An invitation to Poisson geometry and its applications

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Contents:

- Poisson brackets and Hamiltonian dynamics
- Poisson manifolds
- Local Poisson geometry
- Global Poisson geometry
- Deformation quantization

Definition

A Poisson bracket on a manifold M is a **Lie bracket**

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the **Leibniz identity**:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

The pair $(M, \{\cdot,\cdot\})$ is called a Poisson manifold.

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A Poisson map $\phi : (M_1, \{\cdot,\cdot\}_1) \to (M_2, \{\cdot,\cdot\}_2)$ is a smooth map such that pullback is a Lie algebra morphism:

$$\{f \circ \phi, g \circ \phi\}_2 = \{f, g\}_1 \circ \phi, \quad \forall f, g \in C^\infty(M_2).$$

Hamiltonian Dynamics

On a Poisson manifold $(M, \{\cdot, \cdot\})$ a function $h \in C^\infty(M)$ determines a hamiltonian vector field X_h by:

$$X_h(f) := \{h, f\}, \quad \forall f \in C^\infty(M).$$

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Basic Properties

- I is a first integral of X_h if and only if $\{h, I\} = 0$;
- h is always a first integral of X_h;
- If I_1 and I_2 are first integrals of X_h, then $\{I_1, I_2\}$ is also a first integral of X_h.

Classical Mechanics (Newton's Equations)

- Motion of a particle $q(t) \in \mathbb{R}^n$ in a potential $V : \mathbb{R}^n \to \mathbb{R}$:

$$m_i \ddot{q}_i(t) = -\frac{\partial V}{\partial q_i} \quad \Leftrightarrow \quad \begin{cases} \dot{q}_i = \frac{p_i}{m_i} \\ \\ \dot{p}_i = -\frac{\partial V}{\partial q_i} \end{cases}$$

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$M = \mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_{2n}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$:

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} \right).$$

$$h = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)$$

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Then Newton's equations are equivalent to:

$$\dot{x}_a = \{h, x_a\}, \quad (a = 1, \ldots, n)$$

Elasticity (Euler's Equation)

- Motion of a top in absence of gravity, moving around its center of mass, with moments of inertia I_1, I_2 and I_3:

$$\begin{cases} \dot{x}_1 = \frac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dot{x}_2 = \frac{I_3 - I_1}{I_3 I_1} x_3 x_1, \\ \dot{x}_3 = \frac{I_1 - I_2}{I_1 I_2} x_1 x_2. \end{cases}$$

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$M = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3):

$$\{f, g\}(x) = (\nabla f(x) \times \nabla g(x)) \cdot x.$$

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Population Dynamics (Lotka-Volterra equations)

- The dynamics of of n biological species $(x_1, \ldots, x_n)$ interacting in a closed ecosystem:

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$M = \mathbb{R}^n_+$ with coordinates $(x_1, \ldots, x_n)$:

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Problems in Hamiltonian Dynamics

- How does the Poisson geometry constrain the dynamics?
- Is the system stable under perturbation?
- What are symmetries of a system? Reduction using symmetries?
- What is a (completely) integrable system?
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Many open questions beyond the *symplectic case*.

Poisson tensors

1:1 correspondence:

$$\left\{\begin{matrix}\text{Poisson brackets } \{\cdot,\cdot\} \\ \text{on a manifold } M\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{bivector fields } \pi \in \Gamma(\wedge^2 TM) \\ \text{satisfying } [\pi,\pi] = 0\end{matrix}\right\}$$

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In a local chart (U, x^i):

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In this language:

- **Hamiltonian vector field:** $X_h = \pi^\sharp(\mathrm{d}f)$ ("gradient of h")
- **rank at $x \in M$:** $\mathrm{rank}_x\, \pi = \dim(\mathrm{Im}(\pi^\sharp))$ (even integer).

Some examples of Poisson manifolds

- **symplectic manifolds:** (M, ω) where $\omega \in \Omega^2(M)$ is closed and non-degenerate:

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- **Oriented 3-manifolds:** (M^3, μ) where $\mu \in \Omega^3(M)$ is a volume form. Every $F \in C^\infty(M)$ determines a Poisson structure:

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- **b-symplectic structures:** A symplectic form with a log-type singularity along a divisor $Z \subset M$, determines a *smooth* Poisson structure. In local coordinates:

$$\omega = \frac{1}{x}\mathrm{d}x \wedge \mathrm{d}y + \sum_{i=1}^{n-1} \mathrm{d}q_i \wedge \mathrm{d}p_i \quad \leftrightarrow \quad \pi = x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \sum_{i=1}^{n-1} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$$

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- **Poisson-Lie groups:** A Lie group G with a Poisson structure π such that the multiplication is a Poisson map:

$$m : (G \times G, \pi \oplus \pi) \to (G, \pi), \quad (g, h) \mapsto gh.$$

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- **Moduli spaces of flat connections:** The moduli space $\mathcal{M}$ of principal G-bundles with a flat connection over a surface Σ with boundary:

$$\mathcal{M} = \mathrm{Hom}(\pi_1(\Sigma), G)/G,$$

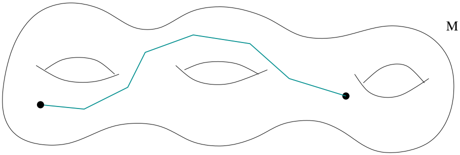
  has a natural Poisson structure (symplectic if $\partial\Sigma = \emptyset$).

Symplectic Foliation

For any two hamiltonian functions h_1 and h_2:

$$[X_{h_1}, X_{h_2}] = X_{\{h_1, h_2\}}$$

Define an **equivalence relation** on M by declaring two points equivalent if they can be joined by trajectories of hamiltonian vector fields.



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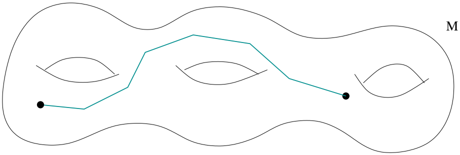

Theorem (Weinstein, 1983)

The equivalence classes form a singular foliation of $(M, \{\cdot,\cdot\})$ by symplectic submanifolds.

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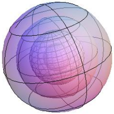

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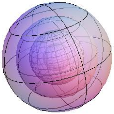

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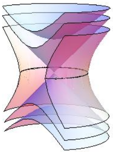

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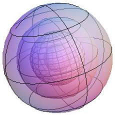

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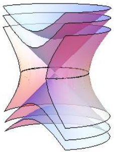

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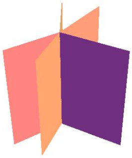

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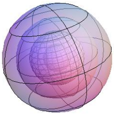

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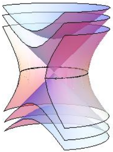

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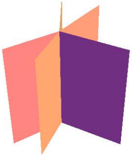

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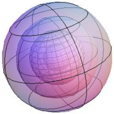

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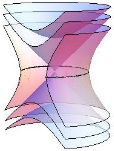

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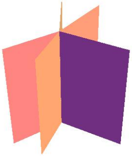
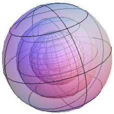


$\mathfrak{b}$

- **Oriented 3-manifolds:** leaves of (M^3, μ, F) are contained in the level sets of $F : M^3 \to \mathbb{R}$.
- **Regular Poisson structures:** Poisson structures whose rank is constant are just **symplectic foliations**:

Examples of symplectic foliations

- **symplectic manifolds:** symplectic leaves of (M, ω) = connected components of M
- **Duals of Lie algebras:** symplectic leaves of $\mathfrak{g}^*$ = coadjoint orbits:



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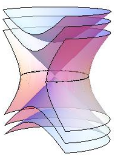

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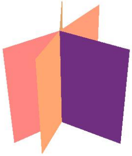

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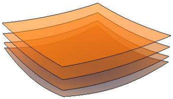

Local Poisson Geometry

A point $x_0 \in M$ where π vanishes is called a **singular point** (so $\{x_0\}$ is a 0-dim symplectic leave).

Definition

The **isotropy Lie algebra** of a singular point x_0 is:

$$\mathfrak{g}_{x_0} := T^*M \quad \text{with } [\mathrm{d}_{x_0}f, \mathrm{d}_{x_0}g] := \mathrm{d}_{x_0}\{f, g\}.$$

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$$\{x^i, x^j\}(x) = \{x^i, x^j\}(x_0) + \sum_k \frac{\partial\{x^i, x^j\}}{\partial x^k}(x_0)\, x^k + o(2).$$
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Linearization Problem: Can one choose coordinates around x_0 where π is linear (no higher order terms)?

Theorem (Conn, 1985)

Let x_0 be a singular point of (M, π). If $\mathfrak{g}_{x_0}$ is a compact semisimple Lie algebra then π can be linearized around x_0: there are local coordinates $(x^1, \ldots, x^m)$ centered at x_0 where the Poisson bracket is linear:

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- For other types of singularities one does not know a complete set of invariants.

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Stability of leaves: In general, one does not expect symplectic leaves to persist under perturbations of π:

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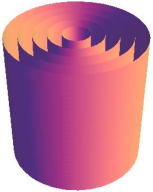

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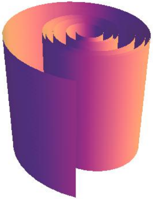

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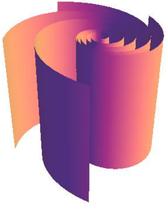

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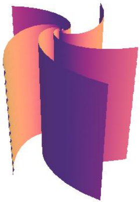

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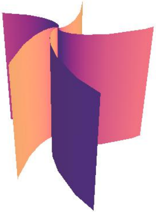

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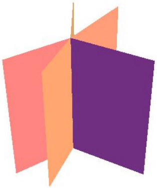

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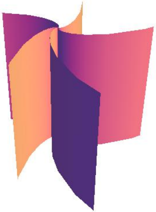

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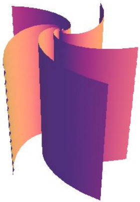

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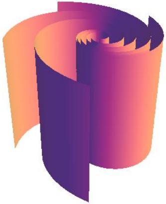

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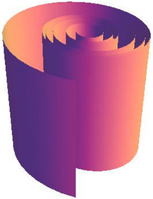

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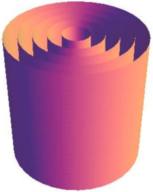

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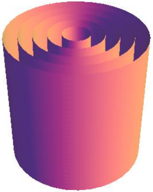

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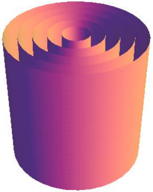

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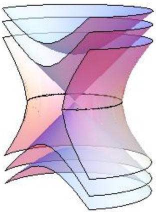

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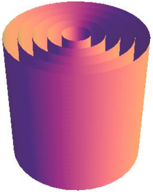

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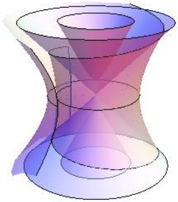

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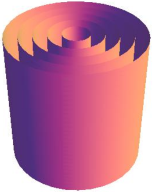

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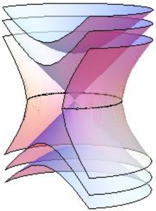

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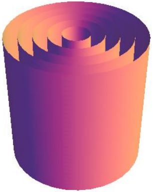

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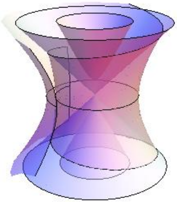

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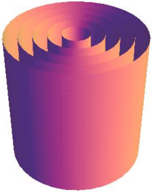

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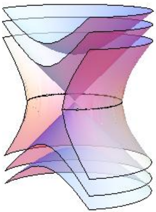

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Let L be a compact symplectic leaf of (M, π) *and assume that*

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- There is also a version for *strong stability* where "diffeomorphic" is replaced by "symplectomorphic".
- The proofs involve some ideas on deforming linear complexes to *non-linear* complexes, that can be traced back to unpublished work of R. Hamilton on deformations of foliations.

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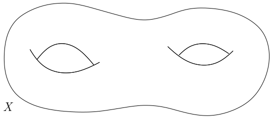
But there are no free meals...

- **Addenda:** (M.Crainic & RLF) This object always exists as a topological groupoid, is finite dimensional, but may fail to be smooth. The precise obstructions to smoothness are known.

Digression into basic topology

X – topological space; look at paths $\gamma : [0, 1] \to X$



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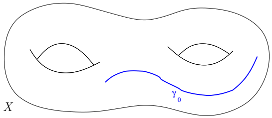

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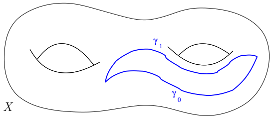

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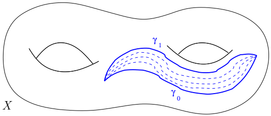

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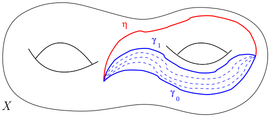

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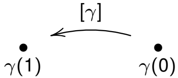

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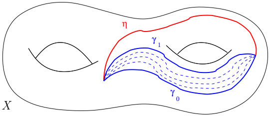

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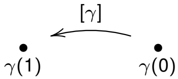
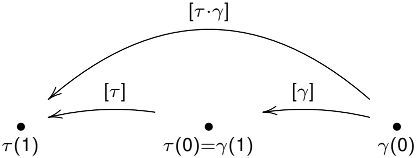


- **product**:



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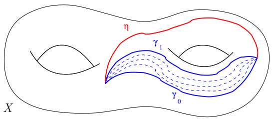

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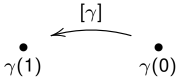

- **identity**:

$$u : X \hookrightarrow \Pi_1(X)$$

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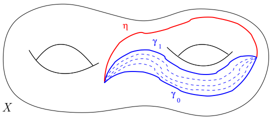

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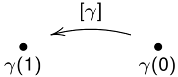
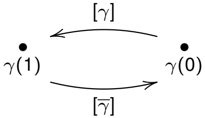


- **inverse**:

$$\iota : G \longrightarrow G$$



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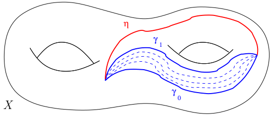

$$\Pi_1(X) = \{[\gamma] \mid \gamma : [0,1] \to X\}$$

$$\mathbf{t} \downarrow\downarrow \mathbf{s}$$

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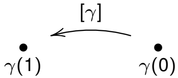

- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.

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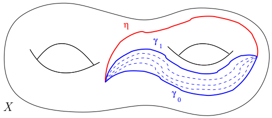

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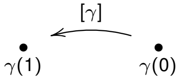

- The space $\Pi_1(X)$ has a natural topology and the source, target, multiplication and inverse are all continous maps: $\Pi_1(X) \rightrightarrows X$ is an example of a topological groupoid.
- If $X = M$ is a manifold, the space $\Pi_1(M)$ is a manifold and the source, target, multiplication and inverse are all smooth maps: then $\Pi_1(M) \rightrightarrows M$ is an example of a Lie groupoid.

(M, π) – Poisson manifold; look at cotangent paths:

$$[0,1] \xrightarrow{a} T^*M$$

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$$\begin{array}{ccc} & & T^*M \\ & \overset{a}{\nearrow} & \downarrow \\ [0,1] & \underset{\gamma}{\dashrightarrow} & M \end{array}$$

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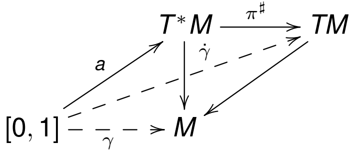

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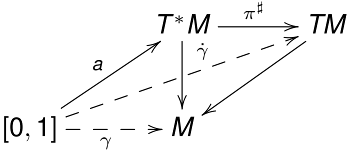

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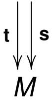

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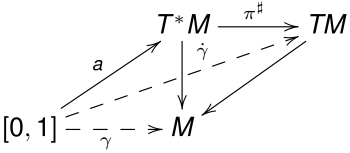

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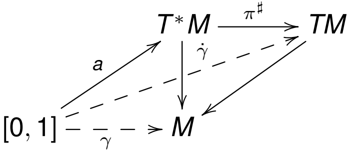

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- For any Poisson manifold (M, π), there is a topological groupoid $\Sigma(M) \rightrightarrows M$ "integrating" it.
- $\Sigma(M) = P(T^*M)//G$ is a symplectic quotient (A. Cattaneo & G. Felder).

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are *Lie groups* (if smooth).

Theorem (Crainic & RLF, 2004)

Let (M, π) be a Poisson manifold and fix a symplectic leaf L. There is a group morphism

$$\partial_x : \pi_2(L, x) \to \nu_x^*(L)$$

controlling integrability: $\Sigma(M)$ is smooth if and only if the groups $Im(\partial_x)$ are uniformly discrete.

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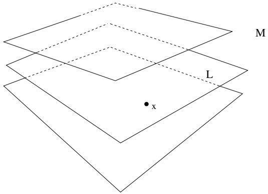

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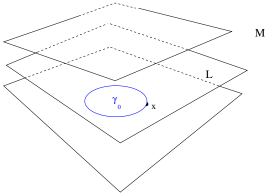

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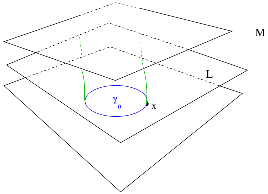

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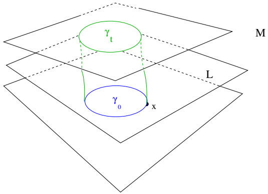

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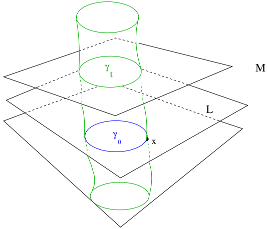

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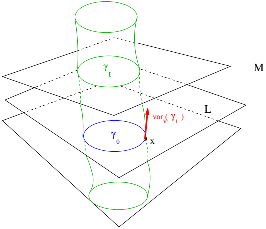

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- There is an explicit local model, which depends on some choices.
- This result can be strengthen by replacing $\Sigma(M)$ by other symplectic groupoids integrating (M, π)
- This result can be generalized by replacing the symplectic leaf L by more general Poisson submanifolds.
- Several proofs are available. The most geometric uses a new notion of *simplicial metric* on the nerve of a groupoid, which has many potential applications (del Hoyo and RLF (2016)).

Deformation quantization

Definition

A **star product** is an associative product $\star_\hbar$ on $C^\infty(M)[[\hbar]]$ deforming the usual product:

$$f \star_\hbar g = \sum_{n=0}^{\infty} B_n(f,g)\hbar^n, \quad \text{where } B_0(f,g) = fg.$$

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- Given $h \in C^\infty(M)$ we have Shrödinger's Equation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{\hbar}[h,f]_{\star_\hbar}$$

Existence of deformation quantizations

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- This theorem is a consequence of a much more general result, Kontsevich's Formality Theorem, which asserts the existence of a certain L_∞-isomorphism between two DGLA.
- Kontsevich gives an explicit formula for $\star_\hbar$.
- Kontsevich's Formality also gives a *classification* of all star products $\star_\hbar$ inducing π.

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> **Conjecture (RLF, 2018)**
> *If there exists a non-formal star product $\star_\hbar$ inducing π, then (M, π) must be integrable by a symplectic groupoid*

Together with Alejandro Cabrera (UFRJ), we have the following strategy to prove this conjecture:

Step 1 From a non-formal star product $\star_\hbar$ construct a *local* symplectic groupoid $G \rightrightarrows M$ integrating (M, π);

Step 2 Associativity of $\star_\hbar$ implies that $G \rightrightarrows M$ satisfies n-associativity for all $n \in \mathbb{N}$, i.e., it is *globally associative*.

Step 3 Use result of RLF & Michiels (2018): if $G \rightrightarrows M$ is globally associative then it extends to a global symplectic groupoid.

Note: This works in the formal case, producing a *formal symplectic groupoid* (Karabegov, Cattaneo & Felder, Contreras)

Many other directions in Poisson geometry

- *b*-symplectic manifolds: Guillemin, Miranda & Pires; Gualtieri, Pelayo & Ratiu; Marcut & Osorno-Torres . . .
- Generalized complex geometry: Hitchin; Bursztyn, Calvacanti & Gualtieri, Baley; . . .
- Poisson-Lie groups and Poisson homogeneous spaces: Drinfeld; Semenov-Tian-Shansky; Lu & Evens; Yakimov; Kosmann-Schwarzbach; Reshetikhin . . .
- Moduli spaces and twisted-Poisson structures: Alekseev & Meinrenken; Boalch; Li-Bland & Severa; . . .
- Cluster algebras: Fomin & Zelevinsky; Gekhtman, Shapiro & Vainshtein; . . .
- Poisson manifolds of compact type: Crainic, RLF, Martinez-Torrres, Zung; . . .
- . . .

. . . there is still a lot of very tasty *Poisson* to be fished!!!

http://poissongeometry.org