\infty)$ if $\alpha = 0$) as well. Thus, $\underline{r}_t$ is a feasible solution. From Proposition 5, we know that the objective function in problem (3) is concave and consequently we have $r_t(q) = \underline{r}_t$. Similarly, if $r_c^*(q) > \overline{r}_t$, then $r_t(q) = \overline{r}_t$.

Finally, let $\underline{q}_r = \max\{\sup\{q : r_c^*(q) < \underline{r}_t\}, \alpha\underline{r}_t + (1-\alpha)L_t\}$ and $\overline{q}_r = \min\{\inf\{q : r_c^*(q) > \overline{r}_t\}, \alpha\overline{r}_t + (1-\alpha)U_t\}$, by monotonicity of $r_c^*(q)$ and the discussion above, it follows that

$$r_t(q) = \begin{cases} \underline{r}_t, & q \in [\alpha\underline{r}_t + (1-\alpha)L_t, \underline{q}_r), \\ r_c^*(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \overline{r}_t, & q \in (\overline{q}_r, \alpha\overline{r}_t + (1-\alpha)U_t], \end{cases} \tag{11}$$

and

$$G_{t+1}(q) = \begin{cases} -\infty, & q \in (-\infty, \alpha\underline{r}_t + (1-\alpha)L_t), \\ \Pi_t(\underline{r}_t, q) + g_t(\underline{r}_t), & q \in [\alpha\underline{r}_t + (1-\alpha)L, \underline{q}_r), \\ f_c(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \Pi_t(\overline{r}_t, q) + g_t(\overline{r}_t), & q \in (\overline{q}_r, \alpha\overline{r}_t + (1-\alpha)U_t] \\ -\infty, & q \in (\alpha\overline{r}_t + (1-\alpha)U_t, +\infty). \end{cases} \tag{12}$$

The above two steps allow us to compute $r_t(q)$ and $G_{t+1}(q)$ from $r^*(q)$ and $f(q)$ if the expressions for the latter two are known. An illustration of $r_2(q)$ is provided in the figure below. In Figure 2, $r_2(q)$ is plotted on its effective domain $[\alpha\underline{r}_2 + (1-\alpha)L, \alpha\overline{r}_2 + (1-\alpha)U] = [\underline{q}_r, \overline{q}_r]$ and different colors correspond to different linear pieces. One can clearly see the structure of the constrained solution $r_c^*(q)$ as characterized in (9) and the structures of the unconstrained solution $r^*(q)$ as demonstrated in Proposition 1 and Proposition 2.

Indeed, the structure demonstrated in Figure 2 is what guarantees an efficient computation for the expressions of $r^*(q)$ and $f(q)$. Next, we will verify that Assumptions 1-3 hold and such structure is preserved through dynamic programming. Assumption 1 (a) clearly holds since $\Pi_t(r,q)$ can be expressed as a minimum of two quadratic functions that is strictly concave in r and from the proof of Lemma 3, we know $\Pi_t(r,q)$ is strictly supermodular as well. Proposition 5 already proves that Assumption 2 (a) is satisfied. In the following, we show that the remaining assumptions also hold.

Proposition 6. *$q - r^*(q)$ satisfies the single crossing property.*

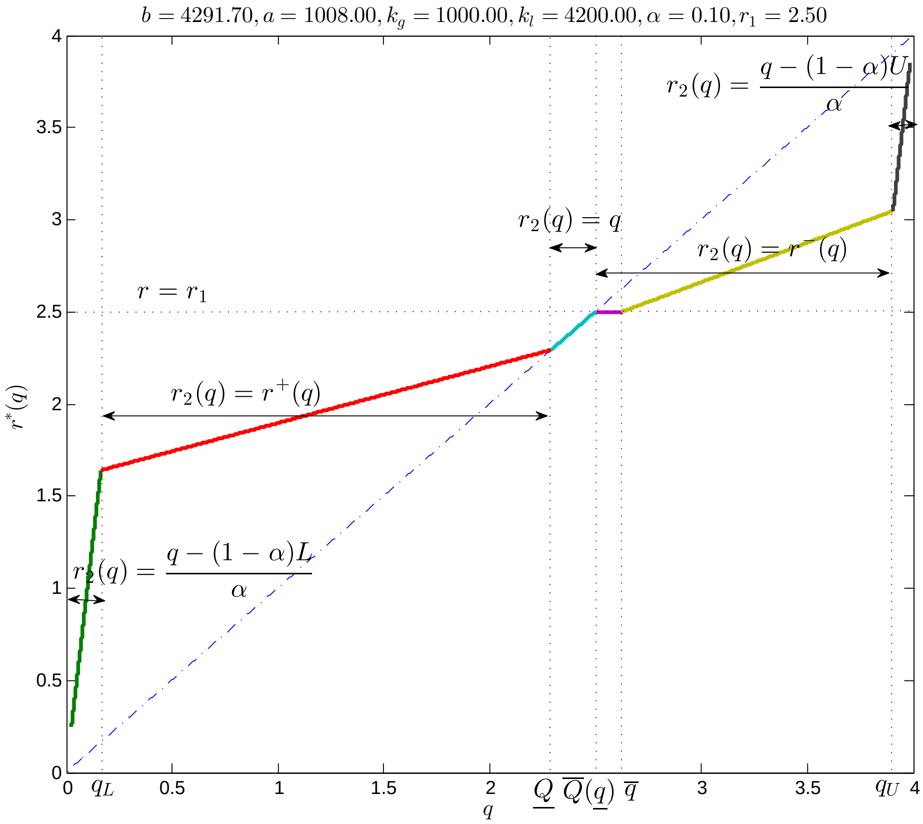


Figure 2 **Illustration of** $r_2(q)$

Proposition 6 verifies that Assumption 2 (b) holds. The proof relies on a careful analysis of both the right and left derivative of the objective function in (4).

Next, we show iteratively that for $2 \le t \le T+1$, $g_t(r)$ consists of finite number of quadratic pieces. Combined with Proposition 5, this shows that Assumption 1 (b) holds. Also, as $\Pi_t(r,q)$ consists of two quadratic functions, Assumption 3 holds as well.

PROPOSITION 7. *If $g_t(r)$ consists of n_t+1 quadratic pieces, and it has n_t breakpoints and m_t kink points ($m_t \le n_t$), then $g_{t+1}(r)$ consists of at most n_t+m_t+7 quadratic pieces, and it has at most n_t+m_t+6 breakpoints and m_t+4 kink points. That is,*

$$n_{t+1} \le n_t + m_t + 6, \quad m_{t+1} \le m_t + 4.$$

Proposition 7 not only shows that $g_t(r)$, $2 \le t \le T$, are all piece-wise quadratic functions, but also provides a bound on the growth of the number of both breakpoints as well as kink points.

With all the assumptions satisfied, we are ready to present Algorithm 2 that solves the dynamic pricing problem (3).

Algorithm 2

Initialize $\underline{r}_2 = \alpha r_1 + (1-\alpha)L_1$, $\overline{r}_2 = \alpha r_1 + (1-\alpha)U_1$, $n_2 = 1$, $m_2 = 1$, $g_2(r) = \Pi_1(r_1, r)$ and

$$G_2(r) = \begin{cases} g_2(r), & r \in [\underline{r}_2, \overline{r}_2], \\ -\infty, & r \in (-\infty, \underline{r}_2) \cup (\overline{r}_2, +\infty). \end{cases}$$

for $t = 2, ..., T$ **do**

step 1: Implement Algorithm 1 to solve problem (4) and output $r^*(q)$ and $f(q)$.

step 2: Compute $r_c^*(q)$ and $f_c(q)$ from $r^*(q)$ and $f(q)$ according to (9) and (10).

step 3: Compute $r_t(q)$ and $G_{t+1}(q)$ from $r_c^*(q)$ and $f_c(q)$ according to (11) and (12).

set $\underline{r}_{t+1} = \alpha \underline{r}_t + (1-\alpha)L_t$, $\overline{r}_{t+1} = \alpha \overline{r}_t + (1-\alpha)U_t$ and

$$g_{t+1}(q) = \begin{cases} \Pi_t(\underline{r}_t, q) + g_t(\underline{r}_t), & q \in (\infty, \underline{q}_r), \\ f_c(q), & q \in [\underline{q}_r, \overline{q}_r], \\ \Pi_t(\overline{r}_t, q) + g_t(\overline{r}_t), & q \in (\overline{q}_r, +\infty). \end{cases}$$

end for

To see the computational complexity, step 1 in Algorithm 2 clearly requires $O(n_{t+1})$ time by our analysis of Algorithm 1 and Proposition 7. Similarly, since the number of quadratic pieces of $g_{t+1}(q)$ and the number of the linear pieces of $r_{t+1}(q)$ are bounded by n_{t+1}, step 2 and step 3 can also be computed in $O(n_{t+1})$ time. Note that by Proposition 7, $m_{T+1} = O(T)$ and $n_{T+1} = O(T^2)$. Therefore, the overall computational complexity is then $O(\sum_{t=2}^{T} n_{t+1}) = O(T^3)$.

4. The Gain-seeking Case

When consumers are gain-seeking, i.e., $\eta^+ > \eta^-$, the per-period profit function $\pi_t(r,p)$ or $\Pi_t(r,q) = \pi_t(r, \frac{q-\alpha r}{1-\alpha})$ is not even component-wise concave, and consequently the value-to-go function $G_t(r)$

in the dynamic programming formulation (3) may not be concave anymore. Non-concavity creates significant difficulties in terms of algorithm design. In this subsection, we will focus instead on a special case by imposing the following assumption.

Assumption 5. *Price of the previous period serves as the reference price ($\alpha = 0$) and demands are insensitive to losses ($\eta^- = 0$).*

Interestingly, this case is similar to the special case, the so-called 2-($K = 1, Q = \infty$) case in Ahn et al. (2007), even though we have a totally different model. Also, Assumption 5 is found in the empirical studies by Hu et al. (2015) to be a good approximation to the demands faced in practice. We now develop a strongly polynomial time algorithm for this special case under certain conditions.

First observe that if a price markup incurs at some period τ, i.e., $p_{\tau-1} < p_\tau$, then the prices before period τ have no impact on prices in later periods and they can be independently determined. Specifically, for a given price sequence $\{p_1, \cdots, p_T\}$, let $1 = \tau_1 < \cdots < \tau_N < \tau_{N+1} = T+1$ be all markup periods, i.e., $p_{t-1} < p_t$ if $t = \tau_n$ for some $1 < n \leq N$ and $p_{t-1} \geq p_t$ otherwise. Here period 1 and the artificial period $T+1$ are counted as price markup periods for convenience. For each pair of consecutive price markup periods $(\tau, \tilde{\tau}) = (\tau_n, \tau_{n+1})$ for some n, one can verify that the accumulated profit from period τ to period $\tilde{\tau} - 1$ is $\Pi_\tau(p_\tau, p_\tau) + \sum_{t=\tau+1}^{\tilde{\tau}-1} \Pi_t(p_{t-1}, p_t)$, which is independent of prices specified before period τ and after period $\tilde{\tau} - 1$.

This observation allows us to partition the planning horizon by the price markup periods, determine prices between each consecutive price markup periods independently, and then find the optimal markup period sequence with maximal total profit. For this purpose, define $\ell(\tau, \tilde{\tau})$ for any $\tau < \tilde{\tau} \leq T+1$ as

$$\begin{aligned} \ell(\tau, \tilde{\tau}) = \max_{p_t : \tau \leq t < \tilde{\tau}} \; & \Pi_\tau(p_\tau, p_\tau) + \Pi_{\tau+1}(p_\tau, p_{\tau+1}) + \cdots + \Pi_{\tilde{\tau}-1}(p_{\tilde{\tau}-2}, p_{\tilde{\tau}-1}), \\ \text{s.t.} \quad & p_\tau \geq p_{\tau+1} \geq \cdots \geq p_{\tilde{\tau}-1}, \quad p_t \in [L_t, U_t], \quad \tau \leq t < \tilde{\tau}. \end{aligned} \tag{13}$$

Note that if τ and $\tilde{\tau}$ turn out to be two consecutive price markup periods in an optimal solution to problem (2), then $\ell(\tau, \tilde{\tau})$ is exactly the maximal profit accumulated from period τ to period $\tilde{\tau} - 1$. From this observation, we construct an acyclic network $(\mathcal{V}, \mathcal{E})$ with the node set and link set

$$\mathcal{V} = \{1, 2, \ldots, T+1\}, \quad \mathcal{E} = \{(\tau, \tilde{\tau}) : 1 \leq \tau < \tilde{\tau} \leq T+1\}.$$

Moreover, let $\ell(\tau, \tilde{\tau})$ be the length of a link $(\tau, \tilde{\tau})$ in $\mathcal{E}$.

To calculate link lengths $\ell(\tau, \tilde{\tau})$, let $G_{\tau,\tau+1}(p) = \Pi_\tau(p,p)$ and recursively define $G_{\tau,t}$ for $t = \tau+1, \cdots, T$ through the optimization problem:

$$G_{\tau,t+1}(p_{t+1}) = \max_{p_t} \Pi_t(p_t, p_{t+1}) + G_{\tau,t}(p_t), \qquad \text{s.t.} \quad p_t \geq p_{t+1}, \quad p_t \in [L_t, U_t], \tag{14}$$

where we set $G_{\tau,t+1}(p_{t+1}) = -\infty$ when the feasible set of the above problem is empty. The function $G_{\tau,t}(p_t)$ can be interpreted as the maximal accumulated profit from period τ to period $t-1$ when the price at period $t-1$ is set at p_t (or equivalently the reference price of period t is set at p_t) and only price markdown is allowed. Observe here the resemblance between the problem (14) and the problem (3) and note that in (14) $\Pi_t(p_t, p_{t+1})$ is now a single quadratic piece on the feasible set, which allows for more efficient computations. Indeed, as we will show in the proof of Proposition 8 that it takes $O(T^3)$ to obtain $G_{\tau,\tilde{\tau}}(p_\tau)$ for all $(\tau, \tilde{\tau})$. The length $\ell(\tau, \tilde{\tau})$ can then be computed by maximizing the function $G_{\tau,\tilde{\tau}}(p_{\tilde{\tau}})$ over $p_{\tilde{\tau}}$, and it again takes $O(T^3)$ to compute the length for all the links.

We illustrate in the following proposition that an optimal solution to problem (2) can be derived by finding a longest path in the acyclic network $(\mathcal{V}, \mathcal{E})$ and present the computational complexity.

PROPOSITION 8. *Suppose Assumption 5 holds and $\eta^+ \leq 2a_t$ in each period. Then solving problem (2) is equivalent to finding a longest path from the origin 1 to the destination $T+1$ in the acyclic network $(\mathcal{V}, \mathcal{E})$, which contains $O(T)$ nodes and $O(T^2)$ links. Moreover, it takes $O(T^3)$ time to construct the network and $O(T^2)$ time to find a longest path.*

Similar idea of converting a dynamic pricing problem to a longest path problem has been used in Ahn et al. (2007) for their 2-($K = 1, Q = \infty$) case, where subproblems similar to (13) are also derived to obtain link lengths of the acyclic network. We point out two important differences with their paper. First, subproblems in their model are automatically concave maximization problems, while for our model, certain technical conditions on the input parameters are required. More importantly, we calculate all link lengths by iteratively obtaining functions $G_{\tau,t}(p_t)$ through (14) for all $1 \leq \tau < t \leq T+1$ in strongly polynomial time, while Ahn et al. (2007) suggest interior point methods whose running time is polynomial but not strongly polynomial in general.

Proposition 8 can be generalized to the case when there are non-zero marginal costs c_t and we refer interested readers to Hu (2012) for more details.

5. The General Problem: Heuristic

In this section, we complement our exact algorithms developed in Section 3 and Section 4 by proposing an approximation heuristic to deal with general inputs of parameters including the cases when Assumption 4 or Assumption 5 fails. In addition, we derive lower and upper bounds to the optimal objective value of problem (2) from the heuristic. To simplify our presentation, we assume a uniform feasible set $[L,U]$ for prices at all periods and $r_1 \in [L,U]$. This assumption is made without loss of generality since we can always let $L = \min_{1\leq t\leq T}\{L_t\}$, $U = \max_{1\leq t\leq T}\{U_t\}$ and assume $\pi_t(r_t,p_t) = -\infty$ when $p_t \notin [L_t,U_t]$. With this assumption, we observe from (1) that $r_t \in [L,U]$ for all $t \geq 2$.

We restrict reference prices to a predetermined finite set $\mathcal{R}_\varepsilon \subset [L,U]$ in our heuristic. To achieve a reasonable accuracy, the set $\mathcal{R}_\varepsilon$ is chosen as below with some positive scalar ε,

$$\max\{\min\{|r - r_\varepsilon| : \ r_\varepsilon \in \mathcal{R}_\varepsilon\} : r \in [L,U]\} \leq \frac{1}{2}\varepsilon,$$

That is, the distance between sets $\mathcal{R}_\varepsilon$ and any point in $[L,U]$ is no more than $\frac{1}{2}\varepsilon$. As a simple instance, we can set $\mathcal{R}_\varepsilon = \{L + (n-1)\varepsilon : n \leq S_\varepsilon\}$, where S_ε is the number of elements in $\mathcal{R}_\varepsilon$ and is given by the integer part of $\varepsilon^{-1}(U-L)$.

Given the finite set $\mathcal{R}_\varepsilon$, we consider the following problem

$$V^\varepsilon = \max_{p_t,r_t:1\leq t\leq T} \sum_{t=1}^{T} \pi_t(r_t,p_t) \tag{15}$$

$$\text{s.t.} \quad \left|r_{t+1} - \alpha r_t - (1-\alpha)p_t\right| \leq \varepsilon, \qquad t = 1,\cdots,T-1, \tag{16}$$

$$r_t \in \mathcal{R}_\varepsilon, \quad p_t \in [L,U], \qquad t = 1,\cdots,T.$$

In the above problem, reference prices are restricted in $\mathcal{R}_\varepsilon$. In addition, the reference price evolution equation (1) is approximated by (16). Clearly, ε controls the accuracy of the approximation and smaller ε leads to better approximation to problem (2). We will construct lower and upper bounds for problem (2) based on problem (15).

Another lower bound to problem (2) is given by the following problem

$$\begin{aligned} V_0^\varepsilon = & \max_{p_t, r_t: 1\le t\le T} \sum_{t=1}^{T} \pi_t(r_t, p_t), \\ \text{s.t.} \quad & r_{t+1} = \alpha r_t + (1-\alpha)p_t, \qquad t = 1, \cdots, T-1, \\ & r_t \in \mathcal{R}_\varepsilon, \ \ p_t \in [L, U], \qquad t = 1, \cdots, T, \end{aligned} \tag{17}$$

where reference prices evolve in the same way as in problem (2) but are restricted to $\mathcal{R}_\varepsilon$. Problem (17) is considered here as a complement to (15) as in many cases it can provide a tighter lower bounds on the optimal value of problem (2).

The advantage of restricting the reference price to a finite set is its tractability. Indeed, we now show that both problems (15) and (17) can be solved in polynomial time in terms of T and the number of elements in $\mathcal{R}_\varepsilon$.

We start with problem (15). Define an acyclic network $(\mathcal{V}, \mathcal{E})$ by

$$\mathcal{V} = \big\{(1, r_1)\big\} \cup \big\{(t, r) : t \le T, r \in \mathcal{R}_\varepsilon\big\} \cup \big\{(T+1, *)\big\}, \quad \mathcal{E} = \big\{\, \langle (t, r), (t+1, \tilde{r})\rangle : (t, r), (t+1, \tilde{r}) \in \mathcal{V}\big\},$$

where $(T+1, *)$ is the artificial node and the symbol "*" denotes an arbitrary value since no reference price is specified at period $T+1$. The length of the link $\langle (t, r), (t+1, \tilde{r})\rangle$ is defined to be the optimal value of the following optimization problem

$$\begin{aligned} & \max_{p_t} \pi_t(r, p_t) \\ \text{s.t.} \ & |\tilde{r} - \alpha r - (1-\alpha)p_t| \le \varepsilon, \ \ p_t \in [L, U], \end{aligned} \tag{18}$$

where we assume the optimal value is $-\infty$ when the feasible set is empty. Observe that in this acyclic network, a longest path from node $(1, r_1)$ to node $(T+1, *)$ gives exactly the optimal solution to (15). Moreover, problem (18) can be solved in $O(1)$ time because its objective function consists of two concave quadratic pieces and its feasible set is either an interval or empty.

Since the acyclic network $(\mathcal{V}, \mathcal{E})$ contains $O(TS_\varepsilon^{\,2})$ links, it takes $O(TS_\varepsilon^{\,2})$ time to find a longest path from $(1, r_1)$ to node $(T+1, *)$.

Similar idea can be applied to problem (17) with the same network $(\mathcal{V}, \mathcal{E})$ but different link lengths. In particular, when computing the length of link $\langle (t,r),(t+1,\tilde{r})\rangle$, we replace problem (18) by

$$\begin{aligned} &\max_{p_t} \pi_t(r,p_t) \\ \text{s.t.}\quad &\tilde{r} = \alpha r + (1-\alpha)p_t, \quad p_t \in [L,U], \end{aligned}$$

which is still solvable in $O(1)$ time. Thus, all link lengths in the network $(\mathcal{V}, \mathcal{E})$ can be constructed in $O(T{S_\varepsilon}^2)$ time as well.

In summary, solving problem (15) or problem (17) is equivalent to finding a longest path in an acyclic network $(\mathcal{V}, \mathcal{E})$, which consists of $O(TS_\varepsilon)$ nodes, $O(T{S_\varepsilon}^2)$ links. Hence we end up with the following result.

Proposition 9. *Problem* (15) *and problem* (17) *can be solved in* $O(T{S_\varepsilon}^2)$ *time.*

Finally, we provide the lower and upper bounds for V^*, the optimal objective value of problem (2), based on V^ε, V_0^ε and the problem parameters.

Proposition 10. $\max\{V^\varepsilon - C_T^- \varepsilon, V_0^\varepsilon\} \le V^* \le V^\varepsilon + C_T\varepsilon$, *where* $C_T^- = 2\min\{(1-\alpha)^{-1}, T\}C_T$ *and*

$$C_T = \frac{T}{2}\max\{\eta^+, \eta^-\}U.$$

It is worth mentioning that our heuristic works for arbitrary demand models $D_t(r_t,p_t)$ besides the piecewise linear demand model. In fact, the procedure is the same by properly replacing the objective functions of optimization problems in this section. Depending on the efficiency to solve the one dimensional optimization problem (18), the computational complexity to solve (15) may be different. Similarly, depending on the specific expression of $D_t(r_t,p_t)$, the bounds for V^* may also differ slightly.

6. Numerical Study

In this section, we implement our exact algorithms and heuristic in a case study to demonstrate how they can be used to solve a practical industry problem with real data. Based on the examples from the case study, we compare the efficiency of our algorithms. Finally, we show that our exact algorithm (Algorithm 2) may still be applied even when Assumption 4 fails.

6.1. Case Study

In the following, we present our case study by utilizing the data set provided by Boatwright et al. (1999) of the Borden sliced cheese in 12 oz packages sold by retailers across the nation in the Bayesm Package of the R software. The data contains the weekly sales as well as prices of the product for up to 68 weeks. As noted by Greenleaf (1995), the linear reference price model has a multi-collinearity problem. One way to alleviate this issue is to perform linear regression with regularization, e.g., ridge regression, lasso or more generally elastic net (see James et al. 2013, for more details). For the purpose of demonstration, we choose lasso in our case study as the regularization method since it usually results in a better fit and fixes the wrong sign in parameter estimates better than ridge regression. Furthermore, we let the minimum historical price be the initial reference price for a more conservative result and we use the estimation procedure employed in Greenleaf (1995) to estimate α. The results for 6 selected retailers are reported in Table 1.

Table 1 Parameter Estimates and Profits Comparison

Stores	$\hat{\alpha}$	$\hat{b}$	$\hat{a}$	$\hat{\eta}^{+}$	$\hat{\eta}^{-}$	Π^*	Π^s	Π^h
Hartford - Stop & Shop	0.93	19811.72	-5271.96	0.00	687.33	1135400	1134700	496360
(average markup of 23.3%)						(108020)	(107800)	(46356)
Boston - Star Market	0.54	6209.50	-1585.68	0.00	1294.39	413380	412050	249740
(average markup of 40.0%)						(79217)	(78162)	(66268)
Indianapolis - Kroger Co	0.93	5019.04	-319.66	2708.75	2946.91	797670	632370	724130
Chicago - Omni	0.00	35082.59	-11799.80	10032.22	0.00	1966400	1774700	878280
(average markup of 23.0%),						(169480)	(150530)	(122640)
Balti/Wash - Giant Food Inc	0.04	13103.80	-2350.83	5259.24	0.00	1788300	1071900	1087700
Jacksonvile - Publix	0.66	3792.61	-765.08	1223.92	352.78	303880	280950	286600

Profits under marginal cost adjustments are in parenthesis.

Π^*: Profits under optimal pricing strategy.

Π^s: Profits under the pricing strategy that ignores reference price effects.

Π^h: Profits under the historical prices.

In addition to the estimates of the parameters b, a, η^+, η^- and α, we also report the profit under optimal pricing strategy Π^* computed through our exact and heuristic algorithms, the profit under the pricing strategy that ignores reference price effects Π^s (computed under the static demand model) and the profit under the historical prices Π^h. For Π^*, we compute it according to our exact algorithms when Assumption 4 or Assumption 5 is satisfied. Otherwise, our heuristic is applied to obtain an approximation.

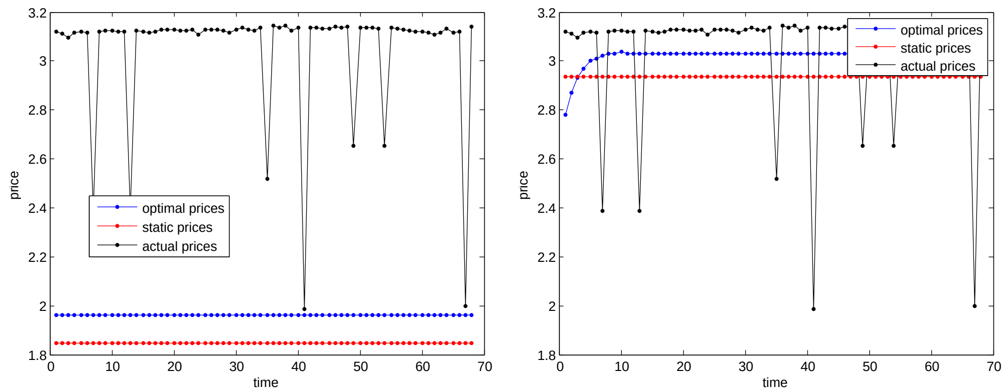
From Table 1, we can see that demands faced by different retailers can have significantly different characteristics. The first three retailers face loss-averse demands while the latter three face gain-seeking demands. Also notice that the demands for the first and second retailers satisfy Assumption 4 while the demands for the fourth retailer satisfy Assumption 5.

By comparing the profits, we can see that retailers' profits can be greatly improved in many scenarios by using the reference price models rather than the static demand models. Note that we are only presenting a basic framework in this paper and many other features such as competitors prices, seasonal effects, marginal costs, etc., are not taken into consideration since they are not available in the data set. This could be the primary reason that in most cases, the optimal pricing strategy results in unbelievable increase in profits.

In Figure 3 and Figure 4, we compare the price paths under the optimal pricing strategy, the pricing strategy that ignores reference price effects (static prices) and that of the historical prices for two retailers respectively. Panel (a) in both figures confirms that ignoring marginal costs can indeed result in an overly underpriced optimal prices for these two retailers. Unfortunately, we do not have the data on marginal costs and it is quite possible that some retailers account for marginal costs when setting up prices while others might employ a "loss-leader" strategy to drive store traffic. As a compromise, for those stores that the average optimal prices are overly underpriced compared to the average historical prices, we test several marginal costs (equivalently average markup levels) and choose the one that results in a similar average optimal prices and historical prices. For instance, for the retailer "Boston - Star Market", we find that an average markup level of 40% produces close average prices as well as price paths (see panel (b) in Figure 3 for illustration).

For the retailer "Indianapolis - Kroger Co", on the other hand, we find that average optimal prices are already above average historical prices and consequently no adjustment is made.

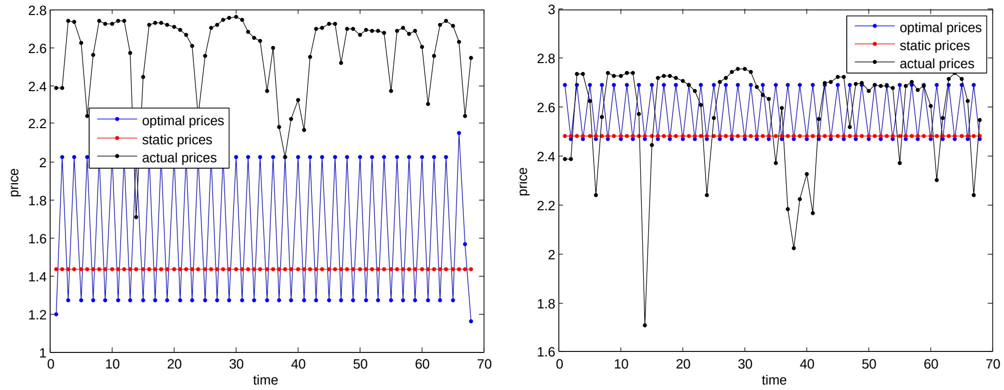
The recomputed profits based on the marginal cost adjustments are reported in the parenthesis in Table 1 and the price paths are shown in panel (b) in Figure 3 and Figure 4 for the respective retailers in panel (a). For both retailers, the optimal prices and static prices are now in close range of the historical prices and the resulting profits are also in a comparable range. With comparable prices, the underlying reasons in the profits gaps may then be explained intuitively using the existing results in the literature. In the case of retailer "Boston - Star Market", since it faces a loss-averse demand, deep price cuts in the historical prices (see Figure 3) can be very costly (Popescu and Wu 2007). On the other hand, the retailer "Chicago - Omni", is not exploiting gain-seeking effects in those periods with flat prices (see Figure 4) and consequently missing the opportunities to increase its profits (Hu et al. 2015).



(a) without marginal cost (b) with 40% average markup

Figure 3 Comparison of the price paths for retailer: Boston - Star Market

Even under marginal cost adjustment, the profit comparisons in Table 1 for the store "Hartford - Stop & Shop" can still be deemed as unrealistic, which requires the attention of additional features. We remark here that, if data available, other features that are independent from the price and reference price can be easily incorporated to improve the reference price model. The addition of other features in the regression model will only result in a time varying intercepts, i.e., b_t, for which the optimal prices can still be computed efficiently by our algorithms.



(a) without marginal cost (b) with 23% average markup

Figure 4 Comparison of the price paths for retailer: Chicago - Omni

6.2. Efficiency and Robustness

In the following, we check the efficiency and robustness of our algorithms based on some of the examples provided in our case study. All experiments below are performed in MATLAB 2014a on a desktop with an Intel Core i5-3770 CPU (3.20 GHz) and 8 GB RAM running 64-bit Windows 7 Enterprise.

In Table 2, we compare the computational efficiency of the exact algorithm (Algorithm 2) developed in Section 3 with the heuristic in Section 5. Specifically, we compare to the heuristic at different accuracy levels, one with $\varepsilon = 0.01$ and another more accurate one with $\varepsilon = 0.001$. For each algorithm, we report the CPU time (in seconds) of the computations under the time horizon ranging from 10 periods to 40 periods. The first thing to note from Table 2 is that Algorithm 2, being an exact algorithm, is much more computationally efficient than the heuristic with the running time far less than a second. Although the theoretical guarantee for the running time of Algorithm 2 is of the order $O(T^3)$, in our computational studies it scales much better. The computational time for our heuristic, on the other hand, scales linearly with respect to time horizon for a given ε as Proposition 9 shows. While this has an advantage in computing long time horizon problems, errors may accumulate over time and reducing such errors would require relatively significant amount of computational time.

Next, we show that our exact algorithm can still be applied even if the technical condition in Assumption 4 fails. Indeed, Assumption 4 is merely a sufficient condition to guarantee that

Table 2 **Comparison of the Computational Time for retailer: Boston - Star Market**

Length of Horizon	Algorithm 2 Time (s)	Heuristic ($\varepsilon = 0.01$) Time (s)	Heuristic ($\varepsilon = 0.001$) Time (s)
$T = 10$	0.066	0.505	10.913
$T = 20$	0.077	0.996	20.161
$T = 30$	0.093	1.428	30.084
$T = 40$	0.139	1.922	40.154

Assumption 2 holds in our dynamic pricing problem and in order to present a neat and interpretable condition we have been quite conservative in the derivation. In actual implementation, it is not necessary to verify Assumption 4 beforehand. Instead, since explicit expressions are computed during each iteration, one can verify Assumption 2 directly in an online fashion. For instance, we have applied Algorithm 2 to the retailer "Indianapolis - Kroger Co" for which Assumption 4 fails. We find that Assumption 2 is not violated while implementing the algorithm. As a result, Proposition 7 still holds and the efficiency of the algorithm is guaranteed. Table 3 summarizes the growth of breakpoints of the value function $G_t(q)$ for the first 20 iterations. As one can see that the growth is quadratic in t.

Table 3 **Growth of the number of breakpoints of $G_t(q)$**

	t = 2	t = 4	t = 6	t = 8	t = 10	t = 12	t = 14	t = 16	t = 18	t = 20
Indiannapolis - Kroger Co	1	9	21	39	68	104	151	211	299	446
Boston - Star Market	1	9	17	25	33	41	49	57	65	73

In comparison, the growth of breakpoints of $G_t(q)$ for retailer "Boston - Star Market", where Assumption 4 holds, is also illustrated in Table 3. Interestingly, the growth is linear instead of quadratic. That is, in addition of being a simple sufficient condition, the stronger property imposed by Assumption 4 seems to have the potential of eliminating many breakpoints, which also explains the efficiency of Algorithm 2 demonstrated in Table 2.

7. Conclusion

In this paper we examine the computational side of a dynamic pricing problem with a memory-based reference price model. In this model, demand depends on both current selling price and reference price, where the latter evolves according to an exponentially smoothing process of past prices. We identify several key structural properties to ensure such non-smooth dynamic optimization problems can be solved exactly in strongly polynomial time. In the loss-averse case, we characterize the sufficient conditions to guarantee those structural properties and develop a strongly polynomial time algorithm to solve for the optimal prices. In the gain-seeking case, we also develop a strongly polynomial time algorithm to solve the problem for a special case. For the general parameter settings, we propose a heuristic and establish lower bounds and upper bounds to the optimal objective value from the heuristic.

To complement our theoretical results, a case study is presented to demonstrate how our algorithms can be applied in a practical setting. Based on the examples in the case study, we show that our exact algorithm can be very efficient and may still be applied even when the technical conditions are violated.

Our work is only a start in looking at the computational side of the optimization models that incorporate reference price effects. We are exploring several related topics. First, further improvement of the current algorithms and heuristic is important and interesting. Efficient algorithms become pertinent as we incorporate these models into decision support systems which usually involve many products and other operations decisions.

Secondly, there is a rapidly growing literature on integrated inventory and pricing models (see Federgruen and Heching 1999, Chen and Simchi-Levi 2004a,b, Huh and Janakiraman 2008, Geunes et al. 2009, Chen and Simchi-Levi 2012). Incorporating reference price effects into such models may significantly complicate algorithm design. Efficient algorithms in these settings may then provide even more benefits to the firms facing inventory as well as pricing decisions.

Finally, we would like to extend our model to settings with multi-products and/or stochastic demand. Such models are quite relevant in practice yet challenging. Indeed, in settings with multiple products, it is not clear how dynamic pricing strategies affect the aggregated demand of a category of products and more empirical studies are necessary. In settings with stochastic demand, we have the additional challenge to deal with complex multi-dimensional stochastic dynamic programs.

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References

Ahn, H., M. Gumus, P. Kaminsky. 2007. Pricing and manufacturing decisions when demand is a function of prices in multiple periods. *Operations Research* **55**(6) 1039–1057.

Anderson, B. D. O., J. B. Moore. 2007. *Optimal control: linear quadratic methods*. Courier Corporation.

Boatwright, P., R. McCulloch, P. Rossi. 1999. Account-level modeling for trade promotion: An application of a constrained parameter hierarchical model. *Journal of the American Statistical Association* **94**(448) 1063–1073.

Briesch, R.A., L. Krishnamurthi, T. Mazumdar, S.P Raj. 1997. A comparative analysis of reference price models. *Journal of Consumer Research* **24**(2) 202–214.

Chen, X., P. Hu, S. Shum, Y. Zhang. 2015. Dynamic stochastic inventory management with reference price effects. *Working Paper* .

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research* **52**(6) 887–896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research* **29**(3) 698–723.

Chen, X., D. Simchi-Levi. 2012. Pricing and inventory management. *The Oxford handbook of pricing management. Oxford University Press, Oxford* .

Clausen, A., C. Strub. 2012. Envelope theorems for non-smooth and non-concave optimization. *Working Paper* .

Edlin, A. S., C. Shannon. 1998. Strict monotonicity in comparative statics. *Journal of Economic Theory* **81**(1) 201–219.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Operations Research* **47**(3) 454–475.

Fibich, G., A. Gavious, O. Lowengart. 2003. Explicit solutions of optimization models and differential games with nonsmooth (asymmetric) reference-price effects. *Operations Research* **51**(5) 721–734.

Geunes, J., Y. Merzifonluoğlu, H. E. Romeijn. 2009. Capacitated procurement planning with price-sensitive demand and general concave-revenue functions. *European Journal of Operational Research* **194**(2) 390–405.

Gimpl-Heersink, D. I. L. 2008. Joint pricing and inventory control under reference price effects. Ph.D. thesis, Vienna University of Economics and Business Administration.

Greenleaf, E.A. 1995. The impact of reference price effects on the profitability of price promotions. *Marketing Science* **14**(1) 82–82.

Hardie, B.G.S., E.J. Johnson, P.S. Fader. 1993. Modeling loss aversion and reference dependence effects on brand choice. *Marketing Science* **12**(4) 378–394.

Hu, P. 2012. Coordinated pricing and inventory management. Ph.D. thesis, University of Illinois at Urbana-Champaign.

Hu, Z., X. Chen, P. Hu. 2015. Dynamic pricing with gain-seeking refernece price effects. *Working Paper* .

Huh, W. T., G. Janakiraman. 2008. (s, S) optimality in joint inventory-pricing control: An alternate approach. *Operations Research* **56** 783–790.

James, G., D. Witten, T. Hastie, R. Tibshirani. 2013. *An Introduction to Statistical Learning with Applications in R.* Springer, New York.

Kalyanaram, G., J.D.C. Little. 1994. An empirical analysis of latitude of price acceptance in consumer package goods. *Journal of Consumer Research* **21**(3) 408–418.

Kopalle, P.K., A.G. Rao, J.L. Assuno. 1996. Asymmetric reference price effects and dynamic pricing policies. *Marketing Science* **15**(1) 60–85.

Kopalle, P.K., R.S. Winer. 1996. A dynamic model of reference price and expected quality. *Marketing Letters* **7**(1) 41–52.

Krishnamurthi, L., T. Mazumdar, S. P. Raj. 1992. Asymmetric response to price in consumer brand choice and purchase quantity decisions. *The Journal of Consumer Research* **19**(3) 387–400.

Mazumdar, T., S. P. Raj, I. Sinha. 2005. Reference price research: Review and propositions. *Journal of Marketing* **69**(4) 84–102.

Milgrom, P., I. Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica* **70**(2) 583–601.

Nasiry, J., I. Popescu. 2011. Dynamic pricing with loss-averse consumers and peak-end anchoring. *Operations Research* **59**(6) 1361–1368.

Ok, E. A. 2007. *Real analysis with economic applications*, vol. 10. Princeton University Press.

Popescu, I., Y. Wu. 2007. Dynamic pricing strategies with reference effects. *Operations Research* **55**(3) 413–429.

Putler, D.S. 1992. Incorporating reference price effects into a theory of consumer choice. *Marketing Science* **11**(3) 287–309.

Raman, K., F.M. Bass. 2002. A general test of reference price theory in the presence of threshold effects. *Tijdschrift Voor Economie En Management* **47**(2) 205–226.

Simchi-Levi, D., X. Chen, J. Bramel. 2014. *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management, Third Edition*. Springer.

Talluri, K. T., G. J. Van Ryzin. 2006. *The Theory and Practice of Revenue Management*. Springer Science & Business Media.

Tversky, A., D. Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *The Quarterly Journal of Economics* **106** 1039–1061.

Appendix

Proof of Proposition 1

Define $\underline{Q} = \sup\{q : q - r^*(q) < 0\}$ and $\overline{Q} = \inf\{q : q - r^*(q) > 0\}$. By the single crossing property of $q - r^*(q)$ in Assumption 2, we have $\underline{Q} \le \overline{Q}$. Furthermore, when $q < \underline{Q}$, $q - r^*(q) < 0$, when $q > \overline{Q}$, $q - r^*(q) > 0$ and when $\underline{Q} \le q \le \overline{Q}$, $q - r^*(q) = 0$.

As a result, when $q < \underline{Q}$,

$$
\begin{aligned}
r^*(q) &= \arg\max_{r}\{\Pi(r,q)+g(r)\} \\
&= \arg\max_{r>q}\{\Pi(r,q)+g(r)\} \\
&= \arg\max_{r>q}\{\Pi^+(r,q)+g(r)\} \\
&= r^+(q).
\end{aligned}
$$

Similarly, when $q > \overline{Q}$, it holds $r^*(q) = r^-(q)$.

Proof of Lemma 1

First note that the strict supermodularity of $\Pi(r,q)$ in Assumption 1 implies the strict supermodularity of $\Pi^+(r,q)$. Thus, $r^+(q)$ is increasing in q. For $1 \le j \le m$, define $\underline{q}_j = \sup\{q : r^+(q) < r_j\}$ and $\overline{q}_j = \inf\{q : r^+(q) > r_j\}$.

By definition, it is clear that when $\underline{q}_j \le q \le \overline{q}_j$, $r^+(q) = r_j$.

For $1 \le j \le m+1$ and any $\overline{q}_{j-1} < q < \underline{q}_j$, we then have $r_{j-1} < r^+(q) < r_j$. Thus, there is no loss of optimality in considering the problem

$$r^+(q) = \arg\max_{r}\{\Pi^+(r,q)+g_j(r)\}.$$

Note here that the objective function is now continuously differentiable and strictly supermodular. There is no constraint imposed on r and consequently, $r^+(q)$ is always an interior solution. By Strict Monotonicity Theorem in Edlin and Shannon (1998), we then have $r^+(q)$ is strictly increasing in q on $\overline{q}_{j-1} \le q \le \underline{q}_j$, $1 \le j \le m+1$.

Proof of Proposition 2

Clearly, $r^*(q) = q$ on $[\underline{Q}, \overline{Q}]$ and is consequently strictly increasing.

On $(-\infty, \underline{Q}]$, $r^*(q) = r^+(q)$ and by Lemma 1, $r^+(q)$ can only take constant values $\{r_j | r_j \le \underline{Q}, 1 \le j \le m\}$ on the corresponding interval $[\underline{q}_j, \overline{q}_j]$ and strictly increasing elsewhere.

Similarly, on $[\overline{Q}, +\infty)$, $r^*(q) = r^-(q)$ and by Lemma 1, $r^-(q)$ can only take constant values $\{r_j | r_j \ge \overline{Q}, 1 \le j \le m\}$ on the corresponding interval $[\underline{q}_j, \overline{q}_j]$ and strictly increasing elsewhere.

Proof of Lemma 2

By Assumption 2 (a), $f(q)$ is concave. Combined with Proposition 1, this implies that $f^+(q)$ and $f^-(q)$ are concave on $(-\infty, \underline{Q})$ and $(\overline{Q}, +\infty)$ respectively.

Now that $f^+(q)$ is concave and $\Pi^+(r,q)$ is continuously differentiable. By Theorem 1 in Milgrom and Segal (2002), $f^+(q)$ is differentiable and

$$\frac{df(q)}{dq} = \Pi_q^+(r^+(q), q).$$

That is, $f^+(q)$ is continuously differentiable on $(-\infty, \underline{Q})$. The same argument applies to $f^-(q)$.

Proof of Proposition 3

By Proposition 1 and Lemma 2, we have already proved that $f(q)$ is continuously differentiable when $q < \underline{Q}$ and $q > \overline{Q}$. Also, note that when $\underline{Q} < q < \overline{Q}$, $f(q) = \pi(q) + g(q)$. Since $\pi(q)$ is continuously differentiable, whether $f(q)$ is continuously differentiable or not only depends on whether $g(q)$ is continuously differentiable or not on $\underline{Q} < q < \overline{Q}$. However, $r_1, r_2, ..., r_m$ are all the kink points of $g(q)$. Therefore, we can conclude that the only possible kink points of $f(q)$ are $\underline{Q}, \overline{Q}$ and $r_1, r_2, ..., r_m$. Next, we conclude that $r_1, r_2, ..., r_m$ are the only possible kink points of $f(q)$.

To show that $r_1, r_2, ..., r_m$ are the only possible kink points, we only need to show that when $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$ and $\overline{Q} \notin \{r_1, r_2, ..., r_m\}$, $f(q)$ is continuously differentiable at $\underline{Q}$ and $\overline{Q}$. Without loss of generality, we next show for the case when $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$, $f(q)$ is continuously differentiable at $\underline{Q}$.

Since $\underline{Q} \notin \{r_1, r_2, ..., r_m\}$ and $-\infty = r_0 < r_1 < ... < r_m < r_{m+1} = +\infty$, we know there exists $1 \le j \le m+1$, such that $r_{j-1} < \underline{Q} < r_j$. By continuity of $r^*(q)$ and Proposition 1, $r_{j-1} < r^*(\underline{Q}) = r^+(\underline{Q}) = \underline{Q} < r_j$. Thus, $r^+(\underline{Q}) = \underline{Q}$ is the solution to:

$$f^+(\underline{Q}) = \max_r \{\Pi^+(r, \underline{Q}) + g_j(r)\}.$$

By the first order condition, $\underline{Q}$ satisfies

$$\Pi_r^+(\underline{Q}, \underline{Q}) + \frac{dg_j(r)}{dr}(\underline{Q}) = 0. \tag{19}$$

Now, since $r_{j-1} < \underline{Q} < r_j$, the right derivative at $\underline{Q}$ is

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \frac{d\pi(\underline{Q})}{dq} + \frac{dg_j(\underline{Q})}{dq}.$$

On the other hand, the left derivative at $\underline{Q}$ is

$$\lim_{q \uparrow \underline{Q}} \frac{df(q)}{dq} = \Pi_q^+(\underline{Q}, \underline{Q}).$$

Using equation (19), we then have

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \frac{d\pi(\underline{Q})}{dq} - \Pi_r^+(\underline{Q}, \underline{Q}).$$

Using total derivative, we then have $\Pi_r^+(q,q) + \Pi_q^+(q,q) = \frac{d\pi(q)}{dq}$. That is,

$$\lim_{q \downarrow \underline{Q}} \frac{df(q)}{dq} = \lim_{q \uparrow \underline{Q}} \frac{df(q)}{dq},$$

which shows that $f(q)$ is continuously differentiable at $\underline{Q}$. The differentiability at $\overline{Q}$ can be proved similarly.

Proof of Proposition 5

Suppose that $G_t(r)$ is strongly concave with concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. We will next show first that $G_{t+1}(r)$ is also strongly concave. The argument we use also implies the base case $G_2(r)$ is strongly concave.

Let $\hat{G}_t(r) = G_t(r) + A_t r^2$ and $B_t = \frac{2a_t + \eta^+}{2(1-\alpha)}$, then the Bellman equation (3) can be rewritten as

$$\hat{G}_{t+1}(q) = \max_r \{\Pi_t(r,q) - A_t r^2 + B_t q^2 + (A_{t+1} - B_t)q^2 + \hat{G}_t(r)\}.$$

By inductive hypothesis, $\hat{G}_t(r)$ is concave. To prove $\hat{G}_{t+1}(q)$ is also concave, it is sufficient to prove that the objective function is jointly concave in r and q. We prove this by showing that $\Pi_t(r,q) - A_t r^2 + B_t q^2$ is a jointly concave function and $A_{t+1} - B_t \leq 0$.

Note that $\Pi_t(r,q) = \min\{\Pi_t^+(r,q), \Pi_t^-(r,q)\}$, where

$$\Pi_t^{\pm}(r,q) = -\frac{a_t + \eta^{\pm}}{(1-\alpha)^2} q^2 + \frac{2\alpha a_t + (1+\alpha)\eta^{\pm}}{(1-\alpha)^2} qr - \frac{\alpha^2 a_t + \alpha\eta^{\pm}}{(1-\alpha)^2} r^2 + \frac{b_t}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} r.$$

From the expression above, we can obtain the Hessian of $\Pi_t^{\pm}(r,q) - A_t r^2 + B_t q^2$ as

$$\frac{1}{(1-\alpha)^2}\begin{bmatrix} -2(a_t+\eta^{\pm})+2(1-\alpha)^2 B_t & 2\alpha a_t + (1+\alpha)\eta^{\pm} \\ 2\alpha a_t + (1+\alpha)\eta^{\pm} & -2(\alpha^2 a_t + \alpha\eta^{\pm}) - 2(1-\alpha)^2 A_t \end{bmatrix}.$$

Substitute the expressions of A_t and B_t, it can be verified that above matrices are diagonally dominant and consequently, $\Pi_t^{\pm}(r,q) - A_t r^2 + B_t q^2$ are jointly concave. Thus, $\Pi_t(r,q) - A_t r^2 + B_t q^2$ is also jointly concave.

To guarantee $A_{t+1} - B_t \le 0$, we need

$$\frac{2\alpha a_{t+1} + \eta^-}{2(1-\alpha)} \le \frac{2a_t + \eta^+}{2(1-\alpha)},$$

which is guaranteed by Assumption 4.

Finally, note that $\hat{G}_2(r) = \Pi_1(r_1, r) + B_1 r^2 + (A_2 - B_1) r^2$. By the fact that $\Pi_1(r_1,r) - A_1 r_1^2 + B_1 r^2$ is jointly concave as shown above, $\Pi_1(r_1,r) + B_1 r^2$ is concave in r. Thus, $\hat{G}_2(r)$ is also concave in r.

To show the second claim, first recall that

$$G_2(r) = \begin{cases} g_2(r), & r \in [\alpha r_1 + (1-\alpha)L_1, \alpha r_1 + (1-\alpha)U_1], \\ -\infty, & r \in (-\infty, \alpha r_1 + (1-\alpha)L_1) \cup (\alpha r_1 + (1-\alpha)U_1, +\infty), \end{cases}$$

which shows that the claim holds for the base case. Now suppose for $t \ge 2$,

$$G_t(r) = \begin{cases} g_t(r), & r \in [\underline{r}_t, \overline{r}_t], \\ -\infty, & r \in (-\infty, \underline{r}_t) \cup (\overline{r}_t, +\infty). \end{cases}$$

When $\alpha > 0$, the price constraint is equivalent as $r \in [\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$. If $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}] \cap [\underline{r}_t, \overline{r}_t] = \emptyset$, then $G_{t+1}(q) = -\infty$. That is, we can let $\underline{r}_{t+1} = \alpha \underline{r}_t + (1-\alpha)L_t$ and $\overline{r}_{t+1} = \alpha\overline{r}_t + (1-\alpha)U_t$. When $\alpha = 0$, clearly, $\underline{r}_{t+1} = L_t$ and $\overline{r}_{t+1} = U_t$. In summary, we then have

$$G_{t+1}(r) = \begin{cases} g_{t+1}(r), & r \in [\underline{r}_{t+1}, \overline{r}_{t+1}], \\ -\infty, & r \in (-\infty, \underline{r}_{t+1}) \cup (\overline{r}_{t+1}, +\infty). \end{cases}$$

Proof of Lemma 3

Since the objective function is strict concave in r and continuous, Maximum Theorem (Ok 2007) guarantees that $r_c^*(q)$ is single-valued and continuous in q.

For the second part, it is shown in Chen et al. (2015) that $\Pi(r,q)$ is supermodular. Also, note that the constraint set $[\frac{q-(1-\alpha)U_t}{\alpha}, \frac{q-(1-\alpha)L_t}{\alpha}]$ is ascending with q. Therefore, by Theorem 2.2.8 in Simchi-Levi et al. (2014), $r_c^*(q)$ is increasing in q. To show the monotonicity for $p_c^*(q)$, by variable transformation, problem (5) is equivalent to the following problem

$$f_c(q) = \max_p \{\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) + g_t(\frac{q-(1-\alpha)p}{\alpha}) : p \in [L_t, U_t]\},$$

$$p_c^*(q) = \arg\max_p \{\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) + g_t(\frac{q-(1-\alpha)p}{\alpha}) : p \in [L_t, U_t]\},$$

where $\Pi_t(\frac{q-(1-\alpha)p}{\alpha}, q) = p(b_t - a_t p + \eta^+(\frac{q-p}{\alpha})^+ + \eta^-(\frac{q-p}{\alpha})^-)$ can also be shown to be supermodular by Lemma 2 in Chen et al. (2015). Meanwhile, by concavity of $g_t(\cdot)$, $g_t(\frac{q-(1-\alpha)p}{\alpha})$ is supermodular in (p,q). Thus, $p_c^*(q)$ is monotonically increasing in q.

Proof of Proposition 6

Define

$$\begin{aligned}\underline{\theta}(r,q) &= \partial_r^+[\Pi_t(r,q) + g_t(r)] \\ &= -\frac{2(\alpha^2 a_t + \alpha\eta)}{(1-\alpha)^2} r + \frac{2\alpha a_t + (1+\alpha)\eta}{(1-\alpha)^2} q - \frac{b_t \alpha}{1-\alpha} + \partial^+ g_t(r),\end{aligned}$$

where $\eta = \eta^+$ if $r \geq q$ and $\eta = \eta^-$ if $r < q$, and

$$\begin{aligned}\overline{\theta}(r,q) &= \partial_r^-[\Pi_t(r,q) + g_t(r)] \\ &= -\frac{2(\alpha^2 a_t + \alpha\eta)}{(1-\alpha)^2} r + \frac{2\alpha a_t + (1+\alpha)\eta}{(1-\alpha)^2} q - \frac{b_t \alpha}{1-\alpha} + \partial^- g(r),\end{aligned}$$

where $\eta = \eta^+$ if $r > q$ and $\eta = \eta^-$ if $r \leq q$. Note that by concavity of the objective function, $\underline{\theta}(r,q)$ and $\overline{\theta}(r,q)$ are both decreasing in r and $\underline{\theta}(r,q) \leq \overline{\theta}(r,q)$.

Following the notation in the proof of Proposition 5, let $\hat{g}_t(q) = g_t(q) + A_t q^2$. By Proposition 5, $\hat{g}_t(q)$ is concave as well. It follows that

$$\begin{aligned}
\underline{\theta}(q,q) &= \frac{2\alpha a_t + \eta^+}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} + \partial^+ g_t(q), \\
&= \frac{2\alpha a_t + \eta^+}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} - 2A_t q + \partial^+ \hat{g}_t(q) \\
&= -\frac{\eta^- - \eta^+}{1-\alpha} q + \partial^+ \hat{g}_t(q) - \frac{b_t \alpha}{1-\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\theta}(q,q) &= \frac{2\alpha a_t + \eta^-}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} + \partial^- g_t(q) \\
&= \frac{2\alpha a_t + \eta^-}{1-\alpha} q - \frac{b_t \alpha}{1-\alpha} - 2A_t q + \partial^- \hat{g}_t(q) \\
&= \partial^- \hat{g}_t(q) - \frac{b_t \alpha}{1-\alpha},
\end{aligned}$$

where in the last equations we substituted the strong concavity constant $A_t = \frac{2\alpha a_t + \eta^-}{2(1-\alpha)}$. As a result, both $\underline{\theta}(q,q)$ and $\overline{\theta}(q,q)$ are decreasing in q and $\eta^- > \eta^+$ implies $\underline{\theta}(q,q)$ is actually strictly decreasing and $\underline{\theta}(q,q) < \overline{\theta}(q,q)$.

Let $\underline{Q} = \sup\{q : \underline{\theta}(q,q) > 0\}$ and $\overline{Q} = \inf\{q : \overline{\theta}(q,q) < 0\}$. By $\underline{\theta}(\underline{Q},\underline{Q}) \geq 0 \geq \overline{\theta}(\overline{Q},\overline{Q}) > \underline{\theta}(\overline{Q},\overline{Q})$, we have $\underline{Q} < \overline{Q}$. We consider the following three cases.

Case 1: If $q < \underline{Q}$, then $\underline{\theta}(q,q) > 0$ and $\underline{\theta}(r,q) > 0$ for $r \leq q$. Thus, $r^*(q) > q$.

Case 2: If $q > \overline{Q}$, then $\overline{\theta}(q,q) < 0$ and $\overline{\theta}(r,q) < 0$ for $r \geq q$. Thus, $r^*(q) < q$.

Case 3: If $\underline{Q} \leq q \leq \overline{Q}$, then $\underline{\theta}(q,q) \leq 0$ and $\overline{\theta}(q,q) \geq 0$. That is $\underline{\theta}(r,q) \leq 0$ for $r \geq q$ and $\overline{\theta}(r,q) \geq 0$ for $r \leq q$. Thus, $r^*(q) = q$.

In summary, for $q' > q''$, if $q'' > r^*(q'')$, then $q' > q'' > \overline{Q}$ and $q' > r^*(q')$. That is, the single-crossing property holds for $q - r^*(q)$.

Proof of Proposition 7

By Proposition 4, $f(q)$ has at most $n_t + m_t + 2$ breakpoints. On top of this, q_L, q_U, $\underline{q}_r$ and $\overline{q}_r$ are the new breakpoints by computing $g_{t+1}(q)$ from $f(q)$. Thus, $n_{t+1} \leq n_t + m_t + 6$.

By Proposition 3, $f(q)$ has at most m_t kink points. On top this, q_L, q_U, $\underline{q}_r$ and $\overline{q}_r$ are the only new candidate kink points by computing $g_{t+1}(q)$ from $f(q)$. Thus, $m_{t+1} \le m_t + 4$.

Proof of Proposition 8

We first verify the computational complexity to construct the network $(\mathcal{V}, \mathcal{E})$. To simplify notations, we rewrite the problem (14) as

$$G_{\tau,t+1}(q) = \max_r \Pi_t(r,q) + G_{\tau,t}(r), \qquad (20)$$
$$\text{s.t.} \quad r \ge q, \quad r \in [L_t, U_t].$$

Note here the resemblance between the problem (20) and the problem (3). However, $\Pi_t(r,q)$ in (20) is now a quadratic function on the feasible region instead of a piece-wise quadratic function in (3), which simplifies the analysis a lot.

It is straightforward to verify that $\Pi_t(r,q) - \frac{1}{2}\eta^+ r^2 + \frac{1}{2}\eta^+ q^2$ is jointly concave on $\{(r,q) : r \ge q\}$ when $\eta^+ \le 2a_t$. For any $t > \tau$, define $F_{\tau,t}(q) = G_{\tau,t}(q) + \frac{1}{2}\eta^+ q^2$ and rewrite (20) as

$$F_{\tau,t+1}(q) = \max_r F_{\tau,t}(r) - \frac{1}{2}\eta^+ r^2 + \Pi_t(r,q) + \frac{1}{2}\eta^+ q^2,$$
$$\text{s.t.} \quad r \ge q, \quad r \in [L_t, U_t].$$

Similar to the proof of Proposition 5, we know both $F_{\tau,t}(q)$ and $G_{\tau,t}(q)$ are concave. Moreover, since $\Pi_t(r,q)$ is now continuously differentiable and $G_{\tau,\tau+1}$ is also continuously differentiable, by Lemma 2, it is straightforward to verify that $G_{\tau,t}(q)$ is also continuously differentiable on its effective domain for all $\tau + 1 \le t \le \tilde{\tau}$, i.e., there are no kink points in $G_{\tau,t}(q)$. Consequently, by Proposition 7, $G_{\tau,t+1}(q)$ consists of $O(t-\tau)$ quadratic pieces and can be obtained in $O(t-\tau)$ time after $G_{\tau,t}(r)$ becomes available. Therefore, it takes $O(T^3)$ time to obtain all $G_{\tau,\tilde{\tau}}(q)$, and an additional $O(T^3)$ to obtain $\ell(\tau,\tilde{\tau})$ by maximizing $G_{\tau,\tilde{\tau}}(q)$ over q for all $1 \le \tau < \tilde{\tau} \le T+1$. Note that $G_{\tau,t}(q)$ is strictly concave when $\eta^+ > 0$, which implies that problem (13) yields a unique optimal solution.

We next show the equivalence of solving problem (2) and finding a longest path in $(\mathcal{V}, \mathcal{E})$. That is, we need to prove that the total profit incurred by an optimal price sequence is no more than the total length of some path in $(\mathcal{V},\mathcal{E})$, and there exists a longest path in $(\mathcal{V},\mathcal{E})$ with the total length no more than the profit associated with some feasible price sequence of problem (2).

On the one hand, given an optimal price sequence $\{p_1,\cdots,p_T\}$, let $1=\tau_1<\tau_2<\cdots<\tau_{N+1}=T+1$ be its price markup periods. Then for any pair of consecutive price markup periods $(\tau,\tilde{\tau})$, we have $p_{\tau-1}\le p_\tau, p_{\tilde{\tau}-1}\le p_{\tilde{\tau}}$ and $p_t\ge p_{t+1}$ when $\tau\le t<\tilde{\tau}-1$. Clearly $\{p_\tau,\cdots,p_{\tilde{\tau}-1}\}$ is feasible to problem (13), which implies the accumulated profit from period τ to $\tilde{\tau}-1$ is no more than $\ell(\tau,\tilde{\tau})$. Therefore the total profit associated with the feasible price sequence $\{p_1,\cdots,p_T\}$ is no more than total length of the path $\{\tau_1\tau_2\cdots\tau_{N+1}\}$ with $\tau_1=1$ and $\tau_{N+1}=T+1$.

On the other hand, suppose $\tau_1\tau_2\cdots\tau_N\tau_{N+1}$ with $1=\tau_1<\cdots<\tau_N<\tau_{N+1}=T+1$ is a longest path in $(\mathcal{V},\mathcal{E})$. Without loss of generality, we can assume $\ell(\tau_{n-1},\tau_n)+\ell(\tau_n,\tau_{n+1})>\ell(\tau_{n-1},\tau_{n+1})$ for any $1<n\le N$; otherwise we can replace links (τ_{n-1},τ_n) and (τ_n,τ_{n+1}) by a new link (τ_{n-1},τ_{n+1}). Let $\{p_1,\cdots,p_T\}$ be the price sequence related to the longest path, i.e., $\{p_\tau,\cdots,p_{\tilde{\tau}-1}\}$ solves problem (13) for any $(\tau,\tilde{\tau})=(\tau_n,\tau_{n+1})$, $n=1,\cdots,N$. If we can prove all τ_n are price markup periods, then this price sequence is feasible to problem (2) and its profit is the same as the length of the longest path.

Assume to the contrary that there exists a node τ on the longest path such that $p_{\tau-1}>p_\tau$. Let $\overline{\tau},\tau,\tilde{\tau}$ with $\overline{\tau}<\tau<\tilde{\tau}$ be three consecutive nodes on the longest path. Since we assume $p_{\tau-1}>p_\tau$, $\{p_{\overline{\tau}},\cdots,p_{\tilde{\tau}-1}\}$ is feasible to problem (13) with $(\tau,\tilde{\tau})$ replaced by $(\overline{\tau},\tilde{\tau})$. It follows from $\Pi_\tau(r,p)=\Pi_\tau(p,p)+p\eta(r-p)$ that

$$\begin{aligned}\ell(\overline{\tau},\tilde{\tau}) &\ge \Pi_{\overline{\tau}}(p_{\overline{\tau}},p_{\overline{\tau}})+\sum_{t=\overline{\tau}+1}^{\tau-1}\Pi_t(p_{t-1},p_t)+\Pi_\tau(p_{\tau-1},p_\tau)+\sum_{t=\tau+1}^{\tilde{\tau}-1}\Pi_t(p_{t-1},p_t)\\ &=\ell(\overline{\tau},\tau)+\ell(\tau,\tilde{\tau})+\eta^+\left[p_\tau(p_{\tau-1}-p_\tau)\right]\end{aligned}$$

where the equality follows from the definition of $\{p_{\overline{\tau}},\cdots,p_{\tau-1}\}$ and $\{p_\tau,\cdots,p_{\tilde{\tau}-1}\}$. Since $p_\tau(p_{\tau-1}-p_\tau)\ge 0$, $\ell(\overline{\tau},\tilde{\tau})\ge\ell(\overline{\tau},\tau)+\ell(\tau,\tilde{\tau})$, which contradicts the assumption that $\ell(\overline{\tau},\tilde{\tau})<\ell(\overline{\tau},\tau)+\ell(\tau,\tilde{\tau})$. It then follows that $p_{\tau-1}\le p_\tau$ and τ is indeed a price markup period.

Proof of Proposition 10

We first prove $V^*\le V^\varepsilon+C_T\varepsilon$. Suppose $\{p_t^*:t=1,\cdots,T\}$ is the optimal price sequence to problem (2). Then a feasible solution to problem (15) is constructed by keeping the price sequence unchanged (i.e. $p_t = p_t^*$). In addition, let $r_t = \arg\min\{|r - r_t^*| : r \in \mathcal{R}_\varepsilon\}$ be the nearest element in $\mathcal{R}_\varepsilon$ to r_t^* for each $t \geq 2$, which implies $|r_t - r_t^*| \leq \frac{1}{2}\varepsilon$ and

$$|r_{t+1} - \alpha r_t - (1-\alpha)p_t^*| = |r_{t+1} - r_{t+1}^* + \alpha(r_t^* - r_t)| \leq \frac{\varepsilon}{2} + \frac{\alpha\varepsilon}{2} \leq \varepsilon,$$

Therefore such sequence $\{(p_t, r_t)\}$ is feasible to problem (15). Let V be the profit associated with the feasible solution $\{(p_t, r_t) : t = 1, \cdots, T\}$. Because $|r_t - r_t^*| \leq \frac{\varepsilon}{2}$ and $p_t \leq U$, we have

$$|D_t(r_t^*, p_t^*) - D_t(r_t, p_t)| = |\eta(r_t^* - p_t^*) - \eta(r_t - p_t)| \leq \frac{1}{2}\max\{\eta^+, \eta^-\}\varepsilon.$$

which in turn implies $|V - V^*| \leq \sum_{t=1}^{T} p_t |D_t(r_t, p_t) - D_t(r_t^*, p_t^*)| \leq C_T\varepsilon$. Hence $V^* \leq V + C_T\varepsilon \leq V^\varepsilon + C_T\varepsilon$.

Since $V_0^\varepsilon \leq V^*$ is trivial, the remaining part is to show $V^\varepsilon \leq C_T^-\varepsilon + V^*$. Suppose $\{r_t^\varepsilon : t = 1, \cdots, T\}$ is the optimal reference price sequence to problem (15). Then a feasible solution to problem (2) is constructed by keeping the pricing unchanged (i.e. $p_t = p_t^\varepsilon$), and generating r_t through $r_{t+1} = \alpha r_t + (1-\alpha)p_t$. Let V be the profit associated with the feasible pricing sequence $\{(r_t, p_t) : t = 1, \cdots, T\}$ in problem (2). Since $p_t = p_t^\varepsilon$, $r_1^\varepsilon = r_1$ and

$$|r_{t+1}^\varepsilon - r_{t+1}| = |r_{t+1}^\varepsilon - \alpha r_t^\varepsilon - (1-\alpha)p_t^\varepsilon + \alpha(r_t^\varepsilon - r_t)| \leq \varepsilon + \alpha|r_t^\varepsilon - r_t|,$$

the following inequality can be proved recursively.

$$|r_t^\varepsilon - r_t| \leq \sum_{\tau=1}^{t-1} \alpha^\tau \varepsilon \leq \varepsilon \min\left\{(1-\alpha)^{-1}, (t-1)\right\},$$

which in turn implies $|D_t(r_t^\varepsilon, p_t^\varepsilon) - D_t(r_t, p_t)| \leq \max\{\eta^+, \eta^-\}\max\{(1-\alpha)^{-1}, t-1\}\varepsilon$. Thus, similar to the argument in the above paragraph, we have $V^\varepsilon \leq V + C_T^-\varepsilon \leq V^* + C_T^-\varepsilon$.