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Stabilization of Markovian Jump Linear Systems With Log-Quantized Feedback

This paper is concerned with mean-square stabilization of single-input Markovian jump linear systems (MJLSs) with logarithmically quantized state feedback. We introduce the concepts and provide explicit constructions of stabilizing mode-dependent logarithmic quantizers together with associated controllers, and a semi-convex way to determine the optimal (coarsest) stabilizing quantization density. An example application is presented as a special case of the developed framework, that of feedback stabilizing a linear time-invariant (LTI) system over a log-quantized erasure channel. A hardware implementation of this application on an inverted pendulum testbed is provided using a finite word-length approximation. [DOI: 10.1115/1.4026133]

1 Introduction

The advance of networking, sensing, and computing technologies make it possible to implement more complex and aggressive control algorithms over data and sensor networks. In order to successfully deploy these control schemes, especially in cases where embedded computing handles both control and communication alongside other tasks, such as in robotics with distributed sensing, issues such as limited resolution, packet loss, and latency must be systematically integrated into the control design process. Multi-modal system models provide a convenient way to capture many of these issues.

Markovian Jump Linear Systems (MJLSs) are frequently used to mathematically represent multi-modal stochastic systems, where the structure of the plant is subject to random changes. This is a commonly used abstraction for the hybrid automata, where the continuous state of the plant changes according to an underlying discrete-time stochastic process. Stabilization of MJLSs have been intensively studied in Refs. [1–4], and references therein, where Lyapunov type convex tests are developed for the second order *mean-square* stabilizability when measurements are received by controllers *precisely*. In this paper, we investigate the quantized control problem of MJLSs; more specifically, we study the stabilization problem of a discrete-time single-input MJLS with logarithmically quantized state feedback. The logarithmic quantization of measurements can be regarded as a convenient idealization of finite resolution or word-length effects in sensing or communication channels when the control objective is stabilization around a setpoint or minimization of a norm-based performance criterion, but is also of intrinsic analytical interest.

We start with the quadratic stabilization of a discrete-time LTI system (a single-mode MJLS) with logarithmically quantized state feedbacks. We develop a linear matrix inequality (LMI) based optimization method to approach the optimal (coarsest) stabilizing quantization density. The result coincides with that developed in Ref. [5] via a Riccati inequality approach. Then, we investigate the mean-square stabilization of MJLSs with logarithmically quantized state feedbacks. We give explicit constructions of stabilizing *mode-dependent* logarithmic quantizers and associated controllers. Necessary and sufficient conditions on stabilizing quantization densities are given along with the semi-convex optimization algorithm to approach the optimal (coarsest) value. In addition, it can also be shown (see Ref. [6]) that, in the special case where the system mode is independently identically distributed (i.i.d.), mean-square stabilization is equivalent to *stochastic quadratic stabilization* studied in Ref. [7], which is a special second order stabilizability.

The framework we developed here provides a systematic approach to model multiple channel constraints when studying control problems over communication and sensing networks, and is an extension of our work in Ref. [8]. Most past literature in this area focuses either entirely on the finite bandwidth issue, for example, Refs. [5] and [9–16]; or exclusively on the unreliable transmission problem, for example, Refs. [17–26]. It is interesting to note that the coarsest stabilizing quantization density for an LTI system with logarithmically quantized state feedbacks is only determined by unstable eigenvalues of the system matrix [5]. On the other hand, in order to stochastically stabilize an LTI system over an unreliable channel with *infinite* bandwidth, over which real numbers can be transmitted but are subject to random loss, the upper bound of packet loss probability is also characterized solely by unstable eigenvalues of the system matrix [17, 18]. Concurrent treatment of these intrinsically related constraints only appears in a few recent works; to our knowledge [7], [6], and [27]. The development of results in all these works relies heavily on the existence of one or multiple control Lyapunov function(s) (CLF). The work in Ref. [27] provides particularly sharp results for the finite word-length case, and gives the relationship between the digital channel property and system unstable modes. The approach used in the current paper can be more practical in many situations, as it can be implemented more simply and directly in hardware; namely, static analog-to-digital conversion devices can be used, as opposed to devices that must dynamically change their range.

As an example application of the framework developed here, we employ it to consider the stabilization problem of a discrete-time LTI system with logarithmically quantized state feedback over a *lossy* channel, in which packets get dropped according to a Markovian process. This particular problem is directly studied in Ref. [7], and is a composite problem possessing the combined features of Refs. [5,17], and [18]. This example setup models the situation in which measurements of the system are distorted not only by quantization due to finite bandwidth or resolution but also by packet loss due to latency or unreliability of the communication channel. Intuition says that for statistical criteria, such as mean-square stabilization, if packets get dropped more frequently, more information should be required in each packet, and vice versa. Our control policies are time-invariant and use log-quantization; practical finite-bit control laws can be obtained from these by

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proper truncation, based on the expected signal range and desired zero-tolerance. One way of deciding the truncation value is given in Ref. [7]. We use the arrival of measurements as the index of the system mode and further model the system as an MJLS with logarithmically quantized state feedback. We illustrate the trade-off between the packet loss probability and the quantization density, which supports our above intuition. We are able to recover results in Refs. [7,5,17], and [18] as extremal cases. We apply these results to an explicit laboratory testbed involving a Furuta Pendulum, demonstrating the use of our algorithms for digital control design on a nonlinear physical process. Both simulation and real Furuta pendulum test results are provided, and illustrate how the packet loss probability and quantization density may affect a real system.

The rest of the paper is organized as follows. Section 2 formulates the problem studied and introduces notation. Section 3 solves the quadratic stabilization problem of an LTI system with logarithmically quantized state feedbacks. The convex tools developed in this section is the foundation of later sections. Section 4 investigates the mean-square stabilization problem for MJLSs and develops the semi-convex algorithm to approach the coarsest quantization density. We also demonstrate the equivalence between the mean-square and the stochastic quadratic stability in this section. Then, Sec. 5 studies the example of quantized control of an LTI system over a lossy channel, and applies the results to a physical laboratory testbed. Some concluding comments are given in the final section.

2 Problem Formulation

The set of integers, non-negative integers, natural numbers, and real numbers is denoted by $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{R}$, respectively. Variables are represented by lower case letters, such as x. Matrices are denoted in upper case letters such as A, B, etc. Notation $(\star)$ in a matrix denotes the conjugate transpose of the corresponding term.

2.1 Markovian Jump Linear Systems. Given a number $N \in \mathbb{N}$, let the matrix $\mathbf{P} \in \mathbb{R}^{N\times N}$ be a *row stochastic matrix*; that is, $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for $i,j \in \{1,\cdots,N\}$. Let the vector p be a *row stochastic vector* such that $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$.

Denote Θ as a random process with finite state space $\{1,\cdots,N\}$, the transition probability matrix $\mathbf{P}$, and the initial distribution p. Let $\theta(t), t \geq 0$ be a realization of Θ. Denote Ω as the sample space of all infinite sequences $\{\theta(t), t \geq 0\}$ and define $\mathbb{P}$ as the unique consistent measure on it that satisfies

$$\begin{aligned} &\mathbb{P}\{\theta(t+1) = j | \theta(t) = i\} = p_{ij} \\ &\mathbb{P}\{\theta(0) = i\} = p_i \end{aligned} \tag{1}$$

for all $t \geq 0$, and $i,j \in \{1,\ldots,N\}$. Then the process Θ is called a discrete-time Markovian process.

Define the following finite set:

$$\mathscr{S} = \{(A_1,B_1,C_1),\cdots,(A_N,B_N,C_N)\}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n_u}$, and $C_i = \mathbb{R}^{n_y\times n}$ for all $i \in \{1,\cdots,N\}$.

The discrete-time Markovian jump linear system defined by the tuple $(\mathscr{S},\mathbf{P},p)$ has the following state-space representation

$$\begin{aligned} &x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t), \quad x(0) = x_0 \\ &y(t) = C_{\theta(t)}x(t) \end{aligned} \tag{2}$$

When $\theta(t) = i$, the system is said to be in *mode* i at time t.

We assume all random variables are on the same probability space.

If $B_i \neq 0$ for some $i \in \{1,\ldots,N\}$, then the MJLS $(\mathscr{S},\mathbf{P},p)$ is called a controlled Markovian jump linear system (C-MJLS).

Remark 1. In this paper, we restrict ourselves to scalar-input systems with state feedbacks; that is, $B_i \in \mathbb{R}^{n\times 1}$ and $C_i = I_n$, the n-dimensional identity matrix, for all $i = 1,\cdots,N$.

Given the following state feedback controller $\mathscr{H}$,

$$u(t) = H(t)x(t)$$

the closed-loop system $\Sigma(\mathscr{S},\mathscr{H},\mathbf{P},p)$ is given by

$$x(t+1) = \big(A_{\theta(t)} + B_{\theta(t)}H(t)\big)x(t), \quad x(0) = x_0 \tag{3}$$

We define an *autonomous Markovian jump linear system* (A-MJLS) to be of the following form

$$x(t+1) = A_{\theta(t)}x(t), \quad x(0) = x_0 \tag{4}$$

where the matrix A_i and the transition sequence θ are defined previously.

Definition 1. *A transition sequence θ of an MJLS $(\mathscr{S},\mathbf{P},p)$ is* admissible *if it satisfies*

1 $p_{\theta(0)} > 0$
2 $p_{\theta(t)\theta(t+1)} > 0$ for any $t \geq 0$

Any finite transition sequence $(\theta(t_0),\cdots,\theta(t))$ is admissible if θ is.

2.2 Logarithmic Quantizers

Definition 2. *A logarithmic quantizer with density $\rho \in (0,1)$ is a function $\mathcal{Q} : \mathbb{R} \to \mathbb{Z} \times \{-1,0,1\}$, given by*

1 If $x = 0$, $\mathcal{Q}(0) = (0,0)$.
2 If $x \neq 0$, $\mathcal{Q}(x) = (n, \mathrm{sgn}(x))$, when $\rho^{n+1} < |x| \leq \rho^n$, and $n \in \mathbb{Z}$.

When $\rho = 1$ or $\rho = 0$, we define $\mathcal{Q}(x) = x$ or $\mathcal{Q}(x) = 0$, respectively, to extend the above definition. The logarithmic quantizer defined here does not saturate; in other words, it is able to quantize the whole real line. The second element of the output identifies the sign of x.

Definition 2 defines a one-dimensional logarithmic quantizer on the real line. We can extend this concept to the n-dimensional space $\mathbb{R}^n$ as well by defining a one-dimensional logarithmic quantizer in the following way: first choose a vector $H_q \in \mathbb{R}^n$; then, for all $x \in \mathbb{R}^n$, define $\mathcal{Q}(x) = \mathcal{Q}(H_q^{\mathrm{T}}x)$. This is equivalent to quantize the projection of $x \in \mathbb{R}^n$ on the one-dimensional subspace generated by the vector H_q, which is called the *quantization direction*.

It is also possible to design higher-order quantizers which quantize the projection of $x \in \mathbb{R}^n$ on a higher order subspace. However, we are not interested in such quantizers in this paper since for a scalar-input system, one-dimensional quantizers are sufficient and optimal [5].

We now define the notion of an exponential decoder.

Definition 3. *An exponential decoder with density $\rho \in (0,1)$ is a function $\hat{x} : \mathbb{Z} \times \{-1,0,1\} \to \mathbb{R}$, which satisfies*

(1) $\hat{x}(0,0) = 0$
(2) $\hat{x}(n,\pm 1) = \mp\rho^n$, for $n \in \mathbb{Z}$

By combining a logarithmic quantizer $\mathcal{Q}$ with density ρ, an exponential decoder with the same ρ, and a constant gain β_u, we define the *log-controller* as follows.

Definition 4. *A static log-controller with density $\rho \in (0,1)$ and constant gain $\beta_u > 0$ is a function $u : \mathbb{R} \to \mathbb{R}$, such that*

$$u(x) = -\mathrm{sgn}(x)\beta_u\rho^{\lfloor \log_\rho |x| \rfloor}$$

where $\lfloor\cdot\rfloor$ is the standard floor function, rounding down to the closest smaller integer.

A schematic view and an example of the static log-controller are given in Figs. 1 and 2, respectively.

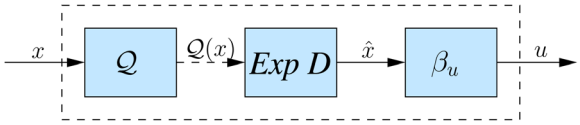

Fig. 1 A log-controller

Remark 2. In later sections, we make the constant gain β_u a function of the quantization density ρ; then the parameter ρ completely specifies the log-controller.

In this paper, we will investigate the stabilization problem of a C-MJLS with a set of log-controllers. The coarsest stabilizing quantization density is solved via semi-convex optimization.

3 Quadratic Stabilization of Linear Time-Invariant Systems

The quadratic stabilization problem of a scalar-input LTI system with logarithmically quantized state feedbacks is studied in Ref. [5]. The coarsest quantization density is approached by solving an optimization problem associated with an *algebraic Riccati equation* (ARE). In this section, we provide a convex alternative by solving an LMI powered optimization problem. This is the cornerstone of the semi-convex optimization in later sections.

Consider the following discrete-time LTI system

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{5}$$

where $x \in \mathbb{R}^n$ is the state variable, and $u \in \mathbb{R}$ is the scalar control. Matrices A, B are of compatible dimensions and (A, B) is controllable. State feedback is assumed. Initial state $x_0 \in \mathbb{R}^n$ is assumed unknown but deterministic. We note this is a special C-MJLS where there is only a single mode and the initial state is deterministic.

Definition 5. *A dynamical system $x(t+1) = f(x(t))$ with the origin as the equilibrium point is quadratically stable if there exists a positive definite matrix P such that the quadratic function $V(x) = x^T Px$ is a valid Lyapunov function for the system; that is, for all $x \neq 0$, $V(x) > 0$, and*

$$(\Delta V)(x) = V(f(x)) - V(x) < 0$$

Definition 6. *Given a feedback control system $x(t+1) = f(x(t), u(t))$ with the origin as the equilibrium point, where u is the feedback control; a function $V(x)$ is a CLF for this system if and only if for all $x \neq 0$, $V(x) > 0$, and*

$$\inf_u (\Delta V)(x, u) < 0$$

where u is an admissible control and $(\Delta V)(x, u) = V(f(x, u)) - V(x)$.

In short, a CLF is a Lyapunov function for the closed-loop system.

For LTI systems, it is well known that stabilizability is equivalent to quadratic stabilizability. Thus we have the following lemma on stabilizing system (5) with quantized feedbacks,

Lemma 1. *Suppose system (5) is stabilizable. There exists a quadratically stabilizing log-controller with density $\rho \in (0, 1)$ if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P - A^T PA + \lambda A^T PB(B^T PB)^{-1} B^T PA \succ 0 \tag{6}$$

where $0 < \lambda = 4\rho/(1+\rho)^2 < 1$.

Proof. ($\Leftarrow$) Suppose there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the Riccati inequality (6) is satisfied. Then consider the following log-controller with density $\rho \in (0, 1)$

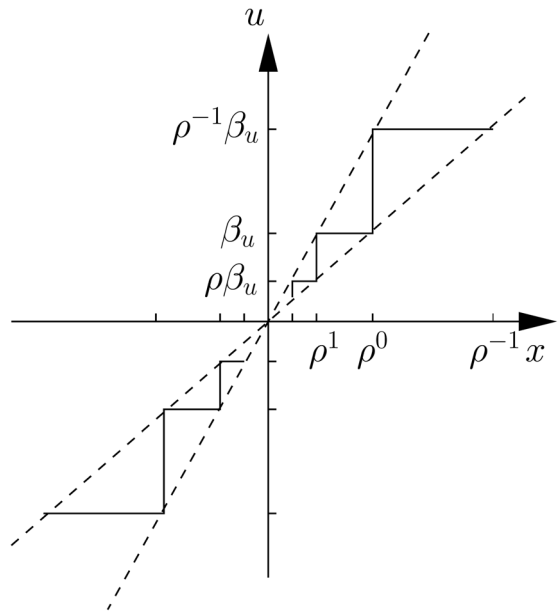

Fig. 2 An example of a 1-D log-controller

$$u = \begin{cases} -\beta_u \rho^\ell & \rho^{\ell+1} < Hx \le \rho^\ell \\ 0 & Hx = 0 \\ \beta_u \rho^\ell & -\rho^\ell \le Hx < -\rho^{\ell+1} \end{cases} \tag{7}$$

where $\beta_u = 2\rho/(1+\rho)$ and $H = B^T PA/(B^T PB)$.

It is sufficient to show that the function $V(x) = x^T Px$ is a CLF for system (5) with controls defined in Eq. (7). Clearly for all $x \neq 0$ and all $t \ge 0$, we have $V(x(t)) > 0$; therefore, we only need to show $(\Delta V)(x(t), u(t)) = V(x(t+1)) - V(x(t)) < 0$.

The difference $(\Delta V)(x(t), u(t))$ can be computed as

$$\begin{aligned}(\Delta V)(x(t), u(t)) &= (Ax(t) + Bu(t))^T P(Ax(t) + Bu(t)) - x(t)^T Px(t) \\ &= x(t)^T (A^T PA - P)x(t) + 2x(t)^T A^T PBu(t) + u(t)^T B^T PBu(t)\end{aligned} \tag{8}$$

$$\begin{aligned}&< \lambda x(t)^T A^T PB(B^T PB)^{-1} B^T PAx(t) + 2x(t)^T A^T PBu(t) \\ &\quad + u(t)^T B^T PBu(t)\end{aligned} \tag{9}$$

where the last inequality comes from by adding and subtracting the term $\lambda x(t)^T A^T PB(B^T PB)^{-1} B^T PAx(t)$ to the right-hand side (RHS) of Eq. (8) and then using the Riccati inequality (6).

We now only need to show that the RHS of Eq. (9) is nonpositive. Divide it by $B^T PB$, which is a scalar due to the scalar input assumption, and notice that $Hx(t) = (B^T PA/(B^T PB))x(t)$ is also a scalar; we only need to show

$$\lambda(Hx(t))^2 + 2Hx(t)u(t) + u(t)^2 \le 0 \tag{10}$$

Simply plug in control values $u(t)$ defined in Eq. (7) for each segment of $Hx(t)$, it is easy to verify the above inequality holds for all x.

Thus, we conclude that $V(x)$ is a legitimate CLF. Therefore, system (5) is quadratically stabilizable with the log-controller defined in Eq. (7).

($\Rightarrow$) If system (5) is quadratically stabilizable, then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that $V(x) = x^T Px$ is a CLF when linear state feedbacks are allowed. From Ref. [5], we know that if we restrict admissible controls to those generated by static log-controllers whose quantization densities $\rho > (\gamma - 1/(\gamma+1))$, where $\gamma = \sqrt{B^T PAQ^{-1} A^T PB/(B^T PB)}$ and $Q = P - A^T PA +$

$A^TPB(B^TPB)^{-1}B^TPA \succ 0$, then the function $V(x)$ remains a valid CLF. Thus quadratic stability of the closed-loop system is preserved. The constraint on ρ is equivalent to the Riccati inequality (6) by some arithmetic manipulations. See Ref. [5] for details. ■

Remark 3. Notice that ρ is strictly greater than $\gamma - 1/(\gamma+1)$ since we need the function $V(x)$ to strictly decrease. The infimum itself is not achievable.

LEMMA 2. *The Riccati inequality (6) is solvable for some positive definite matrix $P \in \mathbb{R}^{n\times n}$ with $0 < \lambda < 1$ if and only if the following LMI is feasible for some positive definite matrix $Y \in \mathbb{R}^{n\times n}$ and some matrix $Z \in \mathbb{R}^{n_u\times n}$,*

$$\begin{bmatrix} Y & \sqrt{1-\lambda}YA^T & \sqrt{\lambda}(YA^T + Z^TB^T) \\ (\star) & Y & 0 \\ (\star) & 0 & Y \end{bmatrix} \succ 0 \tag{11}$$

where $(\star)$ denotes the conjugate transpose of the corresponding term.

This follows directly from Ref. [18], Theorem 5.

Now, by combining the previous two lemmas, we state the first result of this paper. It states that the existence problem of quadratically stabilizing log-controllers for the LTI system (5) is equivalent to the feasibility problem of the LMI (11).

THEOREM 1. *Suppose system (5) is stabilizable. There exists a quadratically stabilizing log-controller with density $\rho \in (0,1)$ if and only if there exist a positive definite matrix $Y \in \mathbb{R}^{n\times n}$ and a matrix $Z \in \mathbb{R}^{n_u\times n}$ such that the LMI (11) is feasible with $0 < \lambda = 4\rho/(1+\rho)^2 < 1$.*

In order to find the coarsest stabilizing quantization density, we can solve the following semi-convex optimization problem

$$\inf \rho \in (0,1)$$

$$\text{subject to LMI (11)}$$

The optimal value of ρ can then be approached via bi-section to any degree of desired precision. For the sake of simplicity, we call this non-achievable infimum as the coarsest stabilizing quantization density. From Ref. [5], we know that the closed-form solution of it is given by

$$\rho_{\inf} = \frac{\prod \mathbf{eig}^u(A) - 1}{\prod \mathbf{eig}^u(A) + 1} \tag{12}$$

where $\mathbf{eig}^u(A)$ denotes all unstable eigenvalues of A.

4 Second-Order Stabilization of Markovian Jump Linear Systems

In this section, we consider the mean-square stabilization problem of MJLSs with a mode-dependent static log-controller (defined later). Necessary and sufficient conditions on stabilizing quantization densities are given along with the semi-convex algorithm to approach the optimal value.

DEFINITION 7. *An A-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is mean-square stable (MSS) if for any $\theta(0) \in \{1,\cdots,N\}$ and initial state $x_0 \in \mathbb{R}^n$ we have*

$$\mathbb{E}[||x(t)||^2|x_0] \to 0 \text{ as } t \to \infty$$

DEFINITION 8. *Given a set of matrices $H_i \in \mathbb{R}^{n_y\times n_u}$, where $i = 1,\cdots,N$, the zeroth-order time varying controller $\mathscr{H}$ defined by*

$$H(t) = H_{\theta(t)}$$

is called a mode-dependent static gain controller.

The corresponding mean-square stabilizability of a C-MJLS is defined as follows:

DEFINITION 9. *A C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is mean-square stabilizable if there exists a mode-dependent static gain controller $\mathscr{H}$ such that the closed-loop A-MJLS $\Sigma(\mathscr{S}, \mathscr{H}, \mathbf{P}, p)$ is mean-square stable.*

From Refs. [1],[3], and [4], we have the following lemma on the mean-square stabilizability of a C-MJLS,

LEMMA 3. *A C-MJLS is mean-square stabilizable if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n_y}$, $i = 1,\cdots,N$, such that for all i, we have*

$$(A_i - B_iH_i)^T\left(\sum_{j=1}^N p_{ij}P_j\right)(A_i - B_iH_i) - P_i \prec 0 \tag{13}$$

Clearly, if $B_i = 0$ for all $i = 1,\ldots,N$, then Lemma 3 provides a Lyapunov type test on the mean-square stability for an A-MJLS.

We define the following stochastic version Lyapunov function for a switched dynamical system:

DEFINITION 10. *Given a switched dynamical system $x(t+1) = f(x(t),\theta(t))$ with the origin as the equilibrium point, where $\theta \in \{1,\cdots,N\}$ is the system mode; a process $V(x,\theta)$ is a stochastic Lyapunov function (SLF) for the system if it is a positive super-Martingale; that is, for all $x \neq 0$ and all $\theta \in \{1,\cdots,N\}$, we have $V(x,\theta) > 0$, and*

$$(\Delta V)(x,\theta) = \mathbb{E}[V(f(x,\theta),\theta^+) - V(x,\theta)|x,\theta] < 0$$

where θ^+ denotes the system mode at the next step.

Notice that since the system is Markovian, instead of conditioning on the *filtration*,[1] we only need to condition on random variables of the current time.

DEFINITION 11. *Given a switched feedback control system*

$$x(t+1) = f(x(t),\theta(t),u(t))$$

with the origin as the equilibrium point, where $\theta \in \{1,\cdots,N\}$ is the system mode, and u is the feedback control; a process $V(x,\theta)$ is a stochastic control Lyapunov function *(SCLF) for the system if for all $x \neq 0$ and $\theta \in \{1,\cdots,N\}$, we have $V(x,\theta) > 0$, and*

$$\inf_u (\Delta V)(x,\theta,u) < 0$$

where u is an admissible control and $(\Delta V)(x,\theta,u) = \mathbb{E}[V(f(x,\theta,u),\theta^+) - V(x,\theta)|x,\theta]$. Notation θ^+ denotes the system mode at the next step.

In short, it is an SLF for the closed-loop system.

The following lemma connects the mean-square stabilizability of a C-MJLS to the existence of an SCLF.

LEMMA 4. *A C-MJLS $(\mathscr{S}, \mathbf{P}, p)$ is mean-square stabilizable if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$ such that the following quadratic process (14) is an SCLF for the system*

$$V(x(t),\theta(t)) = x(t)^TP_{\theta(t)}x(t) \tag{14}$$

where $P_{\theta(t)} = P_i$ when $\theta(t) = i$.

Proof. ($\Rightarrow$) From Lemma 3, we know that if the system is mean-square stabilizable, then there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n_y}$, $i = 1,\cdots,N$, such that LMIs (13) are satisfied. For all $x(t) \neq 0$, $\theta(t) \in \{1,\cdots,N\}$ and $t \geq 0$, the process $V(x(t),\theta(t)) > 0$. We only need to show it is a super-Martingale; that is

$$\mathbb{E}[V(x(t+1),\theta(t+1)) - V(x(t),\theta(t))|x(t),\theta(t)] < 0 \tag{15}$$

[1]Simply put, the union of algebras generated by random variables up to the current time.

Using P_i and H_i given in Lemma 3, we can equivalently check the point-wise property,

$$\begin{aligned}&\mathbb{E}[V(x(t+1),\theta(t+1))-V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i]\\&=x_t^T\left\{(A_i-B_iH_i)^T\mathbb{E}[P_{\theta(t+1)}|\theta(t)=i](A_i-B_iH_i)-P_i\right\}x_t\\&=x_t^T\left\{(A_i-B_iH_i)^T\left(\sum_{j=1}^N p_{ij}P_j\right)(A_i-B_iH_i)-P_i\right\}x_t<0\end{aligned}$$

Therefore, Eq. (15) is valid, which means the process (14) is a legitimate SCLF.

($\Leftarrow$) If there exist positive definite matrices $P_i\in\mathbb{R}^{n\times n}$, $i=1,\cdots,N$ such that process (14) is an SCLF; then from the definition of it, we know

$$\begin{aligned}&\mathbb{E}\Big[(A_{\theta(t)}x(t)+B_{\theta(t)}u(t))^TP_{\theta(t+1)}(A_{\theta(t)}x(t)+B_{\theta(t)}u(t))\\&\quad-x(t)^TP_{\theta(t)}x(t)|x(t),\theta(t)\Big]<0\end{aligned}\tag{16}$$

This means the point-wise difference $V(x(t),\theta(t))<0$ for any $x(t)=x_t$ and $\theta(t)=i$. The control that makes the difference most negative is $u_i(t)=-H_ix_t$, where $H_i=B_i^T\tilde{P}_iA_i/(B_i^T\tilde{P}_iB_i)$ with $\tilde{P}_i=\sum_{j=1}^N p_{ij}P_j$.[2]

It is now straightforward to verify that state feedbacks H_i together with matrices P_i satisfy Eq. (13). Thus the system is mean-square stabilizable. ■

Similar to the mode-dependent controller, we define the *mode-dependent logarithmic quantizer* as follows:

Definition 12. *Let $\mathcal{Q}_1,\cdots,\mathcal{Q}_N$ be a set of logarithmic quantizers; then the time varying quantizer $\mathcal{Q}$ defined by*

$$\mathcal{Q}(t)=\mathcal{Q}_{\theta(t)}$$

is called a mode-dependent logarithmic quantizer. Apparently when $N=1$, it degenerates to a regular logarithmic quantizer.

Notice that for a mode-dependent quantizer, both the quantization density and the quantization direction can change from mode to mode.

Similarly, we define the *mode-dependent static log-controller (with density ρ_i and gain β_i)* as the combination of the mode-dependent quantizer with density ρ_i, the obviously defined exponential decoder, and the constant gain β_i.

Now consider the quantized stabilization problem: Suppose the C-MJLS is mean-square stabilizable; then we have the following lemma on stabilizing it when state feedbacks are logarithmically quantized.

Lemma 5. *Suppose an N-mode C-MJLS $(\mathcal{S},\mathbf{P},p)$ is mean-square stabilizable. There exists a mean-square stabilizing mode-dependent log-controller with density $\rho_i\in(0,1)$, $i=1,\cdots,N$ if and only if there exist positive definite matrices $P_i\in\mathbb{R}^{n\times n}$, $i=1,\cdots,N$ such that*

$$P_i-A_i^T\tilde{P}_iA_i+\lambda_iA_i^T\tilde{P}_iB_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i\succ 0\tag{17}$$

where $0<\lambda_i=4\rho_i/(1+\rho_i)^2<1$ and $\tilde{P}_i=\sum_{j=1}^N p_{ij}P_j$ for all $i=1,\cdots,N$.

Proof. ($\Leftarrow$) Suppose Riccati inequalities (17) are solvable for some positive definite matrices $P_i\in\mathbb{R}^{n\times n}$, $i=1,\cdots,N$. Define the following mode-dependent log-controller with density $\rho_i\in(0,1)$

$$u_i(x)=\begin{cases}-\beta_i\rho_i^\ell & \rho_i^{\ell+1}<H_ix\le\rho_i^\ell\\ 0 & H_ix=0\\ \beta_i\rho_i^\ell & -\rho_i^\ell\le H_ix<-\rho_i^{\ell+1}\end{cases}\tag{18}$$

where $\beta_i=2\rho_i/(1+\rho_i)$ and $H_i=B_i^T\tilde{P}_iA_i/(B_i^T\tilde{P}_iB_i)$.

[2]We denote H_i as the *average gradient descendent direction* for mode i.

It is sufficient to show that the stochastic process defined by Eq. (14) is a legitimate SCLF. Obviously, $V(x(t),\theta(t))>0$ for all $x(t)\neq 0$, $\theta(t)\in\{1,\cdots,N\}$, and $t\ge 0$. We only need to show

$$\mathbb{E}[V(x(t+1),\theta(t+1))-V(x(t),\theta(t))|x(t),\theta(t)]<0$$

We can again check the point-wise property

$$\begin{aligned}&\mathbb{E}[V(x(t+1),\theta(t+1))-V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i]\\&=\mathbb{E}[x(t)^TA_{\theta(t)}^TP_{\theta(t+1)}A_{\theta(t)}x(t)+2x(t)^TA_{\theta(t)}^TP_{\theta(t+1)}B_{\theta(t)}u(t)\\&+u(t)^TB_{\theta(t)}^TP_{\theta(t+1)}B_{\theta(t)}u(t)-x(t)^TP_{\theta(t)}x(t)|x(t)=x_t,\theta(t)=i]\\&=x_t^T(A_i^T\tilde{P}_iA_i-P_i)x_t+2x_t^TA_i^T\tilde{P}_iB_iu_i(x_t)+u_i(x_t)^TB_i^T\tilde{P}_iB_iu_i(x_t)\\&<x_t^T(\lambda_iA_i^T\tilde{P}_iB_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i)x_t+2x_t^TA_i^T\tilde{P}_iB_iu_i(x_t)\\&\quad+u_i(x_t)^TB_i^T\tilde{P}_iB_iu_i(x_t)\end{aligned}\tag{19}$$

where $\tilde{P}_i=\mathbb{E}[P_{\theta(t+1)}|\theta(t)=i]=\sum_{j=1}^N p_{ij}P_j$. Again the last inequality comes from by adding and subtracting the term $\lambda_iA_i^T\tilde{P}_iB_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i$ to the RHS of the previous equation and then using Riccati inequalities (17).

Notice now, Eq. (19) is in the same form as Eq. (9) in the single-mode case. Thus, by simply following the proof of Lemma 1, we conclude that $V(x(t),\theta(t))$ is a valid SCLF. Therefore, the C-MJLS is mean-square stabilizable with the log-controller defined in Eq. (18).

($\Rightarrow$) Assume the C-MJLS is mean-square stabilizable with a mode-dependent linear feedback controller $\mathscr{H}$. Then by Lemma 4, there exists a legitimate SCLF $V(x(t),\theta(t))=x(t)^TP_{\theta(t)}x(t)$, such that

$$\begin{aligned}&\mathbb{E}[V(x(t+1),\theta(t+1))-V(x(t),\theta(t))|x(t)=x_t,\theta(t)=i]\\&=x_t^T(A_i^T\tilde{P}_iA_i-P_i)x_t+2x_t^TA_i^T\tilde{P}_iB_iu(t)+u(t)^TB_i^T\tilde{P}_iB_iu(t)<0\end{aligned}\tag{20}$$

The only difference here from the single-mode case is instead of $\tilde{P}_i$, we have P_i in the first term in Eq. (20); however, this does not affect the choice of the optimal control u_i, which is $(B_i^T\tilde{P}_iA_i/(B_i^T\tilde{P}_iB_i))x_t$. (The partial derivative is taken against u.)

Furthermore, define $\tilde{Q}_i:=P_i-A_i^T\tilde{P}_iA_i+A_i^T\tilde{P}_iB_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i\succ 0$ (since $V(x(t),\theta(t))$ is an SCLF). By mimicking the derivation in Ref. [5], it is easy to verify that if we restrict admissible controls to those generated by mode-dependent log-controller whose densities $\rho_i>1-\gamma_i/(1+\gamma_i)$, where $\gamma_i=\sqrt{B_i^T\tilde{P}_iA_i\tilde{Q}_i^{-1}A_i^T\tilde{P}_iB_i/(B_i^T\tilde{P}_iB_i)}$, the process $V(x(t),\theta(t))$ remains a valid SCLF. Therefore, the C-MJLS can still be stabilized in the mean-square sense. The constraint on ρ_i is equivalent to Riccati inequalities (17) by some arithmetic manipulations. ■

We now show that the feasibility of coupled Riccati inequalities (17) can be converted to an equivalent convex problem.

Given an MJLS $(\mathcal{S},\mathbf{P},p)$ and positive definite matrices $P_i\in\mathbb{R}^{n\times n}$, $i=1,\cdots,N$ with the following relationship:

$$\mathbb{P}[P_{\theta(t+1)}=P_j|\theta(t)=i]=p_{ij}\tag{21}$$

$$\mathbb{P}[P_{\theta(0)}=P_i]=p_i\tag{22}$$

we introduce the following auxiliary functions for convenience

$$g_i(\mathbf{P},P_1,\cdots,P_N)=A_i^T\tilde{P}_iA_i-\lambda_iA_i^T\tilde{P}B_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i\tag{23}$$

$$\phi_i(\mathbf{P},H_i,P_1,\cdots,P_N)=(1-\lambda_i)A_i^T\tilde{P}_iA_i+\lambda_iF_i^T\tilde{P}_iF_i\tag{24}$$

where for $i=1,\cdots,N$, the constant $\lambda_i\in(0,1)$, matrix $F_i=A_i-B_iH_i$ for some $H_i\in\mathbb{R}^{n_u\times n}$ and $\tilde{P}_i=\sum_{j=1}^N p_{ij}P_j$.

LEMMA 6. *There exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, such that $P_i \succ g_i$ is feasible for all $i = 1,\cdots,N$ if and only if there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n}$, such that $P_i \succ \phi_i$ is feasible for all $i = 1,\cdots,N$.*

Proof. ($\Rightarrow$) If $P_i \succ g_i$ is feasible for some positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$, then simply choose $H_i = (B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i$. We have

$$F_i = A_i - B_iH_i = A_i - B_i(B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i$$

and then

$$\phi_i(\mathbf{P},H_i,P_1,\cdots,P_N) = g_i(\mathbf{P},P_1,\cdots,P_n) \prec P_i \tag{25}$$

($\Leftarrow$) If there exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n}$, $i = 1,\cdots,N$ such that $P_i \succ \phi_i$ is feasible, then consider

$$\min_{H_i} \phi_i(\mathbf{P},H_i,P_1,\cdots,P_N)$$

By taking the derivative of ϕ_i to H_i and setting it to 0, that is

$$0 = \frac{\partial\phi_i}{\partial H_i}$$

we have

$$\text{argmin}H_i = (B_i^T\tilde{P}_iB_i)^{-1}B_i^T\tilde{P}_iA_i$$

But from Eq. (25) we know that, with this choice of H_i, function $g_i = \phi_i$; thus,

$$g_i(\mathbf{P},P_1,\cdots,P_n) = \min_{H_i} \phi_i(\mathbf{P},H_i,P_1,\cdots,P_N) \prec P_i$$

■

Using this result, we are able to convert Riccati inequalities (17) into a set of computable LMIs.

LEMMA 7. *There exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$, such that Riccati inequalities (17) are satisfied if and only if the following coupled LMIs are feasible for some positive definite matrices $Y_i \in \mathbb{R}^{n\times n}$, and matrices $Z_i \in \mathbb{R}^{n_u\times n}$, $i = 1,\cdots,N$,*

$$\begin{bmatrix} Y_i & \mathscr{L}_i^1 & \cdots & \mathscr{L}_i^N & \mathscr{M}_i^1 & \cdots & \mathscr{M}_i^N \\ (\star) & Y_1 & & & & & \\ \vdots & & \ddots & & & & \\ (\star) & & & Y_N & & & \\ (\star) & & & & Y_1 & & \\ \vdots & & & & & \ddots & \\ (\star) & & & & & & Y_N \end{bmatrix} \succ 0 \tag{26}$$

where for $i,j = 1,\cdots,N$,

$$\mathscr{L}_i^j = \sqrt{(1-\lambda_i)p_{ij}}Y_iA_i^T$$
$$\mathscr{M}_i^j = \sqrt{\lambda_ip_{ij}}(Y_iA_i^T + Z_i^TB_i^T)$$

Notation $(\star)$ denotes the conjugate transpose of the corresponding term.

Proof. Riccati inequalities (17) are satisfied for positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\cdots,N$ is equivalent to $P_i \succ g_i(\mathbf{P},P_1,\cdots,P_N)$ with g_i defined in Eq. (23). From the previous lemma, this is further equivalent to the existence of positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, and matrices $H_i \in \mathbb{R}^{n_u\times n}$, $i = 1,\cdots,N$ such that

$$P_i \succ \phi_i(\mathbf{P},H_i,P_1,\cdots,P_N) \tag{27}$$
$$= (1-\lambda_i)A_i^T\tilde{P}_iA_i + \lambda_iF_i^T\tilde{P}_iF_i$$
$$= \sum_{j=1}^{N}\left\{(1-\lambda_i)p_{ij}A_i^TP_jA_i + \lambda_iF_i^TP_jF_i\right\} \tag{28}$$

By Schur complement, the last inequality is equivalent to

$$\begin{bmatrix} P_i & \beta_{i1}A_i^T & \cdots & \beta_{iN}A_i^T & \mu_{i1}F_i^T & \cdots & \mu_{iN}F_i^T \\ (\star) & P_1^{-1} & & & & & \\ \vdots & & \ddots & & & & \\ (\star) & & & P_N^{-1} & & & \\ (\star) & & & & P_1^{-1} & & \\ \vdots & & & & & \ddots & \\ (\star) & & & & & & P_N^{-1} \end{bmatrix} \succ 0 \tag{29}$$

where for $i,j \in \{1,\cdots,N\}$, $\beta_{ij} = \sqrt{(1-\lambda_i)p_{ij}}$ and $\mu_{ij} = \sqrt{\lambda_ip_{ij}}$. It is further equivalent to

$$\begin{bmatrix} P_i^{-1} & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & I & & \\ & & & & I & \\ & & & & & \ddots \\ & & & & & & I \end{bmatrix}^T (\text{LHS of (29)}) \begin{bmatrix} P_i^{-1} & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & I & & \\ & & & & I & \\ & & & & & \ddots \\ & & & & & & I \end{bmatrix} \succ 0 \tag{30}$$

Then define $Y_i = P_i^{-1}$ and $Z_i = Y_iH_i^T$, the last inequality is feasible if and only if LMIs (26) are feasible. ■

Lemmas 5 and 7 can now be combined to give a complete convex solution to determine the mean-square stabilizability of a C-MJLS $(\mathscr{S},\mathbf{P},p)$ with logarithmically quantized state feedbacks.

THEOREM 2. *Suppose an N-mode C-MJLS $(\mathscr{S},\mathbf{P},p)$ is mean-square stabilizable. There exists a mean-square stabilizing mode-dependent log-controller with density $\rho_i \in (0,1)$, $i = 1,\cdots,N$ if and only if there exist positive definite matrices $Y_i \in \mathbb{R}^{n\times n}$, and matrices $Z_i \in \mathbb{R}^{n_u\times n}$, $i = 1,\cdots,N$, such that coupled LMIs (26) are feasible with $0 < \lambda_i = 4\rho_i/(1+\rho_i)^2 < 1$.*

In general, the coarsest quantizer may not exist for the MJLS due to the lack of a proper definition of *total ordering* for sets $(\rho_1,\cdots,\rho_N)$. However, one reasonable alternative is to solve the following *min–max* problem, which gives the infimum of the upper bound of the quantization density of stabilizing mode-dependent quantizers.

$$\inf \rho \in (0,1)$$
$$\text{subject to } \rho_i \le \rho \text{ and LMI (26)}$$

This is a semi-convex problem. The optimal value can be found to any desired degree of accuracy via bi-section.

The stabilizing mode-dependent log-controller can then be designed as follows:

(1) Choose any $\rho \in (\rho_{\inf},1)$, and set $\rho_i = \rho$.
(2) Solve coupled LMIs (26) for Y_i. Let $P_i = Y_i^{-1}$.
(3) The quantization direction is given by $H_i = B_i^T\tilde{P}_iA_i/(B_i^T\tilde{P}_iB_i)$.
(4) The log-controller for each mode can be designed according to the definition in Sec. 2.2.

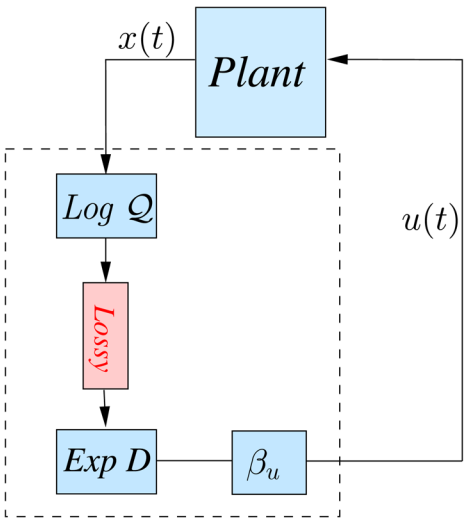

Fig. 3 An LTI system over an unreliable channel with logarithmically quantized feedbacks

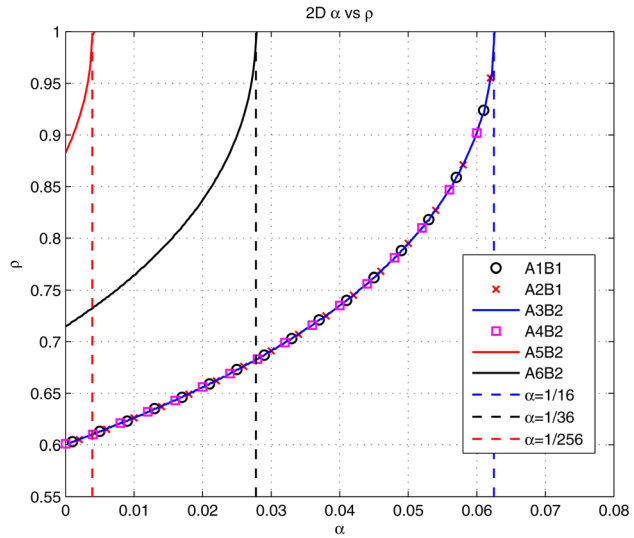

Fig. 4 Trade-off between α and ρ

Remark 4. Of course, we can choose other optimization objective functions. For example, we can minimize the weighted average quantization density $\sum p_i \rho_i$ if the system mode is i.i.d.; this is in some sense trying to minimize the average bandwidth requirement.

5 Application: Bandwidth Limited Lossy Channels

In this section, we will consider a special case of the theory that arises in networked control. In the following subsection, we will introduce the setup of control over bandwidth limited lossy channels. And later, we will demonstrate an explicit experiment of our controller working on a robot.

5.1 The Basic Formulation. Let us now revisit the motivating example: stabilization of an LTI system with logarithmically quantized state feedbacks transmitted over an unreliable communication channel, which is depicted in Fig. 3.

The plant has the following state space representation:

$$x(t+1) = Ax(t) + Bu(t) \tag{31}$$

where $x \in \mathbb{R}^n$, and scalar control $u \in \mathbb{R}$. We assume state-feedback and (A, B) is controllable.

The state feedback is first quantized by a logarithmic quantizer with density ρ and then transmitted over a lossy channel. Control u is computed according to the log-controller (4). When the packet is lost, it is effectively to set $B = 0$. Using the arrival of measurements as the index of the system mode, we have

$$\mathscr{S} = \{(A, B, I), (A, 0, I)\}$$

First, assume the packet loss follows a Bernoulli process with dropping probability $0 \le \alpha \le 1$. Then the system can be modeled as a C-MJLS $(\mathcal{S}, \mathbf{P}_1, p_1)$ with logarithmically quantized state feedback, where the system mode is i.i.d. with transition probability matrix

$$\mathbf{P}_1 = \begin{bmatrix} 1-\alpha & \alpha \\ 1-\alpha & \alpha \end{bmatrix} \tag{32}$$

without loss of generality, we can let p_1 be the stationary distribution $[\,1-\alpha \quad \alpha\,]^T$. Theorem 2 can then be directly applied to solve the second order stabilization problem.

From Ref. [5], we know that if there is no packet loss, a log-controller with quantization density $\rho > \rho_{\inf}$ defined in Eq. (12) can quadratically stabilize the system. On the other hand, from Refs. [18] and [26], we know that even if the channel can transmit real numbers (infinite bandwidth), in order to mean-square stabilize the system, the packet dropping probability is upper bounded by $1/\prod(\mathbf{eig}^u(A))^2$.

Remark 5. A special feature of this packet dropping problem is that when the packet is lost, the direction and the density of the quantizer become irrelevant, since the control value is set to 0 anyway. Therefore, we only need to implement the logarithmic quantizer designed for the case when the packet is delivered successfully, and there is no need to switch based on the system mode. Then this time-invariant static logarithmic quantizer is sufficient for second-order stabilization. In other words, the min–max optimization problem in Sec. 4 degenerates to a minimizing problem.

In order to find the trade-off between the quantization density and the packet dropping probability, we solve the following semi-convex problem for a fixed packet dropping probability α via bisection, and then grid α from 0 to 1.

$$\inf \rho \in (0, 1)$$

$$\text{subject to LMI (26) and } \mathbf{P} \text{ in (32)}$$

The following numerical examples illustrate the trade-off.

Example 1. Consider a two-dimensional system in the form of Eq. (31) with

$$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 4 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad A_4 = \begin{bmatrix} \frac{1}{4} & 1 \\ 0 & 4 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The trade-off between α and ρ is shown in Fig. 4. Again, the region above the curve in Fig. 4 is second-order stable.

From Fig. 4, we observe that:

(1) A *minimum* amount of information is required to stabilize the system; thus, if the packet dropping probability is high, one needs to include more information in each packet by quantizing finer, and vice versa. This observation supports our intuition.

(2) It is clear that unstable eigenvalues are the only decisive factor on the minimal required stabilizing information. The value of B does not matter as long as (A, B) is controllable.

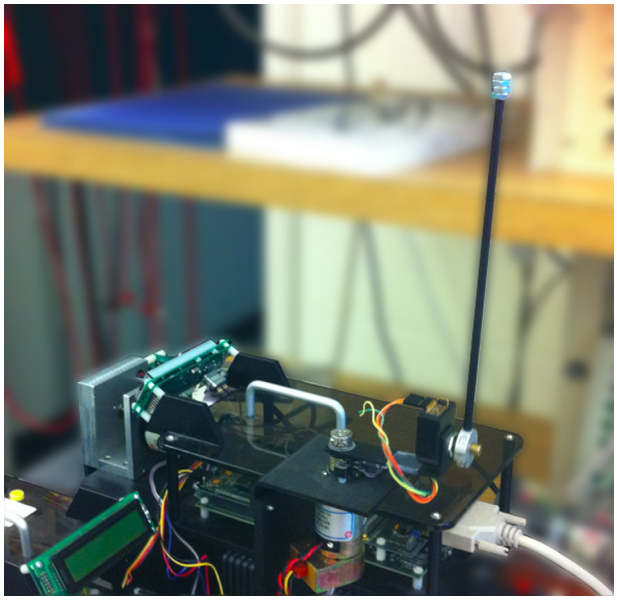

Fig. 5 Furuta pendulum

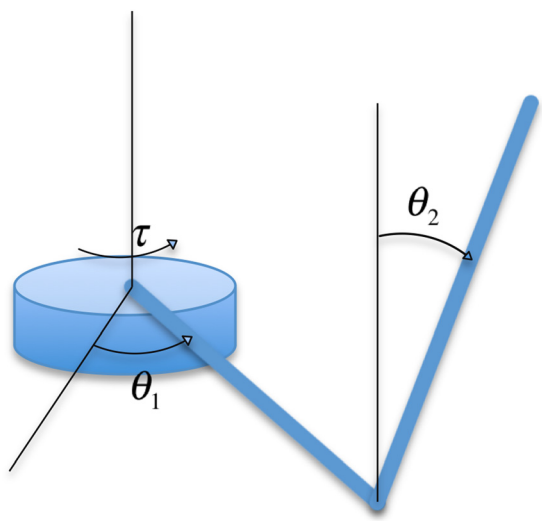

Fig. 6 The model of Furuta pendulum

(3) When $\alpha = 0$, we get the result in Ref. [5]; whereas when ρ explodes (basically no quantization), results in Refs. [17] and [18] are recovered. Thus, our framework solves both problems as special cases. Furthermore, these examples also coincide with results in Ref. [7].

5.2 Experimental Implementation. In this subsection, we apply a truncated version of the log-controller to stabilize a laboratory based Furuta pendulum, illustrating how the quantization density and packet loss affect on a digitally controlled nonlinear process.

5.2.1 Simulation Result. General information about Furuta pendulum can be found in Ref. [28], and about the specific testbed employed here in Ref. [29]. And as shown in Fig. 5, the Furuta pendulum is an inverted pendulum with two links. The first link is driven by a motor and the second one can move freely in the surface orthogonal to the first link. The simplified model is shown in Fig. 6.

The dynamic model of the pendulum used here, with parameter values, can be found in Ref. [29]. The nonlinear state space model is linearized and discretized at 200 Hz to arrive the following discrete-time linear model to be used for controller design

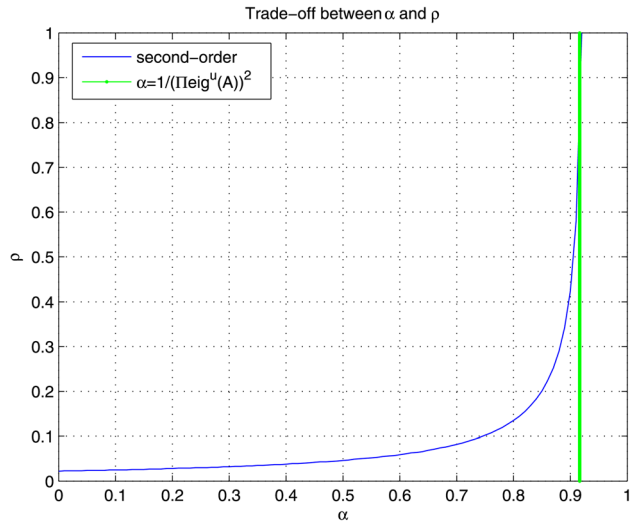

Fig. 7 Trade-off between α and ρ

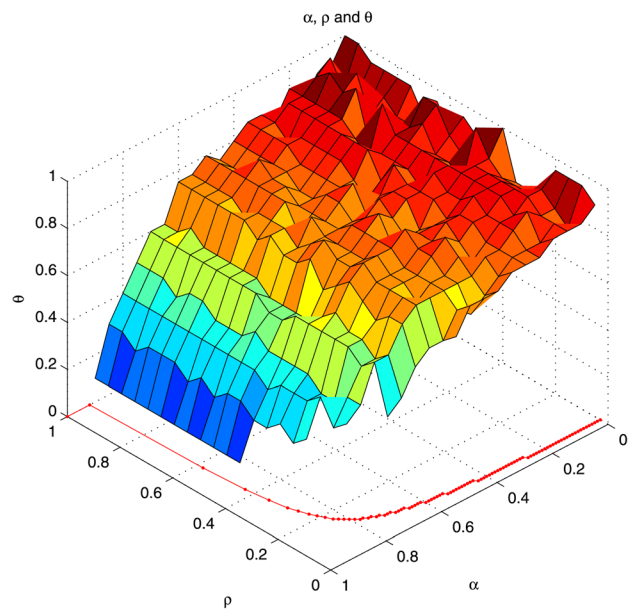

Fig. 8 Trade-off among α, ρ, and θ

$$x(k+1) = \begin{bmatrix} 1 & 0.05 & -0.0007 & -0.0000 \\ 0 & 1 & -0.2604 & -0.0007 \\ 0 & 0 & 1.0010 & 0.0050 \\ 0 & 0 & 0.3810 & 1.0010 \end{bmatrix} x(k) + \begin{bmatrix} 0.0004 \\ 0.1721 \\ -0.0002 \\ -0.0668 \end{bmatrix} u(k)$$
$$y(k) = x(k) \tag{33}$$

The diagram of the system is as Fig. 3. The angle and angular velocity of the pendulum links are measured, quantized by a log-quantizer, and sent to the actuator through a lossy channel. The packet loss follows a Bernoulli process with probability $0 \le \alpha < 1$, and the system can be modeled as a C-MJLS $(\mathcal{S}, \mathbf{P}_1, p_1)$ with $\mathcal{S} = \{(A,B,I),(A,0,I)\}$, $\mathbf{P}_1 = \begin{bmatrix} 1-\alpha & \alpha \\ 1-\alpha & \alpha \end{bmatrix}$, and $p_1 = [\,1-\alpha \quad \alpha\,]^T$. By solving LMI (26), we get the trade-off between packet loss rate α and quantization density ρ shown in Fig. 7.

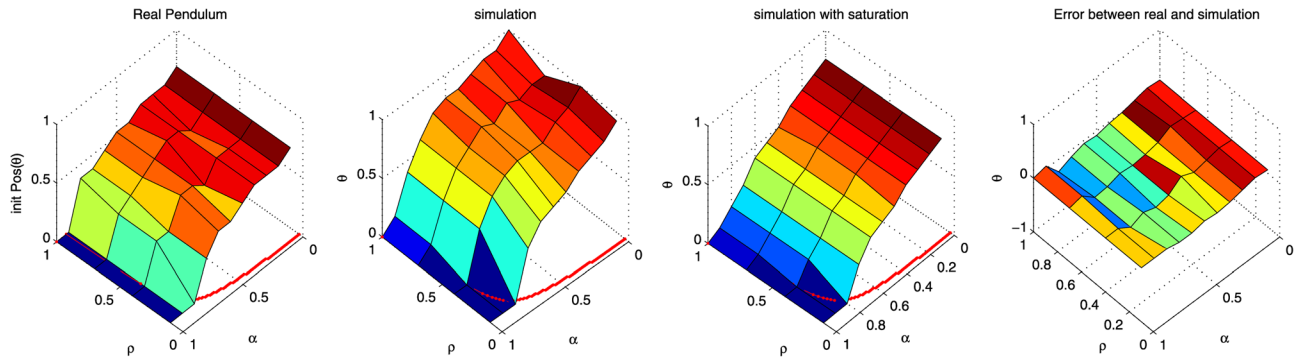

Fig. 9 Comparison of the real test and simulation results

Here, only a finite number of bits are transmitted though the channel in each packet. As shown in Eq. (34), if the absolute value of Hx is smaller than $\rho^{l_{\max}}$, then the feedback is set to zero; if the value is greater than $\rho^{l_{\min}}$, then we use $\rho^{l_{\min}}$ instead. Obviously, if $l_{\min}$ is small enough and $l_{\max}$ is large enough, then the system could be practically stable at the origin for a given error. One way of deciding the dead-zone parameter is provided in Ref. [7, Theorem 3.1], where the special case of the Markovian jump linear system is considered.

$$u = \begin{cases} -\beta_u \rho^{l_{\min}} & Hx > \rho^{l_{\min}} \\ -\beta_u \rho^{l} & \rho^{l+1} < Hx \le \rho^{l} \\ 0 & -\rho^{l_{\max}} \le Hx \le \rho^{l_{\max}} \\ \beta_u \rho^{l} & -\rho^{l} \le Hx < -\rho^{l+1} \\ \beta_u \rho^{l_{\min}} & Hx < -\rho^{l_{\min}} \end{cases} \tag{34}$$

First, the log-controller is used to stabilize the nonlinear model of the Furuta pendulum in the simulation. We find the maximum initial deviation of the link from the upright unstable equilibrium point that is stabilizable when there is no packet loss or quantization. Then the numerical result of percentage of maximum initial position that is stabilizable (θ) is calculated at each α and ρ with a resolution of 0.05. We use randomly generated packet sequences of length 100 to simulate the channel conditions at each packet loss rate; for different quantization densities with the same packet loss rate, the same sequences are always used.

The result of the simulation is shown in Fig. 8. X-axis is the packet drop rate α, y-axis is the quantization density ρ, and the maximum percentage of initial deviation is shown as θ in z-axis. It is obvious to see that the maximum initial position that is stabilizable decreases while the packet loss rate increases. However, it does not decrease monotonously because some packet received/dropped sequences are more favorable for stabilization than others. On the other hand, for those points close to the minimum quantization density, the maximum percentage of the initial position does not seem to converge to 0. If we take a slice of Fig. 8, and keep the quantization direction when there is no solution of LMI (26), we can see the pendulum is still stabilizable with a relatively smaller initial deviation. That is because we attempted to determine the coarsest quantizer when the state vector is in the worst direction that makes the interval of $\Delta V(x) < 0$ smallest for the CLF. So the system might be practically stabilizable even when ρ is smaller than the lower bound shown in the 2D plot. See Theorem 2.1 in Ref. [5] for more precise information.

5.2.2 Real Pendulum Test. We now show results on applying the above control policies on the real laboratory Furuta pendulum. The first graph in Fig. 9 is the test result of the trade-off among ρ, α, and θ of a real Furuta pendulum. Comparing to the simulation result (the second figure), it is apparent that the general tendencies

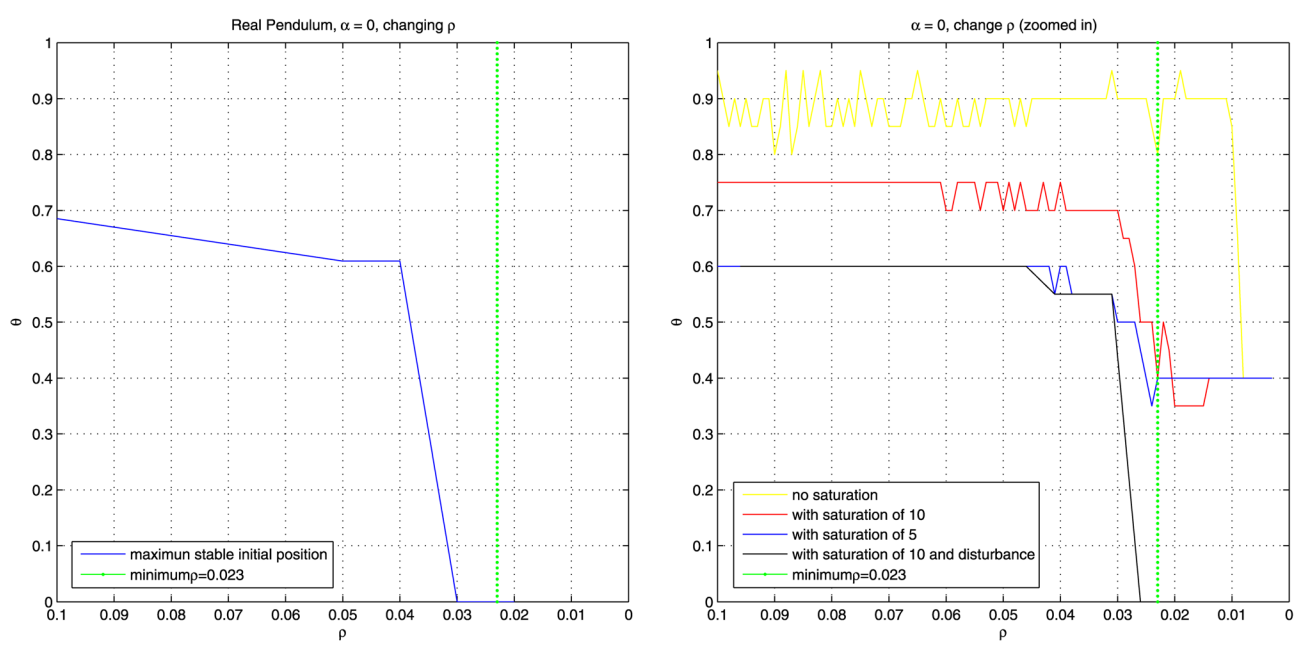

Fig. 10 Zoomed-in comparison with no packet loss

given by two figures are similar. But the maximum stable initial positions are smaller in general in the hardware test. This is believed to be due to a combination of motor saturation, sensor noise and model inaccuracy. The third figure a the simulation result with saturation added and fourth figure shows the error between the real pendulum result and the result of simulation with saturation. We can see the error is reduced and the hardware test results coincide more closely with this augmented simulation.

We are also interested in when will the system be unstable as the quantization density is reduced. Figure 10 shows a zoomed-in view of when the quantization density is small. The figure in the left is the result of the real pendulum and the one in the right shows the results of the simulation. As previously discussed, we find the coarsest quantization density that keeps $\Delta V(x)$ decreasing even when the state vector is in the worst direction and thus as shown in the simulation, the system could be practically stabilizable even the quantization density is below than the theoretical lower bound. It can be seen that it is more difficult to get stable with a smaller saturation value. And when a small disturbance is added to the upright link, the simulation result and real result match each other better.

6 Conclusion

In this paper, we have investigated the log-quantized second order stabilization problem of Markovian jump linear systems. We provide explicit constructions of stabilizing mode-dependent logarithmic quantizers and associated controllers. The coarsest stabilizing quantization density is approached via semi-convex algorithms. In addition, by using tools developed here, we show that the problem of stabilizing an LTI system over bandwidth limited unreliable channels can be solved as a special case. A simple physical testbed demonstration has been used to illustrate how the developed algorithms can be practically applied to a digital control design for a nonlinear system.

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