Math 595 - Poisson Geometry
Chapter 8 - Submanifolds

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- Poisson submanifolds
- Poisson-Dirac submanifolds
- Coisotropic submanifolds
- Pre-Poisson submanifolds

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The different notions of submanifold $N \subset (M, \pi)$ all have to do with how TN interacts with its π-orthogonal:

$$(TN)^{\perp_\pi} := \pi^\sharp(TN)^0.$$

1) Poisson submanifolds

Definition

A **Poisson submanifold** of a Poisson manifold (M, π_M) is a Poisson manifold (N, π_N) together with an injective immersion $i : N \hookrightarrow M$ which a is Poisson map.

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Proposition

Let (M, π_M) be a Poisson manifold. Given an immersed submanifold $N \hookrightarrow M$ there is at most one Poisson structure π_N on N that makes (N, π_N) into a Poisson manifold.

Poisson submanifolds: alternative characterizations

For a submanifold $N \subset (M, \pi_M)$ the following equivalent conditions hold:

(i) N is a Poisson submanifold;

(ii) $\operatorname{Im} \pi^{\#}_{M,x} \subset T_x N$, for all $x \in N$;

(iii) $(TN)^{\perp_\pi} = 0$;

(iv) every hamiltonian vector field $X_H \in \mathfrak{X}(M)$ is tangent to N.

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When N is a closed submanifold, these condition are also equivalent to:

(vi) The vanishing ideal of N

$$\mathscr{I}(N) := \{f \in C^\infty(M) : f(x) = 0, \forall x \in N\}$$

is a Poisson ideal, i.e., for any $f \in \mathscr{I}(N)$, $g \in C^\infty(M)$, one has $\{f, g\} \in \mathscr{I}(N)$.

More about Poisson submanifolds

- A Poisson submanifold $N \hookrightarrow M$ intersects a symplectic leaf S of (M, π_M) in an open subset of S.
- The connected components of the intersections $N \cap S$ are the symplectic leaves of (N, π_N).

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Some Examples:

(i) symplectic leaves;

(ii) $\mathfrak{h}^0 \subset \mathfrak{g}^*$, provided $\mathfrak{h} \subset \mathfrak{g}$ is an ideal;

(iii) the spheres $||v|| = c$ in the dual of a compact Lie algebra $\mathfrak{g}$;

(iv) the singular locus in a log-symplectic submanifold;

(v) (see Lecture notes for more examples)

Symplectic realizations of Poisson submanifolds

Proposition

Let N be a Poisson submanifold of (M,π) and

$$\mu : (S,\omega) \to (M,\pi)$$

a symplectic realization. If $C := \mu^{-1}(N)$, $\omega_C := \omega|_C \in \Omega^2(C)$:

(i) *the kernel of ω_C,*

$$\mathscr{K}_C := \mathrm{Ker}\,\omega_C \subset TC,$$

defines a regular foliation on C.

(ii) *if this foliation is simple and the leaf space is denoted $S_N := C/\mathscr{K}_C$, then ω_C descends to a symplectic form ω_N on S_N and μ descends to a smooth map*

$$\mu_N : (S_N,\omega_N) \to (N,\pi_N)$$

which is a symplectic realization of (N,π_N),

2) Poisson-Dirac submanifolds

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Definition

A **Poisson-Dirac submanifold** of a Poisson manifold (M, π_M) is a Poisson manifold (N, π_N) together with an immersion

$$i : (N, L_{\pi_N}) \hookrightarrow (M, L_{\pi_M})$$

which is a backward Dirac map.

Note that given a submanifold N of a Poisson manifold (M, π_M) there is at most one Poisson structure on N making the inclusion a Poisson-Dirac submanifold.

Poisson-Dirac submflds: ”Dirac-free” characterizations

$N \subset (M, \pi)$ is a Poisson-Dirac submanifold if and only if the following two conditions hold:

(i) $T_x N \cap (T_x N)^{\perp_\pi} = \{0\}$ for all $x \in N$,

(ii) the bivector field $\pi_N \in \Gamma(\wedge^2 TN)$ defined at each point by

$$\pi_N(\xi_x, \eta_x) = \pi(\tilde{\xi}_x, \tilde{\eta}_x) \quad \text{for } \xi_x, \eta_x \in T_x^* N, \tag{1}$$

with $\tilde{\xi}_x, \tilde{\eta}_x \in (T_x^{\perp_\pi} N)^0$ extensions of ξ_x, η_x, is smooth.

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Remarks:

- Extensions in (ii) exist by item (i).
- The proposition says that (i) and (ii) imply that $\pi_N \in \mathfrak{X}(N)$ will automatically be Poisson.

More about Poisson-Dirac submanifolds

- A Poisson-Dirac submanifold $N \hookrightarrow M$ intersects each symplectic leaf S of (M, π_M) in a symplectic submanifold.
- The connected components of the intersections $N \cap S$ are the symplectic leaves of (N, π_N).

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Some Examples:

(i) Poisson submanifolds: $N \cap S$ is open in S;

(ii) Poisson transversals: N intersects each leaf S transversely;

(iii) Any point $\{x\}$ is a Poisson-Dirac submanifold of (M, π).

(iv) $\mathfrak{h}^0 \subset \mathfrak{g}^*$ is Poisson-Dirac if $\mathfrak{h} \subset \mathfrak{g}$ admits a complement $\mathfrak{k} \subset \mathfrak{g}$ such that:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}, \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h}.$$

(v) (see Lecture notes for more examples)

Another example

Take Lotka-Volterra type Poisson structure on $M = \mathbb{R}^4$:

$$\{x, y\} = xy, \quad \{x, z\} = 0, \quad \{x, w\} = xw,$$
$$\{y, z\} = yz, \quad \{y, w\} = 0, \quad \{z, w\} = zw.$$

The embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$, $(u, v) \mapsto (u, v, u, v)$ gives a Poisson-Dirac submanifold.

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This is neither a Poisson submanifold nor a Poisson transversal.

Poisson-Dirac submanflds w/ $(TN)^{\perp_\pi}$ of constant rank

Corollary

Let (M, π) be a Poisson manifold and $N \subset M$ a submanifold with the property that $(TN)^{\perp_\pi}$ has constant rank. Then N is a Poisson-Dirac submanifold if and only if $TN \cap (TN)^{\perp_\pi} = 0$.

Proposition

Given a Poisson manifold (M, π) and a Poisson-Dirac submanifold N, the following are equivalent:

(i) *$(TN)^{\perp_\pi}$ has constant rank,*

(ii) *N is a Poisson submanifold inside some Poisson transversal X of (M, π).*

Moreover, the germ of X around N is unique up to Poisson diffeomorphisms.

Poisson-Dirac submanflds w/ $(TN)^{\perp_\pi}$ of constant rank

Proposition

Let $N \subset (M, \pi)$ be a Poisson-Dirac submanifold with $(TN)^{\perp_\pi}$ of constant rank and assume we have a symplectic realization:

$$\mu : (S, \omega) \to (M, \pi).$$

If $P := \mu^{-1}(N)$ and $\omega_P := \omega|_P \in \Omega^2(P)$, then:

(i) *P is pre-symplectic with ω_P of constant rank;*

(ii) *the kernel of $\mathscr{K}_P := \mathrm{Ker}\,\omega_P \subset TP$ defines a regular foliation on P;*

(iii) *if $\mathscr{K}_P$ is a simple foliation, with leaf space denoted $S_N := P/\mathscr{K}_P$, then ω_P descends to a symplectic form ω_N on S_N and μ descends to a smooth map*

$$\mu_N : (S_N, \omega_N) \to (N, \pi_N)$$

which is a symplectic realization of (N, π_N).

3) Coisotropic submanifolds

The previous classes of submanifolds had the important property that they carry induced Poisson structures. The next class is different:

Definition

A **coisotropic** submanifold of a Poisson manifold (M, π) is any submanifold $C \subset M$ satisfying:

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Proposition

A smooth map $\Phi : M_1 \to M_2$ between two Poisson manifolds (M_1,π_1) and (M_2,π_2) is a Poisson map if and only if its graph,

$$Graph(\Phi) = \{(x_1,\Phi(x_1)) : x_1 \in M_1\} \subset M_1 \times M_2,$$

is a coisotropic submanifold of $(M_1,\pi_1)\times(M_2,-\pi_2)$.

Characterizations of closed coisotropic submanifolds

For a **closed** submanifold $C \subset (M, \pi)$ the following conditions are equivalent:

(i) C is a coisotropic submanifold;

(ii) The vanishing ideal $\mathscr{I}(C)$ is a Poisson subalgebra;

(iii) For all $h \in \mathscr{I}(C)$ the hamiltonian vector field X_h is tangent to C.

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- So there is still a ”Poisson flavor”!

Definition

The **characteristic distribution** of a coisotropic submanifold C of a Poisson manifold (M, π) is:

$$\mathscr{K}_C := (TC)^{\perp_\pi} \subset TC.$$

Coisotropic reduction

Theorem

Let C be a coisotropic submanifold of (M, π) and assume that characteristic distribution $\mathscr{K}_C$ has constant rank. Then:

(i) $\mathscr{K}_C$ *is a regular foliation,*

(ii) *if $\mathscr{K}_C$ is simple then its leaf space $\underline{C} := C/\mathscr{K}_C$ carries a canonical Poisson structure $\underline{\pi} \in \mathfrak{X}^2(\underline{C})$.*

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- **Remark:**

This generalizes the symplectic reduction of a Hamiltonian G-space $\mu : (S, \omega) \to \mathfrak{g}^*$: if 0 is regular value then $\mu^{-1}(0) \subset S$ is a coisotropic submanifold. If action is proper and free then the characteristic distribution of $C = \mu^{-1}(0)$ are the orbits of the action, and:

$$\underline{C} = \mu^{-1}(0)/G = M//G$$

More on coisotropic submanifolds

Many Examples:

(i) $(TC)^{\perp_\pi} = 0$: these are the **Poisson submanifolds**;

(ii) $(TC)^{\perp_\pi} = TC$: these are **Lagrangian submanifolds**;

(iii) Any codimension 1 submanifold $C \subset (M, \pi)$;

(iv) $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ is a coisotropic submanifold if and only if $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra;

(v) (see Lecture notes for more examples)

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Proposition

Let $\Phi : (M, \pi_M) \to (N, \pi_N)$ be a Poisson map and assume that Φ is transverse to a submanifold $C \subset N$. Then $\Phi^{-1}(C) \subset M$ is coisotropic if and only if $C \subset N$ is coisotropic.

Coisotropic embedding theorem

Theorem (Gotay's coisotropic embedding theorem)

Let (C, ω_C) *be a manifold with a closed 2-form. There exists symplectic manifold* (M, ω) *and coisotropic embedding* $i : C \hookrightarrow M$ *such that* $\omega_C = i^* \omega$ *if and only if* $\operatorname{Ker} \omega_C$ *has constant rank.*

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Recall that for a Dirac structure:

$$\ker \mathbb{L} = \mathbb{L} \cap T^*M.$$

Generalizing Gotay's theorem:

Theorem (Coisotropic embedding theorem)

Let $(C, \mathbb{L}_C)$ *be a Dirac manifold. There exists a Poisson manifold* (M, π) *and a coisotropic embedding* $i : C \hookrightarrow M$ *such that* $\mathbb{L}_C = i^*\mathbb{L}_\pi$ *if and only if* $\mathrm{Ker}\,\mathbb{L}_C$ *has constant rank.*

4) Pre-Poisson submanifolds

The previous classes can be related via the following notion:

Definition

A **pre-Poisson submanifold** of (M, π_M) is a submanifold $P \subset M$ with the property that

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Remark:

- For a symplectic manifold (S, ω), a submanifold $P \subset S$ is a pre-Poisson submanifold if and only if $\omega|_P$ has constant rank. In Symplectic Geometry, these are usually called **pre-symplectic submanifolds**.

Relation between submanifolds