Math 595 - Poisson Geometry
Chapter 9 - Poisson Cohomology

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We initiate the study of *global properties* of a Poisson manifold.

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In this chapter we will study

- Lie algebroids
- Poisson cohomology
- Applications of Poisson cohomology

1) Lie algebroids

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Definition

A **Lie algebroid** over a M is a vector bundle $A \to M$, together with a Lie bracket $[\cdot,\cdot]_A$ on the sections $\Gamma(A)$, and a bundle map $\rho_A : A \to TM$ satisfying:

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + \mathscr{L}_{\rho_A(\alpha)}(f)\beta, \quad \forall \alpha, \beta \in \Gamma(A), f \in C^\infty(M).$$

Lie algebroid philosophy

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The **anchor** $\rho_A : A \to TM$, relates this new tangent bundle back to the classical tangent bundle. It satisfies:

$$\rho([\alpha,\beta]_A) = [\rho(\alpha), \rho(\beta)], \quad \forall \alpha, \beta \in \Gamma(A)$$

Examples of Lie algebroids

(i) Tangent bundle of a manifold M:

$$A = TM, \quad [\cdot,\cdot]_A = [\cdot,\cdot], \quad \rho = \mathrm{id};$$

(ii) Cotangent bundle of a Poisson manifold (M,π):

$$A = T^*M, \quad [\cdot,\cdot]_A = [\ ,\ ]_\pi, \quad \rho = \pi^\sharp;$$

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(v) (see Lecture notes for more examples)

2) Poisson cohomology - Poisson differential

Definition

Given Poisson manifold (M,π) the **Poisson differential** is the linear map $\mathrm{d}_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$ given by:

$$\mathrm{d}_\pi \vartheta(\alpha_0, \ldots, \alpha_k) = \sum_{i=0}^{k} (-1)^i \mathscr{L}_{\pi^\sharp(\alpha_i)}(\vartheta(\alpha_0, \ldots, \check{\alpha}_i, \ldots, \alpha_k)) + \\ + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \vartheta([\alpha_i, \alpha_j]_\pi, \alpha_0, \ldots, \check{\alpha}_i, \ldots, \check{\alpha}_j, \ldots, \alpha_k)$$

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It follows that:

$$\mathrm{d}_\pi^2 \vartheta = [\pi,[\pi,\varphi]] = 2[[\pi,\pi],\varphi] = 0.$$

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Definition

The **Poisson cohomology** of (M, π) is the homology of the complex $(\mathfrak{X}^k(M), \mathrm{d}_\pi)$:

$$H^k_\pi(M) := \frac{\ker(\mathrm{d}_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M))}{\mathrm{Im}(\mathrm{d}_\pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M))}.$$

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From:

- $\mathrm{d}_\pi(\vartheta_1 \wedge \vartheta_2) = \mathrm{d}_\pi \vartheta_1 \wedge \vartheta_2 + (-1)^p \vartheta_1 \wedge \mathrm{d}_\pi \vartheta_2,$
- $\mathrm{d}_\pi[\vartheta_1, \vartheta_2] = [\mathrm{d}_\pi \vartheta_1, \vartheta_2] + (-1)^k [\vartheta_1, \mathrm{d}_\pi \vartheta_2],$

It follows that $H_\pi^\bullet(M)$ is:

- a graded commutative algebra;
- a graded Lie algebra (up to a degree shift $p = k+1$);

and the two operations are related by the graded Leibniz identity.

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Most important fact concerning Poisson cohomology:

Almost always impossible to compute!

Poisson cohomology in low degrees - degree 0

$$\mathrm{d}_\pi : \mathfrak{X}^0(M) = C^\infty(M) \to \mathfrak{X}(M), \quad f \mapsto [\pi, f] = X_f.$$

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- $H^0_\pi(M)$ is a ring $\Leftrightarrow$ product of Casimirs is a Casimir.
- It is often infinite dimensional!

Poisson cohomology in low degrees - degree 1

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So degree one Poisson cohomology measures the difference between Poisson vector fields and Hamiltonian ones:

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- $H^1_\pi(M)$ is Lie algebra $\Leftrightarrow$ Poisson vector fields is a Lie algebra, whereHamiltonian vector fields sit as a Lie ideal.

Example - modular class

Question: given (M, π) is there a volume forms $\mu \in \Omega^n(M)$ which invariant under all Hamiltonian diffeomorphisms, i.e., that

$$\mathscr{L}_{X_f}\mu = 0 \quad \forall f \in C^\infty(M)?$$

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A general Poisson manifold (M,π):

(i) need not be orientable;

(ii) if it is orientable, such invariant volume form does not always exist.

The obstruction is given by a degree one Poisson cohomology class.

Example - modular class

Assume (M, π) is orientable and choose volume form μ. If $f \in C^\infty(M)$:

$$\mathscr{L}_{X_f}\mu = X_\mu(f)\,\mu,$$

for some function $X_\mu(f) \in C^\infty(M)$.

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1. *X_μ is a Poisson vector field.*
2. *If $\mu' = \pm e^g \mu$ is some other volume form, the vector fields $X_{\mu'}$ and X_μ differ by a Hamiltonian vector field: $X_{\mu'} = X_\mu - X_g$.*

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Definition

The **modular class** of an orientable Poisson manifold (M, π) is

$$\mathrm{mod}(M, \pi) := [X_\mu] \in H^1_\pi(M).$$

When $\mathrm{mod}(M, \pi) = 0$, one calls (M, π) **unimodular**.

Example - modular class

Corollary

A Poisson manifold (M, π) has an invariant volume form if and only if $\mathrm{mod}(M, \pi) = 0$.

Example

For the linear Poisson structure on $\mathbb{R}^2$ given by:

$$\{x, y\} = x,$$

the modular vector field associated with the standard volume form $\mu = \mathrm{d}x \wedge \mathrm{d}y$ is:

$$X_\mu = -\frac{\partial}{\partial y}.$$

This vector field is not Hamiltonian (it does not vanish along $x = 0$)
Conclusion:

$$\mathrm{mod}(\mathbb{R}^2, \pi) \neq 0,$$

Poisson cohomology in low degrees - degree 2

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Example

The Poisson bivector itself induces a cohomology class called the **fundamental class** of the Poisson manifold:

$$[\pi] \in H^2_\pi(M),$$

One says that (M, π) is an **exact Poisson manifold** when $[\pi] = 0$.
(this rarely happens, but happens, e.g., for linear Poisson structures)

Example - deformation of Poisson structures

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Differentiating $[\pi_t, \pi_t] = 0$ at $t = 0$, one finds that it satisfies:

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [\pi_t, \pi_t] = 2[\pi, \vartheta] = 2\mathrm{d}_\pi \vartheta.$$

Conclusion: elements $\vartheta \in \mathfrak{X}^2(M)$ with $\mathrm{d}_\pi \vartheta = 0$ are “infinitesimal deformations” of π.

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Definition

Two deformations $(\pi_t)_{t\in I}$ and $(\pi_t^{'})_{t\in I}$ of π are **equivalent** if there exists a smooth family $(\phi_t)_{t\in I}$ of diffeomorphisms:

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Differentiating (1), one finds that the variations ϑ' and ϑ satisfy

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Proposition

For any deformation $(\pi_t)_{t\in I}$ of (M,π), its variation at $t=0$ defines a cohomology class

$$[\vartheta] \in H^2_\pi(M)$$

which depends only of the equivalence class of $(\pi_t)_{t\in I}$.

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Example

For any (M,π) Poisson manifold, we have the deformation:

$$\pi_t := e^t\pi.$$

Associated cohomology class is the fundamental class:

$$[\pi] = [\,\frac{\mathrm{d}\pi_t}{\mathrm{d}t}\Big|_{t=0}] \in H^2_\pi(M).$$

This class vanishes if and only if $\mathscr{L}_X\pi = \pi$. If we can choose X to be complete, then:

$$\pi_t = e^t\pi = (\varphi^t_X)_*\pi.$$

Poisson cohomology versus Lie algebroid cohomology

The formula for d_π only depends on $[\cdot,\cdot]_\pi$ and $\pi^\sharp$, so extends to any Lie algebroid $(A,[\cdot,\cdot]_A,\rho)$:

- A-forms: $\Omega^k(A)=\Gamma(\wedge^k A^*)$;
- A-differential: $\mathrm{d}_A:\Omega^k(A)\to\Omega^{k+1}(A)$

$$\mathrm{d}_A\omega(s_0,\ldots,s_k)=\sum_{i=0}^{k}(-1)^i\mathscr{L}_{\rho(s_i)}(\omega(s_0,\ldots,\check{s}_i,\ldots,s_k))+ \\ +\sum_{0\le i<j\le k}(-1)^{i+j}\omega([s_i,s_j]_A,s_0,\ldots,\check{s}_i,\ldots,\check{s}_j,\ldots,s_k)$$

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Examples and properties of Lie algebroid cohomology

- $A = (TM, [\cdot,\cdot], \rho)$: de Rham differential and the de Rham cohomology;
- $A = (T^*M, , [\cdot,\cdot]_\pi, \pi^\sharp)$: the Poisson differential and Poisson cohomology;
- $A = \mathfrak{g}$: Chevalley-Eilenberg differential and Lie algebra cohomology.

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Functoriality: A Lie algebroid morphism $\Phi : A \to B$ induces a pull-back map in cohomology:

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$\implies$ can relate Poisson cohomology with known cohomologies

More applications - see Lectuire Notes

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But remember: Poisson cohomology rarely can be computed!