Control of Linear Switched Systems With Receding Horizon Modal Information

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***Abstract*—We provide an exact solution to two performance problems—one of disturbance attenuation and one of windowed variance minimization—subject to exponential stability. Considered are switched systems, whose parameters come from a finite set and switch according to a language such as that specified by an automaton. The controllers are path-dependent, having finite memory of past plant parameters and finite foreknowledge of future parameters. Exact, convex synthesis conditions for each performance problem are expressed in terms of nested linear matrix inequalities. The resulting semidefinite programming problem may be solved offline to arrive at a suitable controller. A notion of path-by-path performance is introduced for each performance problem, leading to improved system performance. Non-regular switching languages are considered and the results are extended to these languages. Two simple, physically motivated examples are given to demonstrate the application of these results.**

***Index Terms*—H infinity control, H2 control, hybrid systems, linear matrix inequalities, switched systems, uniform exponential stability.**

I. Introduction

We are interested in the stability and control of switched systems; that is, multi-modal systems exhibiting nondeterministic switching between operating modes ([18], [24]–[26], [34]). The system has finitely many modes, each with a corresponding state-space model, and a set of admissible switching sequences which are possible modal trajectories for the system. The system dynamics are given by

$$x_{t+1} = A_{\theta(t)}x_t + B_{\theta(t)}w_t; \quad z_t = C_{\theta(t)}x_t + D_{\theta(t)}w_t. \quad (1)$$

The admissible sequences may be developed as the evolution of a finite automaton (Fig. 1) or may be drawn from a more general switching language (as in Section VII). As a result, the exact sequence of parameters at each time is not known. These systems appear in many contexts, such as networked control systems ([4], [15], [19], [36]), macroeconomic models ([8], [35]), distributed networks of autonomous vehicles ([17], [29]), and biological and chemical processes ([2], [32]).

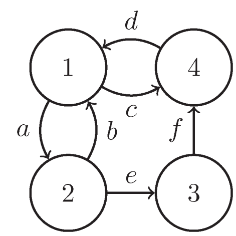

Fig. 1. Automaton with four modes which provides switching logic. The automaton state at each time generates a switching signal for the system.

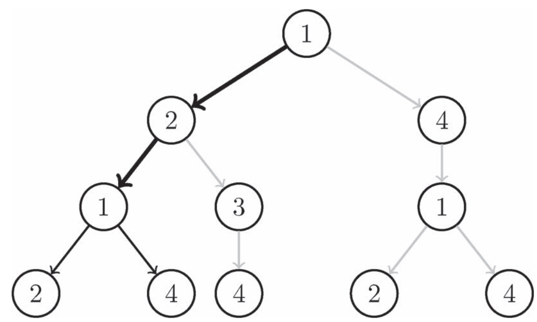

Fig. 2. Tree of possible switching trajectories for the graph of Fig. 1. If the future switching path (bold) is known, some paths (grey) can be discarded.

We present results for control of such systems to achieve two different performance measures. We first consider the ℓ_2-induced norm, and then the average system variance over a finite window. Both performance measures will later be refined to allow for a variable performance level which depends on the particular switching sequence, leading to path-by-path performance. We take a receding-horizon type approach by designing controllers with memory of finitely many past parameters and access to a finite number of future parameters. Such foreknowledge eliminates some system trajectories, but not all (see Fig. 2). This approach resembles previous results on receding-horizon type control of switched systems (such as [1], [3], [8], [13], [26], [31]). However, these results are restricted to controllers which depend only on the current mode, even though controllers with memory of past modes are known to improve performance ([6], [25]).

The dependence of our control law on a finite preview of system modes suggests a comparison with model-predictive control (MPC), also commonly called receding-horizon control. Applications of MPC to switched systems have been considered previously ([5], [27], [28]), but each of these makes different assumptions. In [27], the entire switching signal must be known

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a priori, and in [5] the switching signal is treated as a design choice rather than an exogenous signal; in contrast, [28] assumes no switching knowledge beyond the current mode. There are also no assumptions made on the allowable switching transitions. Our controllers potentially occupy a middle ground between these approaches by permitting a preview of finitely many future modes. Note that the length of this preview may be a design choice, or may be fixed (or bounded) by the problem formulation. In contrast, controller memory is always a design choice.

Our approach also differs from mainstream MPC by computing all control gains offline. In contrast, MPC requires the (repeated) solving of an optimization problem online; our controller requires only selecting the appropriate controller gain based on the observed switching sequence. In addition, we explicitly consider the structure of the allowable switching languages (described by a switching graph). Our approach guarantees infinite-horizon performance levels, but a drawback is the inability to guarantee input or output saturation constraints; this is a key attribute of the MPC approach.

The paper is organized as follows: Section II develops the notation and basic tools needed for our results. In Section III we consider the simpler problem of stabilization without performance to demonstrate the technique used. Section IV considers the ℓ_2-induced norm for performance, while Section V considers the average variance over a finite forward window. Section VI refines these results for path-by-path performance and Section VII extends them to non-regular switching languages. Two physically-motivated examples are presented in Section VIII, and concluding remarks are given in Section IX.

II. Preliminaries

It will be convenient to represent time-varying systems using the notation of block-diagonal operators (following the example of [9]). We will consider the space $\ell_2(\mathbb{R}^n)$, i.e. sequences $x = (x_0, \ldots, x_k, \ldots)$ where each $x_k \in \mathbb{R}^n$ and

$$\sum_{k=0}^{\infty} \|x_k\|^2 < \infty.$$

We will also use the unilateral shift operator Z, defined such that for $x = (x_0, x_1, x_2, \ldots)$ we have

$$Zx = (0, x_0, x_1, \ldots).$$

Any bounded operator $Q : \ell_2(\mathbb{R}^n) \to \ell_2(\mathbb{R}^m)$ is called *block-diagonal* if there exists a sequence of operators $Q_k : \mathbb{R}^n \to \mathbb{R}^m$ such that if $y = Qx$ then $y_k = Q_k x_k$, $k \in \{0, 1, 2, \ldots\}$. Such an operator has the representation

$$Q = \begin{bmatrix} Q_0 & 0 & \cdots \\ 0 & Q_1 & \\ \vdots & & \ddots \end{bmatrix}.$$

We use the notation Q^* to represent the adjoint of Q, and Q^k to denote repeated multiplication. We write $Q \succ 0$ whenever the inner product $\langle x, Qx\rangle \geq c\|x\|$ for some $c > 0$.

Consider an LTV system governed by the equations

$$x_{t+1} = A_t x_t + B_t w_t; \quad z_t = C_t x_t + D_t w_t.$$

The sequences A_t, B_t, C_t, and D_t each describe a block-diagonal operator, and using the shift operator Z we can rewrite the above system as

$$x = ZAx + ZBw; \quad z = Cx + Dw.$$

For an indexed sequence x, let $x_{(a:b)}$ denote the subsequence $(x_a, \ldots, x_b)$ when $a < b$ or x_a when $a = b$. Likewise, for a time-valued sequence $\theta(t)$, we use $\theta_{(a:b)}$ to denote the sequence $(\theta(a), \ldots, \theta(b))$ or $\theta(a)$ if $a = b$. We will use the notation $[N]$ to denote the set of indices $\{1, \ldots, N\}$.

For any symmetric matrix X we use $X \succ 0$ ($X \succeq 0$) when X is positive definite (positive semidefinite); the notation for a negative (semi)definite matrix follows. For any $X \succ 0$, we define the norm $\|y\|_X = y^T X y$. For a matrix X, the image and null spaces are denoted $\mathrm{Im}(X)$ and $\mathrm{Null}(X)$; $N(X)$ represents any full-rank matrix such that $\mathrm{Im}(N(X)) = \mathrm{Null}(X)$. The largest singular value of X is denoted $\sigma(X)$.

Lemma 1: Consider a partitioned matrix given by

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Then the following are equivalent:

(a) $X \succ 0$;
(b) $X_{22} \succ 0$ and $X_{11} - X_{12}X_{22}^{-1}X_{21} \succ 0$;
(c) $X_{11} \succ 0$ and $X_{22} - X_{21}X_{11}^{-1}X_{12} \succ 0$.

Proof: These are the well-known Schur complement formulas; see, e.g., [10] for the proof. ■

Lemma 2: Given matrices R, S and symmetric matrix Q there exists a matrix J of compatible dimensions such that

$$Q + R^T JS + S^T J^T R \prec 0$$

if and only if the following inequalities are satisfied

$$N(R)^T Q N(R) \prec 0; \quad N(S)^T Q N(S) \prec 0.$$

Proof: This result can be found in either [14] or [30]. ■

Lemma 3: Suppose matrices $R, S \in \mathbb{R}^{n\times n}$ are positive definite. Then for $m \geq n$, there is a positive definite $X \in \mathbb{R}^{(n+m)\times(n+m)}$ which satisfies the relationship

$$X = \begin{bmatrix} S & N \\ N^T & \cdot \end{bmatrix}; \quad X^{-1} = \begin{bmatrix} R & L \\ L^T & \cdot \end{bmatrix}$$

if and only if

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} \succeq 0.$$

Proof: See [30]. ■

A switched linear system is described by a finite, indexed set of parameters

$$\{(A_i, B_i, C_i, D_i)\}, \quad i \in [N] \tag{2}$$

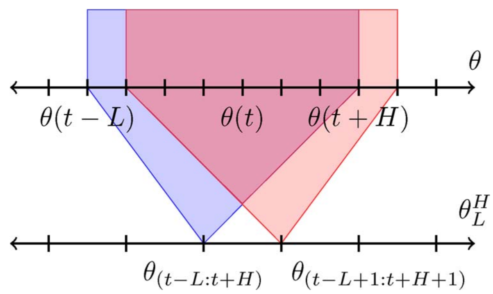

Fig. 3. Finite-length switching paths at time t (blue) and $t+1$ (red), and the resulting induced switching modes at these times.

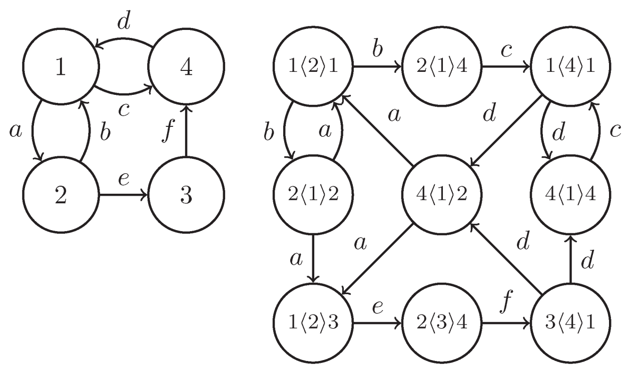

Fig. 4. Switching constraint graph with four modes, and the induced switching graph generated by paths with one past state and one future state. The edge labels identify the corresponding admissible transitions, while the current mode (of the underlying sequence) is marked in angle brackets.

and governed by dynamics of the form of (1). The *switching sequence* $\theta : \mathbb{Z}^+ \to [N]$ selects the system parameters at each time $t \geq 0$. The switching dynamics are described by a directed graph with adjacency matrix $Q \in \{0,1\}^{N\times N}$. An *admissible switching sequence* is a sequence (finite or infinite) representing a valid walk in this graph. In this paper, we assume that Q is a strongly connected graph; i.e. a path exists from any vertex to any other vertex.

Remark 4: A special case of switched systems arises when the adjacency matrix Q is the $N \times N$ matrix of ones. Such a system is also called a discrete linear inclusion.

Our controllers use not only the current switching mode, but also a finite number of past and/or future modes; that is, a finite-length sequence including the current mode. These finite-length switching paths lead to an *induced switched system*. The induced switching modes are the finite-length admissible switching paths (the relationship between the switching sequence and the induced switching mode is shown in Fig. 3). The induced switching graph admits transitions which preserve switching knowledge. The induced graph for a small system is shown in Fig. 4. In the worst case the number of induced modes grows combinatorially in path length; nevertheless finding the induced switching graph is straightforward.

When performing analysis on induced systems formed from switched systems without control inputs, the parameters of each induced mode depend solely on the current mode. We introduce the function $\phi : [N]^{L+H+1} \to [N]$, defined below (as appropriate):

$$\phi\left(\theta_{(t-L:t+H)}\right) = \theta(t), \quad \phi\left((i_{-L}, \ldots, i_0, \ldots, i_H)\right) = i_0. \quad (3)$$

In essence, ϕ selects the current mode from a sequence which includes past and/or future modes. This corresponds to the mode marked in angle brackets in Fig. 4.

III. Uniform Stabilization

We begin with the stability analysis and stabilization of switched linear systems without any performance measure. This simpler setting more clearly demonstrates the intuition behind the performance results in the subsequent sections. We require some results from LTV stability theory; therefore consider first the LTV system described by

$$x_{t+1} = A_t x_t \quad (4)$$

where $A_t \in \mathbb{R}^{n\times n}$ for all $t \geq 0$. We will assume that the A_t are uniformly bounded; in switched systems the A_t take only finitely many possible values so this assumption does not restrict us. In operator notation, this system has the form

$$x = ZAx.$$

Definition 5: The system of (4) is called *uniformly exponentially stable* if there exist constants $c \geq 1$ and $\lambda \in (0,1)$ such that for all $k \geq 0$

$$\left\|(ZA)^k\right\| \leq c\lambda^k. \quad (5)$$

Remark 6: This definition is equivalent to requiring that $\sigma(A_{t+k} \cdot \ldots \cdot A_t) \leq c\lambda^{k+1}$ for every $k \geq 0$ and every $t \geq 0$.

Lemma 7: The system of (4) is uniformly exponentially stable if and only if there exists a block-diagonal $Y \succ 0$ such that

$$(ZA)Y(ZA)^* - Y \prec 0. \quad (6)$$

Proof: This is an operator version of a well-known result; a full proof using this notation can be found in [9]. ■

A solution to the Lyapunov equation (6) leads directly to the bounds c, λ in Definition 5.

Lemma 8: Suppose Y is a solution to (6), and α, β, γ are positive constants such that

$$\alpha I \preceq Y \preceq \beta I; \quad (ZA)Y(ZA)^T - Y \preceq -\gamma I.$$

Then (4) is uniformly exponentially stable with c, λ given by

$$c = \sqrt{\frac{\beta}{\alpha}}; \quad \lambda = \sqrt{1 - \frac{\gamma}{\beta}}. \quad (7)$$

Proof: The inequality above shows, for any x, we have

$$\|(ZA)x\|_Y^2 - \|x\|_Y^2 \leq -\gamma\|x\|^2.$$

Rearranging the above inequality and using the upper bound $\|\cdot\|_Y \leq \beta\|\cdot\|$ shows that $\gamma \leq \beta$ and gives the inequality

$$\|(ZA)x\|_Y^2 \leq \left(1 - \frac{\gamma}{\beta}\right)\|x\|_Y^2.$$

Since the norm $\|\cdot\|_Y$ is submultiplicative, we then have

$$\|(ZA)^k\|_Y^2 \leq \left(1 - \frac{\gamma}{\beta}\right)^k.$$

Using the bounds on Y and rearranging terms gives

$$\frac{\|(ZA)^k x\|}{\|x\|} \le \sqrt{\frac{\beta}{\alpha}} \left(\sqrt{1-\frac{\gamma}{\beta}}\right)^k$$

from which we obtain the desired result. ■

The next lemma gives, for a stable system, a solution to (6) which is finite-path dependent. This solution is central to the stability proof for switched systems that follow.

Lemma 9: Suppose the system in (4) is uniformly exponentially stable. Then there exists a $Y \succ 0$ which solves (6) and whose blocks depend only on a finite number of past parameters.

Proof: Suppose the system is uniformly exponentially stable for c and λ. Pick M such that $c\lambda^M < 1$. Let

$$Y^{(M)} = \sum_{k=0}^{M-1} (ZA)^k \left[(ZA)^*\right]^k .$$

Clearly, $Y^{(M)} \succ 0$. Substituting $Y^{(M)}$ into (6) gives

$$\begin{aligned}(ZA)Y^{(M)}(ZA)^* - Y^{(M)} &= (ZA)^M \left[(ZA)^*\right]^M - I \\ &\le -(1-c^2\lambda^{2M})I\end{aligned}$$

so $Y^{(M)}$ satisfies (6). The block $Y_k^{(M)}$ is given by

$$Y_k^{(M)} = I + \sum_{s=\max\{0,k-M\}}^{k-1} (A_s \cdot \ldots \cdot A_{k-1})(A_s \cdot \ldots \cdot A_{k-1})^T$$

so $Y_k^{(M)}$ depends on at most M past parameters. ■

We turn now to switched systems of the form

$$x_{t+1} = A_{\theta(t)} x_t \tag{8}$$

where $\theta(t)$ is an admissible switching sequence under Q. For a fixed $\theta(t)$, this is an LTV system; we require uniform exponential stability for every admissible sequence.

Definition 10: A switched linear system is uniformly exponentially stable if there exist constants $c \ge 1$ and $\lambda \in (0,1)$ such that for every admissible sequence $\theta(t)$ the system is uniformly exponentially stable in the sense of Definition 5.

Remark 11: In the following theorems, we use the dual form of (6) which is obtained by Schur complements.

Theorem 12: For $H \ge 0$, $L \ge 0$, the system of (8) is uniformly exponentially stable if and only if there exist an integer $M \ge 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L:H)}$ and ϕ given by (3)

$$A^T_{\phi\left(i_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(i_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \prec 0. \tag{9}$$

Remark 13: If $L = H = M = 0$ then the X_j consist of a single matrix X and (9) reads $A_{i_0}^T X A_{i_0} - X \prec 0$ for all $i_0 \in [N]$. In this case, X provides a common Lyapunov function for the system, i.e. $V(x) = x^T X x$.

Proof: For sufficiency, suppose an M and a family of X_j satisfy (9). There are finitely many inequalities, so there exist positive constants α, β such that

$$\begin{aligned}&\alpha I \preceq X \preceq \beta I \\ &A^T_{\phi\left(\theta_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(\theta_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \preceq -\alpha I.\end{aligned}$$

Let $\theta(t)$ be an admissible sequence. Left-extend $\theta(t)$ by picking modes $\psi_{-L-M}, \ldots, \psi_{-1}$ such that $(\psi_{-L-M}, \ldots, \psi_{-1}, \theta(0))$ is admissible. Then define $\theta(-L-M) = \psi_{-L-M}, \ldots, \theta(-1) = \psi_{-1}$ so that $\theta(t)$ is defined for $t \ge -L-M$. Construct a block-diagonal operator X with $X_t = X_{\theta_{(t-L-M:t+H-1)}}$ as well as the operator $(A_\theta)_t = A_{\phi(\theta_{(t-L:t+H)})}$. Then the inequalities of (9) yield

$$\alpha I \preceq X \preceq \beta I; \quad (ZA_\theta)^* X (ZA_\theta) - X \preceq -\alpha I.$$

By Lemma 7, the system is exponentially stable, and by Lemma 8 the bounds c, λ are functions of α, β and not of the particular switching sequence, making them uniform.

For necessity, suppose the system of (8) is uniformly exponentially stable. Lemma 9 gives a block-diagonal X satisfying the dual of (6) whose blocks depend on $A_{t-1}, \ldots, A_{t-M}$. Since $A_t = A_{\theta(t)} = A_{\phi(\theta_{(t-L:t+H)})}$, these blocks are determined by the switching sequence $\theta_{(t-L-M:t+H)}$. Therefore relabel $X_t = X_{\theta_{(t-L-M:t+H)}}$. This works for every admissible sequence, so choose a sequence $\theta(t)$ which is recurrent, i.e., one in which every admissible sequence of length $M+1$ appears infinitely often (such a sequence exists when Q is strongly connected.) Then the blocks of the resulting inequality for $t \ge M$, give every inequality of (9). ■

Remark 14: Theorem 12 characterizes the stability of a switched system in terms of a *finite* collection of matrix inequalities (in fact, the collection of X_j provides a joint Lyapunov function for the system). We use two properties of the Lyapunov inequality for time-varying systems: first, that any solution demonstrates stability, and second, that stability implies a finite-past dependent solution. The proof of Theorem 12 gives a blueprint for the results of Sections IV and V; i.e. developing conditions which share these two properties (see Theorems 26 and 34).

We now apply this result to a closed-loop system by examining the induced switched system it creates. Consider a switched system with controlled input given by

$$x_{t+1} = A_{\theta(t)} x_t + B_{\theta(t)} u_t; \quad y_t = C_{\theta(t)} x_t + D_{\theta(t)} u_t \tag{10}$$

connected in feedback with a controller of the form

$$\hat{x}_{t+1} = \hat{A}_t \hat{x}_t + \hat{B}_t y_t; \quad u_t = \hat{C}_t \hat{x}_t + \hat{D}_t y_t \tag{11}$$

in which the controller state $\hat{x}_t \in \mathbb{R}^n$. Define

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B}_i = \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}; \quad \tilde{C}_i = \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix}$$

for $i \in [N]$, and also

$$K_t = \begin{bmatrix} \hat{A}_t & \hat{B}_t \\ \hat{C}_t & \hat{D}_t \end{bmatrix} \tag{12}$$

from which the following matrix is obtained

$$A_C(t) = \tilde{A}_{\theta(t)} + \tilde{B}_{\theta(t)} K_t \tilde{C}_{\theta(t)}.$$

With $x_C(t) = [x_t^T \; \hat{x}_t^T]^T$, the closed-loop system is

$$x_C(t+1) = A_C(t) x_C(t). \tag{13}$$

The controller has memory of length $L \geq 0$ and horizon of length $H \geq 0$, so write $K_t = K_{\theta_{(t-L:t+H)}}$ (and also $A_C(t) = A_C(\theta_{(t-L:t+H)})$). The closed-loop matrices are the parameters of an induced switched system, giving the following theorem.

Theorem 15: For $L \geq 0$, $H \geq 0$, the system of (13) is uniformly exponentially stable if and only if there exist an $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$

$$A_C\left(i_{(-L:H)}\right)^T X_{i_{(-L-M+1:H)}} A_C\left(i_{(-L:H)}\right) - X_{i_{(-L-M:H-1)}} \prec 0. \tag{14}$$

Proof: Apply Theorem 12 to the closed-loop system. ■

The inequalities of (14) are not linear in both X and K and are treated using the methods of [9] for LTV systems (or those of [14] for LTI systems). Rewrite (14) as

$$H_i + F_{i_0}^T K_{i_{(-L:H)}} G_{i_0} + G_{i_0}^T K_{i_{(-L:H)}}^T F_{i_0} \prec 0 \tag{15}$$

in which

$$F_{i_0} = \begin{bmatrix} \tilde{B}_{i_0}^T & 0 \end{bmatrix}; \quad G_{i_0} = \begin{bmatrix} 0 & \tilde{C}_{i_0} \end{bmatrix}$$
$$H_i = \begin{bmatrix} -X_{i_{(-L-M+1:H)}}^{-1} & \tilde{A}_{i_0} \\ \tilde{A}_{i_0}^T & -X_{i_{(-L-M:H-1)}} \end{bmatrix}.$$

Remark 16: We can only apply Lemma 2 here if each controller variable $K_{i_{(-L:H)}}$ appears in exactly one inequality, i.e. if $M = 0$. Thus, in the results that follow (particularly Theorem 18), we lose the distinction between M and L.

Lemma 17: There exists a controller satisfying (15) if and only if, for every admissible sequence $i_{(-\bar{L}:H)}$

$$N\left(F_{i_0}\right)^T H_i N\left(F_{i_0}\right) \prec 0 \tag{16a}$$

$$N\left(G_{i_0}\right)^T H_i N\left(G_{i_0}\right) \prec 0. \tag{16b}$$

Proof: Apply Lemma 2 to (15). ■

Take a compatible partition of X_i and its inverse by

$$X_i = \begin{bmatrix} S_i & N \\ N^T & \cdot \end{bmatrix}; \quad X_i^{-1} = \begin{bmatrix} R_i & L \\ L^T & \cdot \end{bmatrix}. \tag{17}$$

We now develop equivalent conditions to (16) using matrices R_i and S_i. An explicit representation of $N(F_{i_0})$ is given by

$$N\left(F_{i_0}\right) = \begin{bmatrix} N\left(B_{i_0}^T\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Substituting this into (16a), carrying out the multiplication, and using the partition of X_i^{-1} in (17) yields

$$\begin{bmatrix} -N\left(B_{i_0}^T\right)^T R_{i_{(-L-M+1:H)}} N\left(B_{i_0}^T\right) & N\left(B_{i_0}^T\right)^T A_{i_0} & 0 \\ A_{i_0}^T N\left(B_{i_0}^T\right) & -X_{i_{(-L-M:H-1)}} & \\ 0 & & \end{bmatrix} \prec 0.$$

Applying the Schur complement formula and using the partition again produces (18a) below; applying the same technique to $N(G_{i_0})$ and (16b) produces (18b). These inequalities and Lemma 3 produce Theorem 18.

Theorem 18: There exists a path-dependent controller with horizon $H \geq 0$ such that (13) is uniformly exponentially stable if and only if there exist an integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that for all admissible $i_{(-\bar{L}:H)}$

$$N\left(B_{i_0}^T\right)^T \left(A_{i_0} R_{i_{(-\bar{L}:H-1)}} A_{i_0}^T - R_{i_{(-\bar{L}+1:H)}}\right) N\left(B_{i_0}^T\right) \prec 0 \tag{18a}$$

$$N\left(C_{i_0}\right)^T \left(A_{i_0}^T S_{i_{(-\bar{L}+1:H)}} A_{i_0} - S_{i_{(-\bar{L}:H-1)}}\right) N\left(C_{i_0}\right) \prec 0 \tag{18b}$$

$$\begin{bmatrix} R_{i_{(-\bar{L}:H-1)}} & I \\ I & S_{i_{(-\bar{L}:H-1)}} \end{bmatrix} \succeq 0. \tag{18c}$$

Furthermore, given solutions to the inequalities in (18), a controller may be chosen with memory $L \leq \bar{L}$.

Proof: Given a stabilizing controller such that (14) is feasible for some M, the developments above yield (18) with $\bar{L} = L + M$. If sets of R_j, S_j are found satisfying (18), then one can reconstruct the set of X_j according to (17). Using these X_j in (15) yields LMIs in which the controller matrices (with memory $\bar{L}$ are unknown. Solving this feasibility problem produces the desired controller gains. ■

IV. Disturbance Attenuation

We again require some results for LTV systems. Consider the system described by

$$x_{t+1} = A_t x_t + B_t w_t; \quad z_t = C_t x_t + D_t w_t \tag{19}$$

where $A_t \in \mathbb{R}^{n \times n}$, $B_t \in \mathbb{R}^{n \times m}$, $C_t \in \mathbb{R}^{l \times n}$, and $D_t \in \mathbb{R}^{l \times m}$ for all $t \geq 0$. Once again we assume that these parameters are each uniformly bounded. We will use the operator representation of this system. The uniform exponential stability of this system is the same as in Definition 5.

Definition 19: The LTV system of (19) is called *uniformly strictly contractive* if it satisfies

$$\left\| C(I - ZA)^{-1}(ZB) + D \right\| < 1. \tag{20}$$

Remark 20: This definition is equivalent to the existence of a $\gamma \in (0, 1)$ such that, when $x(0) = 0$, the system has $\|z\|^2 < \gamma^2 \|w\|^2$ for any disturbance w. When this holds for a particular γ, the system achieves attenuation level γ.

Lemma 21: The system of (19) is uniformly exponentially stable and uniformly strictly contractive if and only if there exists a block-diagonal $Y \succ 0$ such that

$$\begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} ZA & ZB \\ C & D \end{bmatrix}^* - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \prec 0. \tag{21}$$

Proof: See [9]. ■

Solutions to (21) are closely related to solutions of the associated Riccati difference equation, defined using the system operators A, B, C, D. For any symmetric operator Y, define

$$\mathcal{V}(Y) := I - DD^* - CYC^*.$$

For any Y such that $\mathcal{V}(Y)$ is invertible, define

$$\mathcal{R}(Y) := (ZA)Y(ZA)^* + (ZB)(ZB)^* + \left[(ZA)YC^* + (ZB)D^*\right] \mathcal{V}(Y)^{-1} \left[CY(ZA)^* + D(ZB)^*\right].$$

Lemma 22: The system of (19) is uniformly exponentially stable and uniformly strictly contractive if and only if there exist positive constants $\epsilon_2, \delta_2, \eta_2$ such that for every $\epsilon \in [0, \epsilon_2]$, the sequence

$$Y^{(\epsilon,0)} = \epsilon I; \quad Y^{(\epsilon,M+1)} = \mathcal{R}\left(Y^{(\epsilon,M)}\right) + \epsilon I \tag{22}$$

satisfies for all $M \geq 0$

$$\mathcal{V}\left(Y^{(\epsilon,M)}\right) \succeq \eta_2 I; \quad \epsilon I \preceq Y^{(\epsilon,M)} \preceq Y^{(\epsilon,M+1)} \preceq \delta_2 I.$$

Then the limit $Y^{(\epsilon)} = \lim_{M\to\infty} Y^{(\epsilon,M)}$ satisfies (21).

A technical lemma, demonstrating algebraic properties of the Riccati operator $\mathcal{R}$, is needed to prove the previous claim.

Lemma 23: Define the operator

$$\mathcal{A}(Y) := (ZA) + [(ZA)YC^* + (ZB)D^*]\,\mathcal{V}(Y)^{-1}C.$$

Then any operators Y_1 and Y_2 satisfy the following:

$$\begin{aligned}\mathcal{R}(Y_1) - \mathcal{R}(Y_2) &= \mathcal{A}(Y_2)[Y_1 - Y_2]\mathcal{A}(Y_2)^* \\ &\quad + \mathcal{A}(Y_2)[Y_1 - Y_2]C^*\mathcal{V}(Y_1)^{-1} \\ &\quad \times C[Y_1 - Y_2]\mathcal{A}(Y_2)^* &\text{(23a)}\\ &= \mathcal{A}(Y_1)[Y_1 - Y_2]\mathcal{A}(Y_2)^*. &\text{(23b)}\end{aligned}$$

Proof: Our proof generalizes the proof of [22, Lemma 2.6] to consider operators in place of individual blocks. From [16], Lemma 11, we may rewrite the Riccati operator $\mathcal{R}$ as

$$\mathcal{R}(Y) = \bar{A}Y\bar{A}^* + \bar{Q} + \bar{A}YC^*\mathcal{V}(Y)^{-1}CY\bar{A}^*$$

where

$$\begin{aligned}\bar{A} &= ZA + (ZB)D^*(I - DD^*)^{-1}C\\ \bar{Q} &= (ZB)(ZB)^* + (ZB)D^*(I - DD^*)^{-1}D(ZB)^*.\end{aligned}$$

We apply [7], Lemma 3.1, to this equation to obtain (23a). It is straightforward to verify that

$$\mathcal{V}(Y_1)^{-1} - \mathcal{V}(Y_2)^{-1} - \mathcal{V}(Y_1)^{-1}C(Y_1 - Y_2)C^*\mathcal{V}(Y_2)^{-1} = 0$$

from which we obtain the equations

$$\begin{aligned}\mathcal{A}(Y_2) &= \mathcal{A}(Y_1)\left(I - [Y_1 - Y_2]C^*\mathcal{V}(Y_2)^{-1}C\right)\\ I &= \left(I - [Y_1 - Y_2]C^*\mathcal{V}(Y_2)^{-1}C\right)\\ &\quad \left(I + [Y_1 - Y_2]C^*\mathcal{V}(Y_1)^{-1}C\right).\end{aligned}$$

Using these identities and (23a) we can derive (23b). ■

A useful result of (23a) is that the Riccati operator $\mathcal{R}$ preserves order; that is, if $Y_1 \succeq Y_2 \succeq 0$, then $\mathcal{R}(Y_1) - \mathcal{R}(Y_2) \succeq 0$.

Proof of Lemma 22: First suppose that the conditions on the Riccati sequence hold. It follows from (22) that

$$Y_k^{(\epsilon,M)} = Y_k^{(\epsilon,M+1)} \Longrightarrow Y_{k+1}^{(\epsilon,M+1)} = Y_{k+1}^{(\epsilon,M+2)}$$

and also for any $M \geq 0$, $Y_0^{(\epsilon,M)} = \epsilon I$. These two facts and a simple induction argument show that for $M \geq k$, $Y_k^{(M)}$ is fixed. Let $Y^{(\epsilon)}$ denote the block-wise (weak) limit of the sequence. Then this operator satisfies $Y^{(\epsilon)} = \mathcal{R}(Y^{(\epsilon)}) + \epsilon I$; rearranging terms produces $\mathcal{R}(Y^{(\epsilon)}) - Y^{(\epsilon)} \preceq -\epsilon I$, from which a Schur complement produces (21).

Now suppose the system is uniformly stable and contractive, and let Y be the necessary solution to (21). Let α, β be constants such that $\alpha I \preceq Y \preceq \beta I$ and that $-\alpha I$ bounds the left side of (21). Taking the Schur complement of $CYC^T + DD^T - I$ produces

$$\mathcal{V}(Y) \succeq \alpha I; \quad \mathcal{R}(Y) + \alpha I \preceq Y.$$

Pick $\epsilon \in [0, \alpha]$ and construct $Y^{(\epsilon,M)}$ as in (22). This gives

$$\epsilon I = Y^{(\epsilon,0)} \preceq \alpha I \preceq Y \preceq \beta I$$

and by the monotonicity of the Riccati equation given by (23a)

$$\begin{aligned}\epsilon I \preceq Y^{(\epsilon,M+1)} &= \mathcal{R}\left(Y^{(\epsilon,M)}\right) + \epsilon I\\ &\preceq \mathcal{R}(Y) + \alpha I \preceq Y \preceq \beta I\end{aligned}$$

and also $\mathcal{V}(Y^{(\epsilon,M)}) \succeq \mathcal{V}(Y) \succeq \alpha I$. Finally, consider

$$Y^{(\epsilon,0)} = \epsilon I \preceq \epsilon I + \mathcal{R}\left(Y^{(\epsilon,0)}\right) = Y^{(\epsilon,1)}$$

so by induction the sequence remains monotone for all $M \geq 0$. Then setting $\epsilon_2 = \alpha$, $\delta_2 = \beta$ and $\eta_2 = \alpha$ the conditions of Lemma 22 are satisfied. ■

The sequence $Y^{(\epsilon,M)}$ captures the evolution of the forward-iterating Riccati difference equation; the block $Y_t^{(\epsilon,M)}$ (for $t > M$) is the value of a Riccati iteration with initial condition ϵI at time $t_0 = t - M$. The key observation is that each block in $Y^{(\epsilon,M)}$ depends on parameters for the past M steps.

Lemma 24: Suppose the system of (19) is both uniformly exponentially stable and uniformly strictly contractive. Let $Y^{(\epsilon,M)}$ be as in (22) for $\epsilon \in (0, \epsilon_2)$. The following are true:

(a) There exist constants $c_\epsilon \geq 1$ and $\lambda_\epsilon \in (0,1)$ such that for all $M \geq M_0 \geq 0$

$$\left\|\mathcal{A}\left(Y^{(\epsilon,M)}\right) \cdot \ldots \cdot \mathcal{A}\left(Y^{(\epsilon,M_0)}\right)\right\| \leq c_\epsilon \lambda_\epsilon^{M-M_0}. \tag{24}$$

(b) There exists an M such that $Y^{(\epsilon,M)}$ satisfies (21).

Proof: To prove (a), fix $\epsilon, \bar{\epsilon} \in (0, \epsilon_2)$ such that $\epsilon < \bar{\epsilon}$, and also define $Y^{(M)} := Y^{(\bar{\epsilon},M)} - Y^{(\epsilon,M)}$. Since $\mathcal{V}(Y^{(\bar{\epsilon},M)}) \succeq 0$, applying (23a) gives

$$\begin{aligned}Y^{(M+1)} &= \mathcal{R}\left(Y^{(\bar{\epsilon},M)}\right) - \mathcal{R}\left(Y^{(\epsilon,M)}\right) + (\bar{\epsilon} - \epsilon)I\\ &\succeq \mathcal{A}\left(Y^{(\epsilon,M)}\right)Y^{(M)}\mathcal{A}\left(Y^{(\epsilon,M)}\right)^* + (\bar{\epsilon} - \epsilon)I.\end{aligned}$$

Since $(\bar{\epsilon} - \epsilon)I \preceq Y^{(M)} \preceq (\delta_2 - \epsilon)I$ and $\bar{\epsilon} - \epsilon > 0$, rearranging the above produces a Lyapunov inequality which establishes the uniform stability of the operators $\mathcal{A}(Y^{(\epsilon,M)})$. A proof similar to that of Lemma 8 shows that the corresponding c_ϵ, λ_ϵ are functions of the bounds on $Y^{(M)}$.

To prove (b), pick an M such that $c_\epsilon^2 \lambda_\epsilon^{2M} < \epsilon/(\delta_2 - \epsilon)$. Then applying (23b) produces

$$\begin{aligned}
&\mathcal{R}\left(Y^{(\epsilon,M)}\right) - Y^{(\epsilon,M)} \\
&\quad= \mathcal{R}\left(Y^{(\epsilon,M)}\right) - \mathcal{R}\left(Y^{(\epsilon,M-1)}\right) - \epsilon I \\
&\quad= \mathcal{A}\left(Y^{(\epsilon,M)}\right)\left[Y^{(\epsilon,M)} - Y^{(\epsilon,M-1)}\right]\mathcal{A}\left(Y^{(\epsilon,M-1)}\right)^* - \epsilon I \\
&\quad= \mathcal{A}\left(Y^{(\epsilon,M)}\right)\ldots\mathcal{A}\left(Y^{(\epsilon,1)}\right)\left[Y^{(\epsilon,1)} - Y^{(\epsilon,0)}\right] \\
&\qquad\times \mathcal{A}\left(Y^{(\epsilon,0)}\right)^*\ldots\mathcal{A}\left(Y^{(\epsilon,M-1)}\right)^* - \epsilon I \\
&\quad\preceq (\delta_2 - \epsilon)c_\epsilon^2\lambda_\epsilon^{2M} I - \epsilon I.
\end{aligned}$$

By the choice of M, the last expression is negative definite. Then the Schur complement formula produces (21). ■

We now examine switched linear systems using the same intuition as in Section III. Consider the switched system described by (1) along with switching constraint Q. Uniform exponential stability is same as in Definition 10.

Definition 25: A switched system is uniformly strictly contractive if for every admissible $\theta(t)$ the system is uniformly strictly contractive in the sense of Definition 19.

Theorem 26: For $H \geq 0$, $L \geq 0$, the system of (1) is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an integer $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$ and ϕ given by (3)

$$\begin{aligned}
&\begin{bmatrix} A_{\phi(i_{(-L:H)})} & B_{\phi(i_{(-L:H)})} \\ C_{\phi(i_{(-L:H)})} & D_{\phi(i_{(-L:H)})} \end{bmatrix}^T \begin{bmatrix} X_{i_{(-L-M+1:H)}} & 0 \\ 0 & I \end{bmatrix} \\
&\times \begin{bmatrix} A_{\phi(i_{(-L:H)})} & B_{\phi(i_{(-L:H)})} \\ C_{\phi(i_{(-L:H)})} & D_{\phi(i_{(-L:H)})} \end{bmatrix} - \begin{bmatrix} X_{i_{(-L-M:H-1)}} & 0 \\ 0 & I \end{bmatrix} \prec 0.
\end{aligned} \tag{25}$$

Proof: The proof follows that of Theorem 12, applying Lemma 21 for sufficiency and Lemma 24 for necessity. ■

We now apply this result to a closed-loop system. Consider a system with controlled input governed by the equation

$$\begin{aligned}
x_{t+1} &= A_{\theta(t)}x_t + B_{1,\theta(t)}w_t + B_{2,\theta(t)}u_t \\
z_t &= C_{1,\theta(t)}x_t + D_{11,\theta(t)}w_t + D_{12,\theta(t)}u_t \\
y_t &= C_{2,\theta(t)}x_t + D_{21,\theta(t)}w_t.
\end{aligned} \tag{26}$$

The system is connected in feedback with a controller of the form in (11). Define

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B}_{1,i} = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}; \quad \tilde{B}_{2,i} = \begin{bmatrix} 0 & B_{2,i} \\ I & 0 \end{bmatrix};$$

$$\tilde{C}_{1,i} = [C_{1,i} \quad 0]; \quad \tilde{C}_{2,i} = \begin{bmatrix} 0 & I \\ C_{2,i} & 0 \end{bmatrix}; \quad \tilde{D}_{21,i} = \begin{bmatrix} 0 \\ D_{21,i} \end{bmatrix}$$

$$\tilde{D}_{12,i} = [0 \quad D_{12,i}]$$

and K_t as in (12) to find closed-loop matrices

$$\begin{aligned}
A_C\left(i_{(-L:H)}\right) &= \tilde{A}_{i_0} + \tilde{B}_{2,i_0}K_{i_{(-L:H)}}\tilde{C}_{2,i_0} \\
B_C\left(i_{(-L:H)}\right) &= \tilde{B}_{1,i_0} + \tilde{B}_{2,i_0}K_{i_{(-L:H)}}\tilde{D}_{21,i_0} \\
C_C\left(i_{(-L:H)}\right) &= \tilde{C}_{1,i_0} + \tilde{D}_{12,i_0}K_{i_{(-L:H)}}\tilde{C}_{2,i_0} \\
D_C\left(i_{(-L:H)}\right) &= D_{11,i_0} + \tilde{D}_{12,i_0}K_{i_{(-L:H)}}\tilde{D}_{21,i_0}
\end{aligned}$$

and the corresponding closed-loop equations

$$\begin{aligned}
x_C(t+1) &= A_C\left(\theta_{(t-L:t+H)}\right)x_C(t) + B_C\left(\theta_{(t-L:t+H)}\right)w(t) \\
z(t) &= C_C\left(\theta_{(t-L:t+H)}\right)x_C(t) + D_C\left(\theta_{(t-L:t+H)}\right)w(t).
\end{aligned} \tag{27}$$

Theorem 27: For $L \geq 0$, $H \geq 0$, the system of (27) is uniformly exponentially stable and uniformly strictly contractive if and only if there exist an $M \geq 0$ and matrices $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$

$$\begin{aligned}
&\begin{bmatrix} A_C\left(i_{(-L:H)}\right) & B_C\left(i_{(-L:H)}\right) \\ C_C\left(i_{(-L:H)}\right) & D_C\left(i_{(-L:H)}\right) \end{bmatrix}^T \begin{bmatrix} X_{i_{(-L-M+1:H)}} & 0 \\ 0 & I \end{bmatrix} \\
&\times \begin{bmatrix} A_C\left(i_{(-L:H)}\right) & B_C\left(i_{(-L:H)}\right) \\ C_C\left(i_{(-L:H)}\right) & D_C\left(i_{(-L:H)}\right) \end{bmatrix} - \begin{bmatrix} X_{i_{(-L-M:H-1)}} & 0 \\ 0 & I \end{bmatrix} \prec 0.
\end{aligned} \tag{28}$$

Proof: Apply Theorem 26 to the closed-loop system. ■

From here we proceed in the same way as Section III by using Lemmas 2 and 3, subject to the limitation discussed in Remark 16. As a generalization based on Remark 20, the system achieves attenuation level γ if and only if the system $\{(A_i, \gamma^{-1/2}B_i, \gamma^{-1/2}C_i, \gamma^{-1}D_i)\}$ is contractive. This fact leads to the next result.

Theorem 28: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable with attenuation level γ if and only if there exist an $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that for all admissible $i_{(-\bar{L}:H)}$

$$\begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_{i_0}R_{i_-}A_{i_0}^T - R_{i_+} & A_{i_0}R_{i_-}C_{1,i_0}^T & B_{1,i_0} \\ C_{1,i_0}R_{i_-}A_{i_0}^T & C_{1,i_0}R_{i_-}C_{1,i_0}^T - \gamma I & D_{11,i_0} \\ B_{1,i_0}^T & D_{11,i_0}^T & -\gamma I \end{bmatrix} \times \begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{29a}$$

$$\begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_{i_0}^T S_{i_+}A_{i_0} - S_{i_-} & A_{i_0}^T S_{i_+}B_{1,i_0} & C_{1,i_0}^T \\ B_{1,i_0}^T S_{i_+}A_{i_0} & B_{1,i_0}^T S_{i_+}B_{1,i_0} - \gamma I & D_{11,i_0}^T \\ C_{1,i_0} & D_{11,i_0} & -\gamma I \end{bmatrix} \times \begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{29b}$$

$$\begin{bmatrix} R_{i_-} & I \\ I & S_{i_-} \end{bmatrix} \succeq 0 \tag{29c}$$

where $i_- = i_{(-\bar{L}:H-1)}$, $i_+ = i_{(-\bar{L}+1:H)}$, and

$$N_{F,i_0} = N\left(\left[B_{2,i_0}^T \; D_{12,i_0}^T\right]\right); \quad N_{G,i_0} = N\left(\left[C_{2,i_0} \; D_{21,i_0}\right]\right).$$

Furthermore, given solutions to the inequalities in (29), a controller may be chosen with memory $L \leq \bar{L}$.

Proof: The proof is similar to that of Theorem 18 by transforming (28). ■

These conditions are also convex in the performance level γ; one can optimize the performance level of the controller by minimizing γ subject to (29).

V. Windowed Output Regulation

Consider again the LTV system of (19). Suppose now that the disturbance input w is an i.i.d. sequence of $\mathbb{R}^m$-valued random variables satisfying for all $s, t \geq 0$

$$\mathbb{E}\left[w(t)\right] = 0; \quad \mathbb{E}\left[w(t)w(s)^T\right] = \begin{cases} I & \text{for } t = s \\ 0 & \text{for } t \neq s. \end{cases} \tag{30}$$

Once again, uniform exponential stability of this system is the same as in Definition 5. Performance of this system is the the average output variance over a finite forward window of length $T \geq 0$ as in the following definition.

Definition 29: Let $T \geq 0$. The system of (19) satisfies *T-step uniform performance level* γ if for $\gamma > 0$, w as in (30) and $x(0) = 0$ the output satisfies (31) for all $t \geq 0$

$$\frac{1}{T+1}\sum_{s=t}^{t+T}\mathbb{E}\left[\|z(s)\|^2\right] < \gamma^2. \tag{31}$$

For a system which is uniformly exponentially stable, there exists a unique solution $Y_0 \succ 0$ to the Lyapunov equation

$$(ZA)Y_0(ZA)^* - Y_0 = -(ZB)(ZB)^*. \tag{32}$$

The block structure of Y_0 is given by

$$\left(Y_0\right)_t = \sum_{s=0}^{t}\Phi(t+1, s+1)B_sB_s^T\Phi(t+1, s+1)^T$$

where $\Phi(t, s)$ is the state transition matrix defined by

$$\Phi(t, s) = \begin{cases} I & \text{for } t = s \\ A(t-1)\cdot \ldots \cdot A(s) & \text{for } t > s. \end{cases}$$

Remark 30: It is immediate from the definition of Y_0 that any block-diagonal operator $X \succ 0$ satisfying

$$(ZA)X(ZA)^* - X \preceq -(ZB)(ZB)^*$$

will also satisfy $X \succeq Y_0$.

From (19) and (30), it is simple to show that

$$\mathbb{E}\left[\|z(t)\|^2\right] = \operatorname{Tr}(CY_0C^* + DD^*)_t. \tag{33}$$

We define a *windowed trace* for block diagonal operators as

$$\operatorname{Tr}_{(t,T)}(X) = \frac{1}{T+1}\sum_{s=t}^{t+T}\operatorname{Tr}(X_t). \tag{34}$$

Then the T-step performance of (31) can be written as

$$\operatorname{Tr}_{(t,T)}(CY_0C^* + DD^*) < \gamma^2 \quad \forall t \geq 0.$$

Remark 31: The windowed trace operator preserves order; that is, if $X \succeq Y$, then $\operatorname{Tr}_{(t,T)}(X) \geq \operatorname{Tr}_{(t,T)}(Y)$ for all $t \geq 0$.

Lemma 32: The following are true:

1) The system of (19) is uniformly exponentially stable and satisfies T-step uniform performance level γ if there exists a $Y \succ 0$ such that

$$(ZA)Y(ZA)^* - Y \prec -(ZB)(ZB)^* \tag{35}$$
$$\operatorname{Tr}_{(t,T)}(CYC^* + DD^*) < \gamma^2 \quad \forall t \geq 0. \tag{36}$$

2) If the system of (19) is uniformly exponentially stable and satisfies T-step uniform performance level γ, then there exists a $Y \succ 0$ satisfying (35) and (36) whose blocks depend on a finite number of past parameters.

Proof: To prove (a), suppose a solution to (35) and (36) exists. Then by Lemma 7 the system is uniformly exponentially stable. Construct Y_0 by solving (32). Then from Remarks 30 and 31 it is clear that

$$\operatorname{Tr}_{(t,T)}(CY_0C^* + DD^*) < \gamma^2$$

and from (33) the system satisfies T-step performance level γ.

To prove (b), let $\epsilon > 0$ and consider the sequence

$$Y^{(\epsilon,M+1)} = (ZA)Y^{(\epsilon,M)}(ZA)^* + (ZB)(ZB)^* + \epsilon I$$

with $Y^{(\epsilon,0)} = \epsilon I$. The block $Y_t^{(\epsilon,M)}$ depends on at most M past parameters. Also, for any $t \geq 0$ the blocks $Y_t^{(\epsilon,M)} = Y_t^{(\epsilon,M+1)}$ for all $M \geq t$. Let the block-wise (weak) limit of this sequence be $Y^{(\epsilon)}$. This limit satisfies

$$Y^{(\epsilon)} = (ZA)Y^{(\epsilon)}(ZA)^* + (ZB)(ZB)^* + \epsilon I$$

and we have $\epsilon I \preceq Y^{(\epsilon,M)} \preceq Y^{(\epsilon,M+1)} \preceq Y^{(\epsilon)}$ for $M \geq 0$.

From the definition of $Y^{(\epsilon)}$ is is clear that

$$Y_t^{(\epsilon)} = Y_0 + \epsilon\tilde{Y}$$

where $\tilde{Y}$ is the solution to the Lyapunov equation

$$(ZA)\tilde{Y}(ZA)^* - \tilde{Y} = -I.$$

Since both Y_0 and $\tilde{Y}$ are bounded operators, there exists a $\beta > 0$ such that $Y^{(\epsilon)} \preceq \beta I$. Further,

$$\operatorname{Tr}_{(t,T)}\left(CY^{(\epsilon)}C^* + DD^*\right) = \operatorname{Tr}_{(t,T)}(CY_0C^* + DD^*) + \epsilon\operatorname{Tr}_{(t,T)}(C\tilde{Y}C^*)$$

because the operators $\tilde{Y}$, C, and D are uniformly bounded, $\operatorname{Tr}_{(t,T)}(C\tilde{Y}C^*)$ is also bounded. Therefore choosing ϵ small will make the rightmost term above as small as desired.

Since the system satisfies performance level γ, we have $\operatorname{Tr}_{(t,T)}(CY_0C^* + DD^*) < \gamma^2$; choose ϵ sufficiently small such that $\operatorname{Tr}_{(t,T)}(CY^{(\epsilon)}C^* + DD^*) < \gamma^2$ also. Now since the system is uniformly exponentially stable, there exist c, λ as in Definition 5. Choose M such that $c^2\lambda^{2M} < \alpha/(\beta - \alpha)$. Then

$$\begin{aligned} &(ZA)Y^{(\epsilon,M)}(ZA)^* - Y^{(\epsilon,M)} \\ &\quad = (ZA)\left[Y^{(\epsilon,M)} - Y^{(\epsilon,M-1)}\right](ZA)^* - \alpha I \\ &\quad = (ZA)^M\left[Y^{(\epsilon,1)} - Y^{(\epsilon,0)}\right]((ZA)^*)^M - \alpha I \\ &\quad \preceq (c^2\lambda^{2M})(\beta - \alpha)I - \alpha I \end{aligned}$$

and because $Y^{(\alpha,M)} \preceq Y^{(\alpha)}$, we have

$$\mathrm{Tr}_{(t,T)}\left(CY^{(\alpha,M)}C^* + DD^*\right) < \gamma^2.$$

∎

We proceed to the switched system of (1) and extend the performance measure to all admissible switching sequences.

Definition 33: Let T be a nonnegative integer and $\gamma > 0$. The system described by (1) satisfies *T-step uniform performance level* γ if for all admissible $\theta(t)$ the system satisfies T-step performance level γ in the sense of Definition 29.

Theorem 34: For $H \geq 0, L \geq 0$, the system of (1) is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists an integer $M \geq 0$ and a collection of matrices $Y_j \succ 0$ for $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$

$$A_{\phi\left(i_{(-L:H)}\right)} Y_{i_{(-L-M:H-1)}} A^T_{\phi\left(i_{(-L:H)}\right)} - Y_{\left(i_{(-L-M+1:H)}\right.} \prec -B_{\phi\left(i_{(-L:H)}\right)} B^T_{\phi\left(i_{(-L:H)}\right)} \tag{37a}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathrm{Tr}\Big(C_{\phi\left(\hat{i}_{(t-L:t+H)}\right)} Y_{\hat{i}_{(t-L-M:t+H-1)}} C^T_{\phi\left(\hat{i}_{(t-L:t+H)}\right)} + D_{\phi\left(\hat{i}_{(t-L:t+H)}\right)} D^T_{\phi\left(\hat{i}_{(t-L:t+H)}\right)}\Big) < \gamma^2. \tag{37b}$$

Proof: Proceed in the same manner as the proof of Theorem 12 using the conditions of Lemma 32. ∎

We now apply this result to a closed-loop system. Consider the system of (26) connected in feedback with a controller of the form of (11). Construct the closed-loop system equation as in (27) to obtain the following result.

Theorem 35: For $H \geq 0$, $L \geq 0$, the system of (27) is uniformly exponentially stable and satisfies T-step uniform performance level γ if and only if there exists an $M \geq 0$ and a collection of matrices $Y_j \succ 0$ for each admissible $j \in [N]^{L+M+H}$ such that for all admissible $i_{(-L-M:H)}$ and $\hat{i}_{(-L-M:H+T)}$

$$A_C\left(i_{(-L:H)}\right) Y_{i_{(-L-M:H-1)}} A_C\left(i_{(-L:H)}\right)^T - Y_{\left(i_{(-L-M+1:H)}\right.} \prec -B_C\left(i_{(-L:H)}\right) B_C\left(i_{(-L:H)}\right)^T \tag{38a}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathrm{Tr}\left(C_C\left(\hat{i}_{(t-L:t+H)}\right) Y_{\hat{i}_{(t-L-M:t+H-1)}} C_C\left(\hat{i}_{(t-L:t+H)}\right)^T + D_C\left(\hat{i}_{(t-L:t+H)}\right) D_C\left(\hat{i}_{(t-L:t+H)}\right)^T\right) < \gamma^2. \tag{38b}$$

Proof: Apply Theorem 34 to the (27). ∎

We now derive equivalent convex conditions by introducing a change of variables similar to that of [21], [33]. A Schur complement argument gives the equivalent conditions

$$\begin{bmatrix} -Y^{-1}_{i_{(-L-M:H-1)}} & A_C^T\left(i_{(-L:H)}\right) & 0 \\ A_C\left(i_{(-L:H)}\right) & -Y_{i_{(-L-M+1:H)}} & B_C\left(i_{(-L:H)}\right) \\ 0 & B_C^T\left(i_{(-L:H)}\right) & -I \end{bmatrix} \prec 0 \tag{39a}$$

$$\begin{bmatrix} -Y^{-1}_{\hat{i}_{(-L-M:H-1)}} & C_C^T\left(\hat{i}_{(-L:H)}\right) & 0 \\ C_C\left(\hat{i}_{(-L:H)}\right) & -Z_{\hat{i}_{(-L-M:H-1)}} & D_C\left(\hat{i}_{(-L:H)}\right) \\ 0 & D_C^T\left(\hat{i}_{(-L:H)}\right) & -I \end{bmatrix} \prec 0 \tag{39b}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathrm{Tr} Z_{\hat{i}_{(t-L-M:t+H-1)}} < \gamma^2. \tag{39c}$$

Partition the matrices Y_j and Y_j^{-1} as

$$Y_j = \begin{bmatrix} R_j & T_j \\ T_j^T & \cdot \end{bmatrix}; \quad Y_j^{-1} = \begin{bmatrix} S_j & U_j \\ U_j^T & \cdot \end{bmatrix} \tag{40}$$

where $R_j, S_j \in \mathbb{R}^{n\times n}$, $U_j, T_j \in \mathbb{R}^{n\times n_k}$. Now define

$$\begin{aligned} W_i = &\begin{bmatrix} S_{i_{(-L-M+1:H)}} A_{j_0} R_{i_{(-L-M:H-1)}} & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} U_{i_{(-L-M+1:H)}} & S_{i_{(-L-M+1:H)}} B_{2,i_0} \\ 0 & I \end{bmatrix} K_{i_{(-L:H)}} \\ &\times \begin{bmatrix} T^T_{i_{(-L-M:H-1)}} & 0 \\ C_{2,i_0} R_{i_{(-L-M:H-1)}} & I \end{bmatrix} \end{aligned} \tag{41}$$

for every $i_{(-L-M:H)}$. Let

$$M_i = \begin{bmatrix} I & S_{i_{(-L-M+1:H)}} \\ 0 & U^T_{i_{(-L-M+1:H)}} \end{bmatrix}; \quad \tilde{M}_i = \begin{bmatrix} I & R_{i_{(-L-M:H-1)}} \\ 0 & T^T_{i_{(-L-M:H-1)}} \end{bmatrix}.$$

Apply a congruence transformation to (39a) with $M_i \oplus \tilde{M}_i \oplus I$, and to (39b) with $M_i \oplus I \oplus I$. Algebraic manipulation leads to Theorem 37.

Remark 36: Because the introduction of W_i eliminates the controller variable, it also costs us the distinction between L and M (similar to Remark 16).

Theorem 37: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable and satisfies T-step uniform performance level γ for the system of (27) if and only if there exist an integer $\bar{L} \geq 0$, matrices $R_j \succ 0, S_j \succ 0$ for $j \in [N]^{\hat{L}+H}$, and matrices Z_i, W_i for $i \in [N]^{\hat{L}+H+1}$ such that for all admissible $i_{(-\hat{L}:H)}$ and $\hat{i}_{(-\bar{L}:H+T)}$

$$H_i + F_{i_0}^T W_i G_{i_0} + G_{i_0}^T W_i^T F_{i_0} \prec 0 \tag{42a}$$

$$\hat{H}_i + \hat{F}_{i_0}^T W_i \hat{G}_{i_0} + \hat{G}_{i_0}^T W_i \hat{F}_{i_0} \prec 0 \tag{42b}$$

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathrm{Tr} Z_{\hat{i}_{(t-\hat{L}:t+H)}} < \gamma^2 \tag{42c}$$

with $i_- = i_{(-\bar{L}:H-1)}$, $i_+ = i_{(-\bar{L}+1:H)}$, and

$$F_{i_0} = \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & B_{2,i_0}^T & 0 & 0 \end{bmatrix}; \quad \hat{F}_{i_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & D_{12,i_0}^T & 0 \end{bmatrix}$$

$$G_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_{2,i_0} & 0 & 0 & 0 & D_{21,i_0} \end{bmatrix}; \ \hat{G}_{i_0} = \begin{bmatrix} 0 & I & 0 & 0 \\ C_{2,i_0} & 0 & 0 & D_{21,i_0} \end{bmatrix}$$

$$H_i = \begin{bmatrix} -S_{i_-} & -I & A_{i_0}^T & A_{i_0}^T S_{i_+} & 0 \\ \cdot & -R_{i_-} & R_{i_-} A_{i_0}^T & 0 & 0 \\ \cdot & \cdot & -R_{i_+} & -I & B_{1,i_0} \\ \cdot & \cdot & \cdot & -S_{i_+} & S_{i_+} B_{1,i_0} \\ \cdot & \cdot & \cdot & \cdot & -I \end{bmatrix}$$

$$\hat{H}_i = \begin{bmatrix} -S_{i_-} & -I & C_{1,i_0}^T & 0 \\ \cdot & -R_{i_-} & R_{i_-} C_{1,i_0}^T & 0 \\ \cdot & \cdot & -Z_i & D_{11,j_0} \\ \cdot & \cdot & \cdot & -I \end{bmatrix}.$$

Furthermore, given solutions to the inequalities in (42), a controller may be constructed with memory $L \leq \bar{L}$.

Proof: The proof is similar to that of Theorems 18 or 28 by manipulating (39). Given the $W_{\hat{\imath}}$ satisfying (42), a controller gains are found algebraically via (41). ■

VI. Path-by-Path Performance Measures

Sections IV and V consider *uniform* performance. We now refine these performance measures by introducing a notion of *path-by-path performance* for each. These refinements allow for improved performance along certain switching sequences.

First recall the notion of uniform disturbance attenuation of Section IV and consider the following modification.

Definition 38: Let $\Gamma = \{\gamma_i : i \in [N]^{T+1}\}$ be an indexed collection of positive parameters γ_i. Then the system of (1) achieves *path-by-path disturbance attenuation levels* Γ if, for every admissible switching sequence θ, it satisfies

$$\sum_{t=0}^{\infty} |z_t|^2 \leq \sum_{t=0}^{\infty} \gamma^2_{\theta_{(t-L-M:t+H)}} |w_t|^2. \tag{43}$$

When a system achieves path-by-path attenuation levels Γ, it also achieves uniform attenuation level $\hat{\gamma}$ if $\hat{\gamma}$ is an upper bound on Γ. By letting the γ_i be time-varying (implicitly though θ), we may improve performance for certain switching paths. The resulting conditions mirror those of Theorem 28.

Theorem 39: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable and achieves *path-by-path attenuation levels* Γ if and only if there exist an integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that for all admissible $i_{(-\bar{L}:H)}$

$$\begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_{i_0} R_{i_-} A_{i_0}^T - R_{i_+} & A_{i_0} R_{i_-} C_{1,i_0}^T & B_{1,i_0} \\ C_{1,i_0} R_{i_-} A_{i_0}^T & C_{1,i_0} R_{i_-} C_{1,i_0}^T - \gamma_i I & D_{11,i_0} \\ B_{1,i_0}^T & D_{11,i_0}^T & -\gamma_i I \end{bmatrix}$$
$$\times \begin{bmatrix} N_{F,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{44a}$$

$$\begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A_{i_0}^T S_{i_+} A_{i_0} - S_{i_-} & A_{i_0}^T S_{i_+} B_{1,i_0} & C_{1,i_0}^T \\ B_{1,i_0}^T S_{i_+} A_{i_0} & B_{1,i_0}^T S_{i_+} B_{1,i_0} - \gamma_i I & D_{11,i_0}^T \\ C_{1,i_0} & D_{11,i_0} & -\gamma_i \end{bmatrix}$$
$$\times \begin{bmatrix} N_{G,i_0} & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{44b}$$

$$\begin{bmatrix} R_{i_-} & I \\ I & S_{i_-} \end{bmatrix} \succeq 0 \tag{44c}$$

with i_-, i_+, N_{F,i_0}, and N_{G,i_0} as in Theorem 28. Furthermore, given solutions to the inequalities in (44), a controller may be chosen with memory $L \leq \bar{L}$.

As with Theorem 28, these conditions are convex in the γ_i. We can pick weightings $\{\lambda_i\}$ and find a Pareto-optimal performance level by minimizing $\sum_i \lambda_i \gamma_i$ subject to (44).

We likewise modify the windowed variance of Section V:

Definition 40: Let $T \geq 0$ and $\Gamma = \{\gamma_j : j \in [N]^{T+1}\}$ be an indexed collection of positive parameters γ_j. Then the system of (1) satisfies *T-step path-by-path performance levels* Γ if for w satisfying (30) and $x(0) = 0$ the output sequence satisfies

$$\frac{1}{T+1} \sum_{s=t}^{t+T} \mathbb{E}\left[\|z(s)\|^2\right] < \gamma^2_{\theta_{(t:t+T)}}$$

for all admissible switching sequences θ and all $t \geq 0$.

Theorem 41: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable and achieves T-step performance levels Γ if and only if there exist an integer $\bar{L} \geq 0$, matrices $R_j \succ 0, S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ admissible, and matrices Z_i, W_i for $i \in [N]^{\bar{L}+H+1}$ admissible such that for all admissible $i_{(-\hat{L}:H)}$ and $\hat{\imath}_{(-\hat{L}:H+T)}$

$$H_i + F_{i_0}^T W_i G_{i_0} + G_{i_0}^T W_i^T F_{i_0} \prec 0 \tag{45a}$$
$$\hat{H}_i + \hat{F}_{i_0}^T W_i \hat{G}_{i_0} + \hat{G}_{i_0}^T W_i \hat{F}_{i_0} \prec 0 \tag{45b}$$
$$\frac{1}{T+1} \sum_{t=0}^{T} \mathrm{Tr} Z_{\hat{\imath}_{(t-\hat{L}:t+H)}} < \gamma^2_{i_{(0:T)}} \tag{45c}$$

with F_{i_0}, G_{i_0}, $\hat{F}_{i_0}$, $\hat{G}_{i_0}$, H_j, and $\hat{H}_j$ defined as in Theorem 37. Furthermore, given solutions to the inequalities in (45), a controller may be constructed with memory $L \leq \bar{L}$.

VII. Extension to Non-Regular Switching Languages

So far we have allowed our admissible switching sequences to be any language developed by a directed graph (in automata theory these are called *regular languages*). We now modify our results to accomodate more general switching languages. Consider again the system given by (2) but instead define

$$\Theta \subset [N]^{\infty} \tag{46}$$

to be the set of (infinite length) admissible switching sequences. We introduce the dummy mode 0 and let $\theta(t) = 0$ for all $t < 0$ when $\theta \in \Theta$. For integers $L \geq 0$, $H \geq 0$, define

$$\mathcal{L}_{(L,H)}(\Theta) := \left\{\theta_{(t-L:t+H)} : \theta \in \Theta,\ t \geq 0\right\}. \tag{47}$$

The set $\mathcal{L}_{(L,H)}$ collects all the finite-length switching paths which may occur, some of which contain the dummy mode. We now define the set $\mathcal{M}_{(L,H)}(\Theta) \subset \mathcal{L}_{(L,H)}(\Theta)$ to be the smallest subset with the following properties:

- For all $\theta \in \Theta$ and $t \geq L$, $\theta_{(-L:H)}(t) \in \mathcal{M}_{(L,H)}(\Theta)$
- For all $j \in \mathcal{L}_{(0:H)}$ there exists $i \in \{0, \ldots, N\}^L$ such that for every $\theta \in \Theta$ and $0 \leq t < L$

$$\left(i_{1+t}^{\theta_{(0:H)}}, \ldots, i_{-1}^{\theta_{(0:H)}}, \theta(0), \ldots, \theta(H+t)\right) \in \mathcal{M}_{(L,H)}.$$

The set $\mathcal{M}_{(L,H)}$ is constructed both to have fewer elements than $\mathcal{L}_{(L,H)}$ and to replace some (but possibly not all) of the zero-led sequences in $\mathcal{L}_{(L,H)}$ with alternate sequences which do not have the dummy zero mode (the second condition is like the left-extending used in the proof of Theorem 12). We can now state a version of Theorem 12 for non-regular languages.

Theorem 42: For a fixed $H \geq 0$, $L \geq 0$, the switched system of (8) with admissible switching sequences Θ is uniformly exponentially stable if and only if there exist an integer $M \geq 0$ and a collection $X_j \succ 0$ for $j \in [N]^{L+M+H}$ such that

$$A^T_{\phi\left(i_{(-L:H)}\right)} X_{i_{(-L-M+1:H)}} A_{\phi\left(i_{(-L:H)}\right)} - X_{i_{(-L-M:H-1)}} \prec 0 \tag{48}$$

for every admissible $i \in \mathcal{M}_{(L+M,H)}(\Theta)$, with ϕ as in (3).

Proof: The intuition is the same as in Theorem 12. Sufficiency is demonstrated by constructing the operator X whose blocks X_t correspond to the finite-length switching sequences $\theta_{(-L-M:H)}(t)$ (the left-extension of the switching sequence is made possible by the properties of the set $\mathcal{M}_{(L+M,H)}(\Theta)$). For necessity, considering the inequalities generated by each admissible sequence $\theta \in \Theta$ produces an inequality for every element of $\mathcal{L}_{(L+M,H)}(\Theta) \supset \mathcal{M}_{(L+M,H)}(\Theta)$. ■

Results analogous to Theorems 26 and 34 follow similarly, as do the corresponding controller existence conditions, with the results stated below.

Theorem 43: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable and achieves attenuation level γ if and only if there exist an integer $\bar{L} \geq 0$ and matrices $R_j \succ 0$, $S_j \succ 0$ for $j \in [N]^{\bar{L}+H}$ such that for all $i \in \mathcal{M}_{(\bar{L},H)}(\Theta)$ the inequalities of (29) hold. Furthermore, given a solution to the inequalities of (29), a controller may be constructed with memory $L \leq \bar{L}$.

To obtain results for the stability case (i.e., Section III), replace (29) with (18).

For the windowed performance measure of Section V, we capture the possible switching sequences over the performance window T by defining

$$\mathcal{W}_T(\Theta) := \left\{(\theta(t), \ldots, \theta(t+T)) : \theta \in \Theta,\ t \geq 0\right\}.$$

Theorem 44: There exists a path-dependent controller with horizon $H \geq 0$ such that (27) is uniformly exponentially stable and achieves T-step performance level γ if and only if there exist an integer $\bar{L} \geq 0$, positive definite matrices R_j, S_j for $j \in [N]^{\bar{L}+H}$, and matrices Z_i, W_i for $i \in [N]^{\bar{L}+H+1}$ such that for all $i \in \mathcal{M}_{(\bar{L},H)}(\Theta)$ and all $\hat{\imath} \in \mathcal{M}_{(\bar{L},H+T)}$ such that $\hat{\imath}_{(0:T)} \in \mathcal{W}_T(\Theta)$ the inequalities of (38) hold. Furthermore, given a solution to the inequalities of (38), a controller may be constructed with memory $L \leq \bar{L}$.

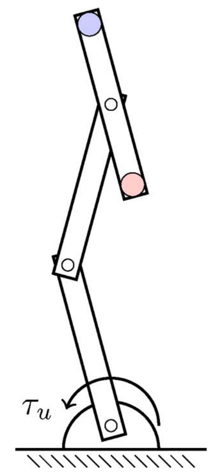

Fig. 5. Barbell double-pendulum system with a movable mass located at either the upper (blue) or lower (red) position.

VIII. Examples and Applications

We present two physically-motivated examples which illustrate how switching arises both from the dynamics of the system as well as from its interaction with the environment. For simplicity, our first example demonstrates switching in dynamics only, while the second demonstrates switching in response to environmental changes.

Example 45: Consider the system of Fig. 5, which depicts a double-pendulum system with a barbell connected to the upper linkage using the operating point where all linkages are vertical. Each link has length 1 m and mass 1 Kg. A small 0.1 Kg mass may jump between the ends of the barbell at each time step, producing two operating modes. The nonlinear dynamics of each operating mode are linearized with all linkages vertical (transitions between the two operating modes correspond to a discontinuous drop or rise in the potential energy of the system via the motion of the small mass). A controlled torque τ_u is applied about the bottom hinge, and a disturbance torque about the top hinge. The continuous dynamics are discretized using an interval of $t = 0.05$ s.

In the configuration shown in Fig. 5, the barbell can be stabilized by driving the topmost hinge (i.e. the upper end of the middle link) left if the jumping mass is at the blue end, or right if the jumping mass is at the red end; the correct control action depends on the position of the jumping mass. However, the intermediate linkage delays the effect of an input choice on the position of this hinge. If the mass switches position after a choice has been made, this action chosen will push the system away from equilibrium. Knowledge of the next switching mode allows the controller to stabilize the system.

Fig. 6 presents the minimal achievable performance level γ for which the conditions of Theorem 28 are feasible. Notice that when $H = 0$, the system cannot be stabilized and therefore the system has infinite gain. Fig. 7 presents the minimal achievable γ using instead the windowed variance of Section V with performance horizon $T = 2$. Once again, when $H = 0$ the system is not stabilized and no finite gain is possible. Also note that once $H = 2$, the minimal gain remains the same even when past states are considered. Since the performance window is

L / H	0	1	2
0	$+\infty$	$+\infty$	1.04
1	$+\infty$	1.84	0.92
2	$+\infty$	1.40	·
3	$+\infty$	·	

Fig. 6. Table of minimal H_∞ gains for varying controller memory and horizon lengths for the system of Fig. 5.

L / H	0	1	2
0	$+\infty$	$+\infty$	1.40
1	$+\infty$	1.66	1.40
2	$+\infty$	1.40	·
3	$+\infty$	·	

Fig. 7. Table of minimal windowed variance performance levels for varying controller memory and horizon lengths for the system of Fig. 5.

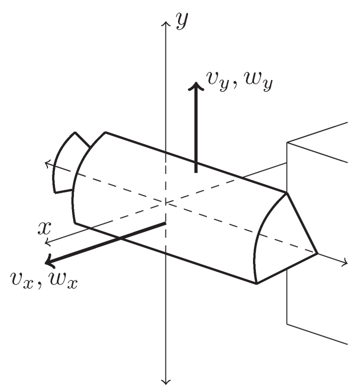

Fig. 8. Two-degrees-of-freedom spacecraft which travels along a straight trajectory past an obstacle.

of length 2, once we have $H \geq 2$ the problem reduces to that of a finite-length LQR optimization, as every parameter in the performance window is known.

Example 46: A small spacecraft shown in Fig. 8 flies on a straight reference trajectory, subject to disturbances w_x, w_y pushing the craft out of alignment. The ship has lateral thrusters v_x, v_y which respond as first-order systems to leaky control efforts u_x, u_y. The resulting system dynamics are

$$\ddot{x} = v_x + w_x; \quad \dot{v_x} = -0.5v_x + u_x + 0.1u_y$$
$$\ddot{y} = v_y + w_y; \quad \dot{v_y} = -0.5v_y + 0.1u_x + u_y.$$

During flight, the craft passes near obstacles in either the x- or y-direction; while near an obstacle, deviations in the direction of the obstacle present a greater risk to the safety of the craft. To this end, the controlled output switches between three modes given by

$$z_1 = [x \quad y \quad 0.5u_x \quad 0.5u_y]$$
$$z_2 = [5x \quad 0.5y \quad 0.5u_x \quad 0.5u_y]$$
$$z_3 = [0.5x \quad 5y \quad 0.5u_x \quad 0.5u_y]$$

where mode 1 represents unobstructed flight, while modes 2 and 3 represent flight near an obstacle in the x- or y-direction respectively. The ship is permitted observation of its x- and y-position. The dynamics are discretized with time interval $t = 0.1$ s. The switching dynamics allow the system to switch between the unobstructed mode and either obstructed modes, or to remain in the current mode (but not to switch directly from one obstructed mode to the other).

$$\begin{array}{llll} \gamma_{111} = 1.65 & \gamma_{211} = 1.48 & \gamma_{311} = 1.48 & \gamma_{121} = 1.85 \\ \gamma_{221} = 1.80 & \gamma_{131} = 1.85 & \gamma_{331} = 1.80 & \gamma_{112} = 1.61 \\ \gamma_{212} = 1.44 & \gamma_{312} = 1.44 & \gamma_{122} = 1.71 & \gamma_{222} = 2.32 \\ \gamma_{113} = 1.61 & \gamma_{213} = 1.44 & \gamma_{313} = 1.44 & \gamma_{133} = 1.71 \\ \gamma_{333} = 2.33 & & & \end{array}$$

Fig. 9. Optimal path-by-path H_∞ gains for controllers with $L = H = 1$ for the system of Example 46.

This example features a mode-dependent performance measure but mode-independent dynamics. The switching in the controlled output z designates either the x- or y- position as "critical" when an obstacle is present (a similar approach is taken in [12]). We consider controllers with $L = H = 1$ and apply the path-by-path performance result of Theorem 39, minimizing the sum of the path-dependent γ_i. The table in Fig. 9 shows the optimal path-by-path H_∞ gains for this system. The largest gains are found for paths 222 and 333, which occur when the craft is near an obstacle over several time-steps. This matches our intuition in setting the output to punish deviations when near an obstacle. Using the same controller, the optimal *uniform* gain is $\bar{\gamma} = 1.75$; comparing this value to those in the table we see that performance is improved on paths away from an obstacle, but suffers near an obstacle. In practice, the performance levels γ_i could be weighted unequally, or a subset of the γ_i given an upper bound to develop path-by-path gains which are minimized on the most likely or important paths.

IX. Concluding Remarks

We have examined receding-horizon-type control problems with different performance criteria and developed exact, convex conditions for the existence and synthesis of controllers achieving these performance goals. Our results provide a sequence of semidefinite programming problems which can be solved offline to search for a suitable controller, or to optimize controller performance. The resulting controllers are finite-path dependent, allowing for controller gains to be selected from a finite collection during runtime with no online computation needed. The examples presented demonstrate that a path-dependent controller can outperform a modal controller, and that controllers with foreknowledge of the system modes can outperform past-dependent controllers.

Of interest for future research is the relationship between the feasibility of any particular SDP and the achievable performance levels for the system (particularly the stability decay rate). While any feasible SDP generated by our results provides a certificate of stability at some decay rate, Lemma 9 suggests that the infeasibility of a particular SDP should also indicate levels of stability which cannot be achieved. The exact relationship for this case, as well as the performance cases, remains unexplored.

REFERENCES

[1] A. Alessandri, M. Baglietto, and G. Battistelli, "Receding-horizon estimation for switching discrete-time linear systems," *IEEE Trans. Autom. Control*, vol. 50, pp. 1736–1748, 2005.

[2] R. Alur, C. Belta, V. Kumar, M. Mintz, G. Pappas, H. Rubin, and J. Schug, "Modeling and analyzing biomolecular networks," *Comput. in Sci. Eng.*, vol. 4, pp. 20–31, 2002.

[3] F. Blanchini, S. Miani, and F. Mesquine, "A separation principle for linear switching systems and parametrization of all stabilizing controllers," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 279–292, 2009.

[4] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, pp. 1279–1289, 2000.

[5] P. Colaneri and R. Scattolini, "Robust model predictive control of discrete-time switched systems," in *Proc. 3rd IFAC Workshop PSYCO*, 2007.

[6] O. Costa and E. Tuesta, "H_2-control and the separation principle for discrete-time Markovian jump linear systems," *Math. Control, Signals Syst.*, vol. 16, pp. 320–350, 2004.

[7] C. De Souza, "On stabilizing properties of solutions of the Riccati difference equation," *IEEE Trans. Autom. Control*, vol. 34, pp. 1313–1316, 1989.

[8] J. B. R. do Val and T. Basar, "Receding horizon control of jump linear systems and a macroeconomic policy problem," *J. Econom. Dynam. Control*, vol. 23, pp. 1099–1131, Aug. 1999.

[9] G. Dullerud and S. Lall, "A new approach for analysis and synthesis of time-varying systems," *IEEE Trans. Autom. Control*, vol. 44, pp. 1486–1497, 1999.

[10] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory*. New York, NY USA: Springer, 1999.

[11] R. Essick, J.-W. Lee, and G. Dullerud, "An exact convex solution to receding horizon control," in *Amer. Control Conf. (ACC)*, 2012.

[12] M. Farhood and E. Feron, "Obstacle-sensitive trajectory regulation via gain scheduling and semidefinite programming," *IEEE Trans. Control Syst. Technol.*, vol. 20, pp. 1107–1115, 2012.

[13] G. Ferrari-Trecate, D. Mignone, and M. Morari, "Moving horizon estimation for hybrid systems," *IEEE Trans. Autom. Control*, vol. 47, pp. 1663–1676, 2002.

[14] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. J. Robust Nonlin. Control*, vol. 4, pp. 421–448, 1994.

[15] V. Gupta, B. Hassibi, and R. M. Murray, "Optimal LQG control across packet-dropping links," *Syst. Control Lett.*, vol. 56, pp. 439–446, 2007.

[16] A. Halanay and V. Ionescu, *Time-Varying Discrete Linear Systems*, vol. 68. Basel, Switzerland: Birkhauser, 1994, ser. Oper. Theory Adv. Appl., vol. 68, Basel, Switzerland: Birkhauser, 1994.

[17] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, pp. 988–1001, 2003.

[18] R. Jungers, *The Joint Spectral Radius: Theory and Applications*. Berlin, Germany: Springer, 2009.

[19] R. Krtolica, U. Özgüner, H. Chan, H. Göktas, J. Winkelman, and M. Liubakka, "Stability of linear feedback systems with random communication delays," *Int. J. Control*, vol. 59, pp. 925–953, 1994.

[20] J.-W. Lee, G. Dullerud, and P. Khargonekar, "An output regulation problem for switched linear systems in discrete time," in *Proc. 46th IEEE Conf. Decision and Control*, 2007.

[21] J.-W. Lee, G. Dullerud, and P. Khargonekar, "Path-by-path optimal control of switched and Markovian jump linear systems," in *Proc. 46th IEEE Conf. Decision Control*, 2008.

[22] J.-W. Lee and G. E. Dullerud, "Optimal disturbance attenuation for discrete-time switched and Markovian jump linear systems," *SIAM J. Control Optim.*, vol. 45, pp. 1329–1358, 2006.

[23] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and Markovian jump linear systems," *Automatica*, vol. 42, pp. 205–218, 2006.

[24] D. Liberzon and A. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst.*, vol. 19, pp. 59–70, 1999.

[25] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.

[26] H. Lin and P. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *Automatic Control, IEEE Trans. Autom. Control*, vol. 54, pp. 308–322, 2009.

[27] P. Mhaskar, N. El-Farra, and P. Christofides, "Predictive control of switched nonlinear systems with scheduled mode transitions," *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1670–1680, 2005.

[28] M. Müller, P. Martius, and F. Allgöwer, "Model predictive control of switched nonlinear systems under average dwell-time," *J. Process Control*, vol. 22, no. 9, pp. 1702–1710, 2012.

[29] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, pp. 215–233, 2007.

[30] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Control Lett.*, vol. 22, pp. 79–92, 1994.

[31] B.-G. Park and W. H. Kwon, "Robust one-step receding horizon control of discrete-time markovian jump uncertain systems," *Automatica*, vol. 38, pp. 1229–1235, 2002.

[32] H. Salis and Y. Kaznessis, "Accurate hybrid stochastic simulation of a system of coupled chemical or biochemical reactions," *J. Chem. Phys.*, vol. 122, p. 054103, 2005.

[33] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Autom. Control*, vol. 42, pp. 896–911, 1997.

[34] Z. Sun and S. S. Ge, *Switched Linear Systems: Control and Design*. London, U.K.: Springer-Verlag, 2005.

[35] F. Zampolli, "Optimal monetary policy in a regime-switching economy: The response to abrupt shifts in exchange rate dynamics," *J. Econom. Dynam. and Control*, vol. 30, pp. 1527–1567, 2006.

[36] L. Zhang, Y. Shi, T. Chen, and B. Huang, "A new method for stabilization of networked control systems with random delays," *IEEE Trans. Autom. Control*, vol. 50, pp. 1177–1181, 2005.

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