Math 595 - Poisson Geometry
Chapter 12 - Complete symplectic realizations

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What is this chapter about:

We are now entering the last part of these lectures:

- **Aim:** What are the group-like objects integrating the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$?

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This chapter starts unveiling these objects by looking at symplectic realizations:

- Discuss classes of symplectic realizations;
- Look at special classes: non-degenerate, zero and linear Poisson structures;
- Extrapolate from these examples to general symplectic realizations.

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(iii) a **complete** symplectic realization if μ is complete i.e., for any smooth family $f_t \in C^\infty(M)$ of time-dependent functions one has:

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Notations: For $\mu : (S,\omega) \to (M,\pi)$ and $\alpha \in \Omega^1(M)$:

- $X_\alpha := \pi^\sharp(\alpha) \in \mathfrak{X}(M)$
- $a(\alpha) \in \mathfrak{X}(S)$ defined by $i_{a(\alpha)}\omega = \mu^*\alpha$.

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Proposition

The following are equivalent characterizations of completeness of $\mu : (S,\omega) \to (M,\pi)$:

(i) *For $f_t \in C^\infty(M)$: X_{f_t} complete $\Rightarrow X_{\mu^*(f_t)} \in \mathfrak{X}(S)$ complete.*

(ii) *For $\alpha_t \in \Omega^1(M)$: X_{α_t} complete $\Rightarrow a(\alpha_t) \in \mathfrak{X}(S)$ complete*

(iii) *For $f_t \in C^\infty_c(M)$, $X_{\mu^*(f_t)} \in \mathfrak{X}(S)$ is complete.*

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Note: given $\alpha_t \in \Omega^1(M)$ and $\gamma : I \to M$ there exists $f_t \in C^\infty_c(M)$:

$$(\alpha_t)|_{\gamma(t)} = (\mathrm{d}f_t)|_{\gamma(t)}, \quad \forall t \in I.$$

2) Study case: linear Poisson structures

Proposition

Assume (S,ω) is symplectic, $\mu : S \to \mathfrak{g}^*$ is smooth map, G the 1-connected Lie group integrating $\mathfrak{g}$, and let:

$$a : \mathfrak{g} \to \mathfrak{X}(S), \qquad i_{a(v)}\omega = \mathrm{d}\mu_v.$$

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$$\begin{array}{c} \mu : (S, \omega) \to (\mathfrak{g}^*, \pi_{\mathfrak{g}}) \\ \text{Poisson map} \end{array} \quad \Leftrightarrow \quad \begin{array}{c} a : \mathfrak{g} \to \mathfrak{X}(S) \text{ is } \mathfrak{g}\text{-action} \\ \text{and } \mu \text{ is } \mathfrak{g}\text{-equivariant} \end{array}$$

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$\mu : (S,\omega) \to (\mathfrak{g}^*, \pi_{\mathfrak{g}})$ *Poisson map* $\Leftrightarrow$ $a : \mathfrak{g} \to \mathfrak{X}(S)$ *is $\mathfrak{g}$-action and μ is $\mathfrak{g}$-equivariant*

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$$\left\{\begin{array}{c}\text{complete symplectic realizations}\\ \mu : (S,\omega) \to (U, \pi_{\mathfrak{g}}|_U)\\ \text{with } U \subset \mathfrak{g}^* \text{ open } G\text{-invariant}\end{array}\right\} \stackrel{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\text{locally free}\\ G\text{-Hamiltonian}\\ \text{spaces } (S,\omega)\end{array}\right\}$$

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Proposition

Assume (S,ω) is symplectic, $\mu : S \to M$ a submersion, and let:

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$$\left\{ \begin{array}{c} \text{Symplectic realization} \\ \mu : (S,\omega) \to (M,0) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{infinitesimal action of abelian} \\ \text{Lie algebra } T_x^*M \text{ on } \mu^{-1}(x) \end{array} \right\}$$

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Lagrangian fibrations

Corollary

Symplectic realizations $\mu : (S, \omega) \to (M, 0)$ *w/* $\dim S = 2 \dim M$ *are the same thing as surjective submersions* $\mu : (S, \omega) \to M$ *with Lagrangian fibers.*

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Definition

A surjective submersion $\mu : (S, \omega) \to M$ with *connected*, Lagrangian fibers is called a (regular) **Lagrangian fibration.**

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Definition

A surjective submersion $\mu : (S, \omega) \to M$ with *connected*, Lagrangian fibers is called a (regular) **Lagrangian fibration.**

If $\mu : (S, \omega) \to M$ is proper Lagrangian fibration:

- $(T_x^* M, +)$ acts on the fiber $\mu^{-1}(x)$: locally free and transitive;
- isotropy groups:

$$\Lambda_x = \{\xi \in T_x^* M : \phi^1_{a(\xi)} = \mathrm{id}_{\mu^{-1}(x)}\}.$$

  This is called the **subgroup of periods** of the Lagrangian fibration at x.

Geometry of proper Lagrangian fibrations

Notations: For $\mu : (S, \omega) \to M$ proper Lagrangian fibration:

- group of periods Λ_x (isotropy of $(T_x^*M, +)$-action on $\mu^{-1}(x)$)
- $\mathscr{T}_{\Lambda_x} := T_x^*M/\Lambda_x$

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Proposition

(i) *each* $\Lambda_x \subset T^*_x M$ *is a lattice so* $\mathscr{T}_{\Lambda_x} \simeq \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$*;*

(ii) $\Lambda = \bigcup_{x \in M} \Lambda_x \subset T^* M$ *is an integrable lattice;*

(iii) *the actions* (1) *are free and transitive, so* $\mu^{-1}(x) \simeq \mathbb{T}^n$*,*

$$\begin{array}{ccc} \mathscr{T}_\Lambda \circlearrowleft & (S, \omega) & \\ \searrow & \downarrow \mu & \\ & (M, 0) & \end{array}$$

Lagrangian fibrations and integral affine structures

Definition

An **integrable lattice** $\Lambda \subset T^*M$ consists of abelian subgroups $\Lambda_x = \Lambda \cap T^*_x M$ forming a lattice and such that Λ is locally spanned by closed 1-forms.

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An **integral affine structure** on M is an atlas $\{(U_i, \phi_i)\}_{i \in I}$ such that the transition functions are integral affine transformations:

$$\phi_i \circ \phi_j^{-1}(x) = Ax + v, \quad A \in \mathrm{GL}(n, \mathbb{Z}), v \in \mathbb{R}^n.$$

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Given $\Lambda \subset T^*M$ define a local chart $(U, \phi) = (U, x^i)$ to be an integral affine chart if:

$$\Lambda|_U = \{k_1 \mathrm{d}x^1 + \cdots + k_n \mathrm{d}x^n : k_i \in \mathbb{Z}\}.$$

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Conversely, given integral affine structure $\{(U_i, \phi_i)\}_{i \in I}$ define a lattice:

$$\Lambda_x := \{k_1 \mathrm{d}x^1|_x + \cdots + k_n \mathrm{d}x^n|_x : k_i \in \mathbb{Z}\},$$

for any integral affine chart $(U, \phi) = (U, x^i)$ containing x

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Conclusion: The base of a proper Lagrangian fibration (=proper symplectic realization of zero Poisson structure with connected fibers) has a natural integral affine structure.

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Proposition

For any $\mu : (S, \omega) \to (M, \pi)$, the integrable distribution

$$\mathscr{H} := \mathscr{F}_\mu^\perp \subset S,$$

defines a flat Ehresmann connection on μ:

$$TS = \operatorname{Ker} d\mu \oplus \mathscr{H}.$$

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Given $p_0 \in \mu^{-1}(\gamma(0))$ there is unique $\tilde{\gamma}^{p_0} : [0,\varepsilon[\to S$:

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Complete + Flat $\Longrightarrow$ τ_γ is invariant under path-homotopy

Homotopy groupoid

The **homotopy groupoid** of M:

$$\Pi(M) := \frac{\{\gamma : [0,1] \to M\}}{\sim} \underset{s}{\overset{t}{\rightrightarrows}} M$$

source/target maps:

$$s : \Pi(M) \to M, \quad [\gamma] \mapsto \gamma(0),$$
$$t : \Pi(M) \to M, \quad [\gamma] \mapsto \gamma(1)$$

multiplication:

$$[\gamma_1] \circ [\gamma_2] := [\gamma_1 \circ \gamma_2] \quad \text{if} \quad s([\gamma_1]) = t([\gamma_2])$$

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We have an "action":

$$\Pi(M, x_0, x_1) \times \mu^{-1}(x_0) \to \mu^{-1}(x_1), \quad ([\gamma], p) \mapsto [\gamma] \cdot p := \tau_\gamma(p),$$

where:

$$\Pi(M, x_0, x_1) := \{[\gamma] \in \Pi(M) : s([\gamma]) = x_0, t([\gamma]) = x_1\}.$$

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(i) the class of the constant path $\gamma(t) = x$ acts as an identity:

$$[x] \cdot p = p$$

(ii) if $[\gamma_1], [\gamma_2] \in \Pi(M)$ are composable one has:

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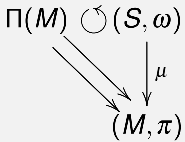
Proposition

If (M, π) is a nondegenerate Poisson manifold then any complete symplectic realization $\mu : (S, \omega) \to (M, \pi)$ carries a canonical action of $\Pi(M)$:



5) General symplectic realizations

$$\mu : (S, \omega) \to (M, \pi)$$

- **vertical foliation**: $\mathscr{F}_\mu = \operatorname{Ker} \mathrm{d}\mu$;
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(a) pointwise: linear map $a_p : T^*_{\mu(p)} M \to T_p S$ for each $p \in S$

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$$a([\alpha, \beta]_\pi) = [a(\alpha), a(\beta)], \quad \forall \alpha, \beta \in \Omega^1(M).$$

Properties of the action

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\begin{array}{ccc}
 & & T_p S \\
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(iii) in the isotropy Lie algebra gives a linear isomorphism

$$a_p : \mathfrak{g}_{\mu(p)} = \mathrm{Ker}\,\pi^\sharp_{\mu(p)} \xrightarrow{\sim} \mathscr{F}_{\mu,p} \cap \mathscr{F}^\perp_{\mu,p} .$$

Geometry of symplectic realizations

Corollary

Each orbit $\mathscr{O}_p \in \mathscr{F}_\mu^\perp$ is a presymplectic manifold and the restriction $\mu_{\mathscr{O}_p} := \mu|_{\mathscr{O}_p}$ is a submersion onto the symplectic leaf containing $\mu(p)$:

$$\mathscr{O}_p \overset{i}{\hookrightarrow} (S, \omega)$$

$$\mathscr{O}_p \xrightarrow{\mu_{\mathscr{O}_p}} (S_{\mu(p)}, \omega_{S_{\mu(p)}})$$

$$i^*\omega = \mu^*_{\mathscr{O}_p}\omega_{S_{\mu(p)}}.$$

Complete symplectic realizations

$$\begin{array}{ccc} \mu^* T^* M & \xrightarrow{a} & TS \\ \downarrow & & \downarrow \mathrm{d}\mu \\ T^* M & \xrightarrow[\pi^\sharp]{} & TM \end{array}$$

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Definition

Given a symplectic realization, a **lift of a cotangent path** $a : [0, 1] \to T^* M$ to S is any path

$$\tilde{\gamma} : [0, \varepsilon[\to S, \quad \text{such that} \quad \begin{cases} \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}(t) = a_{\tilde{\gamma}(t)}(a(t)), \\ \mu(\tilde{\gamma}(t)) = \gamma_a(t). \end{cases}$$

We call $\tilde{\gamma}$ a **complete lift** if it is defined for $t \in [0, 1]$.

Complete symplectic realizations

$$\mu : (S, \omega) \to (M, \pi)$$

Proposition

Given cotangent path $a : [0,1] \to T^*M$ *and* $p_0 \in \mu^{-1}(\gamma_a(0))$*:*

- *there exists a unique lift* $\tilde{\gamma}_a^{p_0} : [0, \varepsilon[\to S$ *starting at* p_0*.*
- *there exist complete lifts* $\tilde{\gamma}_a^{p_0}$ *if and only if* μ *is complete.*

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Complete realization $\Longrightarrow$ parallel transport:

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Proposition

- $a \sim b \Longleftrightarrow$ *if* $\tilde{\gamma}_a^{p_0} \sim \tilde{\gamma}_a^{p_0}$ *in a leaf of* $\mathscr{F}_\mu^\perp$.
- $a \sim b \Longrightarrow \tau_a = \tau_b$.

Groupoid of a Poisson manifold

$$\Pi(M,\pi) := \frac{\{a : [0,1] \to T^*M : \text{cotangent path}\}}{\sim} \underset{s}{\overset{t}{\rightrightarrows}} M$$

source/target:

$$s : \Pi(M,\pi) \to M, \quad [a] \mapsto \gamma_a(0),$$
$$t : \Pi(M,\pi) \to M, \quad [a] \mapsto \gamma_a(1).$$

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Proposition (action on complete realization)

Any complete symplectic realization $\mu : (S,\omega) \to (M,\pi)$ carries a canonical action of $\Pi(M,\pi)$ defined by:

$$[a] \cdot p := \tau_a(p) \quad \textit{if} \quad s([a]) = \mu(p)$$