Lectures on Poisson Geometry

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To our families

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Preface

The aim of this book is to provide an introduction to Poisson geometry. The book grew out of several sets of lecture notes that we have prepared along many years while teaching master and graduate level courses at our home institutions and mini-courses at various Poisson geometry schools. In particular, the writing of the book was influenced by our experiences teaching the material and by the interactions we have had with the students who attended those lectures. Although it is fair to say that the book has grown and includes a bit more material that one can actually hope to cover in class during a one semester course, the aim remains the same: to provide lecture notes for a graduate level course giving an introduction to Poisson geometry, addressed to students and researchers which have some familiarity with classical differential geometry and differentiable manifolds. Some basic knowledge of algebraic topology and symplectic geometry would be a plus, but not a requirement, to fully grasp some parts of the book. Some standard topics from differential geometry that we need, but might be missing from an introductory course, are summarized in the appendices at the end of the text.

Poisson geometry emerged from the mathematical formulation of classical mechanics. Historically, it all started with the work of Siméon Denis Poisson on the mechanics of particles which led him to the discovery in 1809 of the so-called Poisson bracket as a method for obtaining new integrals of the motion. Poisson computations occupied many pages, and his results where rediscovered and simplified two decades later by Carl Gustav Jacob Jacobi, who was the first to realize the fundamental role played by the Poisson bracket in rational mechanics and who identified its main properties: an operation (bracket) which associates to any two observables f and g a new observable $\{f, g\}$, and which satisfies the Leibniz and Jacobi identity. Jacobi's work on Poisson brackets, including the discovery of his famous identity, the commutator of derivations, etc., greatly influenced Sophus Lie in his foundational study at the end of the 19th century of symmetries of partial differential equations, which led him to the discovery of Lie groups and Lie algebras (see [**102**]). *Linear* Poisson structures correspond to Lie algebra structures, so Lie was in fact the first to study them and it is remarkable how deep Lie's work dives into Poisson geometric aspects. For instance, Lie explicitly poses the realization problem for linear Poisson structures, a problem which turns out to be the same as that of searching for a Lie group integrating a Lie algebra. However, perhaps somewhat surprisingly, the first geometric, systematic, study of Poisson structures occurred much more recently in the work of André Lichnerowicz [**108**] in the 1970s, which marks the birth of Poisson *geometry* in its modern formulation.

The spectacular development of Poisson geometry from the last few decades owes much to the foundational work of Alan Weinstein [**146**] in the 80s and his discovery of symplectic groupoids as the global objects behind Poisson structures [**150**]. In retrospective, this discovery follows the same path as in Lie's work: the search for non-degenerate (symplectic) realizations led to the discovery of interesting global structures. In some sense, this book can be seen as an updated and expanded exposition of Weinstein's pioneer work. In particular, our aim here is not to provide a survey of the vast amount of work done in this subject in the last 30-40 years, but rather to provide an introduction to the subject that will allow the reader to plunge into any of these recent exciting developments, some of which are mentioned throughout the text.

We have tried to provide our own insight into the subject while resisting the temptation of concentrating on our contributions. Our philosophy can be summarized as follows: Poisson geometry is an amalgam of foliation theory (partition into leaves), symplectic geometry (along the leaves) and Lie theory (transverse to the leaves). In particular, it provides the framework in which these geometries get to interact with each other in a beautiful symbiosis. While this is already, we believe, the main message in Weinstein's foundational paper [**146**], the full extent of this interaction came to life later with the discovery of the global counterparts to Poisson structures: *symplectic Lie groupoids*. These objects codify all these 3 different aspects and we have organized the book so that one is led naturally to uncover them, giving an upgraded view on Weinstein's and Lichnerowicz's works.

The monograph by Vaisman [**140**] was for a long period of time the only text book on Poisson geometry, apart from an earlier account by Bhaskara and Viswanath [**15**]. The book by Cannas da Silva and Weinstein [**30**] contains a nice elementary introduction to the subject, aimed towards non-commutative geometry and quantization. A more up-to-date account of Poisson geometry, with a strong emphasis on local normal forms, was provided by Dufour and Zung in their research monograph [**59**]. More recently, appeared the beautiful book by Laurent-Gengoux, Pichereau and Vanhaecke [**105**], which is highly recommended for people with an algebraic-geometric background. As the authors point out in the introduction, "The main topic about Poisson structures which is absent from this book is what should be called Poisson geometry." We hope that our book provides an introduction to Poisson geometry, which can be assimilated during a semester-long course, or can be used as material for self-study of the topic.

The main body of the book is divided into four parts, followed by the appendices that were already mentioned. Each part ends with a small set of notes containing brief historical comments and directions for further reading. The best overview of the book is its table of contents. Still, we would like to emphasize that we payed special attention to the way we introduce those basic concepts in the theory that are more complex and require a deeper thought process. Take for example the notion of symplectic leaf: set-theoretically, we introduce them right away in the first chapter as the orbits of Hamiltonian diffeomorphisms, promising the reader that the actual structure (smooth, symplectic) will be discussed later. In the second chapter, we take advantage of the bivector field point of view to indicate how the smooth structure may arise from a Frobenius type theorem. However, the actual local result that is needed, the Weinstein Splitting Theorem, is then dealt with in chapter three. Finally, we discuss properly their smooth and symplectic structure in the fourth chapter of the book. We have also paid special attention to examples and exercises - at the price of increasing the size of the book. Several sections of the book are called "Examples" or "Case study", and there are well over 200 exercises, split into two types: the ones spread throughout the text, called "Exercises", which are helpful in understanding the main material, and the ones listed at the end of each chapter, called "Problems", which are useful in consolidating the material and providing further examples. We have tried to fill in a gap in the existing literature by providing a longer list of concrete examples of symplectic realizations and symplectic groupoids. We have made an effort to include full proofs for all the results we discuss, the exception being Lie's 3rd Theorem for Lie algebroids. Some of the arguments used in the proofs are new, others simplify and fill some gaps in the literature (see the notes and references at the end of each Part).

There are a few topics, which may be now considered standard in Poisson geometry, which we have decided not to include, such as Poisson-Lie groups, deformation quantization, generalized complex structures or integrable systems. They go beyond our purpose here and they deserve a separate volume. We hope that our book will provide a solid background for learning such topics, or for moving to more advanced ones in the cutting edge of research.

Acknowledgments

We would like to express our appreciation to the many students, in particular to our PhD students, who took part in the various courses on Poisson geometry we have taught in Utrecht, Urbana-Champaign and Nijmegen, as well as in various summer schools on Poisson geometry. Their comments and feedback were invaluable to the writing of this book.

There are many colleagues and collaborators with whom we have interacted throughout the years. These interactions have shaped our views of Poisson geometry and mathematics in general, and from them we have learned many ideas which have influenced the writing of this book. We are grateful to all of them!

We would like to thank also our home institutions for providing a stimulating atmosphere and work environment and being welcoming during our mutual visits.

We have been generously supported by the NWO and NSF during the writing of this book, most recently by NWO Vici grant 639.033.312 and NSF grant DMS-2003223.

March 11, 2021

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