Math 595 - Poisson Geometry
Chapter 11 - Contravariant Connections

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We look at the notion of *contravariant connection* in Poisson Geometry.

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- Parallel transport along cotangent paths
- Flat contravariant connections
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We will apply these to prove:

- *every Poisson manifold* (M, π) *admits a symplectic realization* $\mu : (S, \omega) \to (M, \pi)$

1) Contravariant connections on vector bundles

Notation:

- (M,π): Poisson manifold (M,π)
- $E \to M$: vector bundle

Definition

A **contravariant connection** on E is a $\mathbb{R}$-bilinear operation:

$$\Omega^1(M) \times \Gamma(E) \to \Gamma(E), \quad (\alpha, s) \mapsto \nabla_\alpha s,$$

satisfying:

$$\nabla_{f\alpha} s = f\nabla_\alpha s, \quad \nabla_\alpha(fs) = f\nabla_\alpha s + \mathscr{L}_{\pi^\sharp\alpha}(f)s.$$

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Curvature of a contravariant connection $R_\nabla \in \mathfrak{X}^2(M;\mathrm{End}(E))$ is:

$$R_\nabla(\alpha,\beta)s := \nabla_\alpha(\nabla_\beta s) - \nabla_\beta(\nabla_\alpha s) - \nabla_{[\alpha,\beta]_\pi} s.$$

Examples

1) $E \to M$ vector bundle with an ordinary (covariant) connection $\bar{\nabla}$.
   Then:
   $$\nabla_\alpha s := \bar{\nabla}_{\pi^\sharp(\alpha)} s.$$
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2) Assume (M, π) is a regular so we have the bundle $E = \nu^*(\mathscr{F}_\pi)$. Note that $\nu^*(\mathscr{F}_\pi) = \ker \pi^\sharp$, so the **contravariant Bott connection**:

$$\nabla_\alpha \beta := [\alpha, \beta]_\pi, \quad \alpha \in \Omega^1(M), \beta \in \Gamma(\nu^*(\mathscr{F}_\pi)),$$

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3) For any (M, π) the line bundle $L = \wedge^{\mathrm{top}} T^* M$ has a contravariant connection ∇:

$$\nabla_{\mathrm{d}f} \mu := \mathscr{L}_{X_f} \mu,$$

and then extends to any 1-form by requiring $C^\infty(M)$-linearity.
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Aplication: characteristic class of flat line bundles

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For a nowhere vanishing section μ:

$$\nabla_\alpha \mu = c_\mu(\alpha)\mu,$$

for some C^∞-linear map $c_\mu : T^*M \to \mathbb{R}$, i.e., a vector field $c_\mu \in \mathfrak{X}(M)$.

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- $L \to (M,\pi)$: any line bundle with flat contravariant connection ∇

(a) $L^2 = L \otimes L$ is trivial;

(b) L^2 has the flat connection:

$$\tilde{\nabla}_\alpha(\xi \otimes \xi') := \nabla_\alpha \xi \otimes \xi' + \xi \otimes \nabla_\alpha \xi'.$$

Conclusion: $[c(L^2,\tilde{\nabla})] \in H^1_\pi(M)$.

Aplication: characteristic class of flat line bundles

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For a flat line bundle (L, ∇) its **characteristic class** is:

$$c(L, \nabla) = \frac{1}{2} c(L^2, \nabla) \in H^1_\pi(M).$$

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Example (Modular class)

Recall that for any (M, π), the line bundle $\wedge^{\mathrm{top}} T^*M$ is canonically flat.
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Even for non-orientable (M,π), we call $c(\wedge^{\mathrm{top}} T^*M,\nabla)$ the **modular class** of (M,π) and denote it by $\mathrm{mod}(M,\pi)$.

2) Parallel transport along cotangent paths

- $p : E \to (M, \pi)$ vector bundle with contravariant connection ∇,
- $a : I \to T^*M$ cotangent path with base path $\gamma_a : I \to M$.
- $c : I \to E$ path above a: $p(c(t)) = \gamma_a(t)$.

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If $s_t \in \Gamma(E)$ is any time-dependent section with $s_t(\gamma_a(t)) = c(t)$, set:

$$(D_a c)(t) := \nabla_{a(t)} s_t + \frac{\mathrm{d}}{\mathrm{d}t} s_t \Big|_{\gamma_a(t)}.$$

This is independent of choice of extension s_t.

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This is independent of choice of extension s_t.

Definition

$D_a c$ is called the **contravariant derivative** of c along the cotangent path a.

Properties of contravariant derivative D:

(i) Linearity: if $c_1, c_2 : I \to E$ are any two paths above a and $\lambda_1, \lambda_2 \in \mathbb{R}$:

$$D_a(\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 D_a c_1 + \lambda_2 D_a c_2;$$

(ii) Leibniz: if $c : I \to E$ is a path above a and $f \in C^\infty(M)$:

$$D_a(fc) = (f \circ \gamma_a(t))\, D_a c + \pi^\sharp(a)(f)\, c.$$

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Proposition

If $\Phi : T([0,1] \times [0,1]) \to T^*M$ *is a cotangent surface*

$$\Phi(t, \varepsilon) = \Phi_t(t, \varepsilon)\, \mathrm{d}t + \Phi_\varepsilon(t, \varepsilon)\, \mathrm{d}\varepsilon.$$

and $c : I \times I \to E$ *is a map above it, one has:*

$$R_\nabla(\Phi_t, \Phi_\varepsilon)c = D_{\Phi_t} D_{\Phi_\varepsilon} c - D_{\Phi_\varepsilon} D_{\Phi_t} c.$$

Parallel transport

Definition

Given (M, π) be a Poisson manifold and $E \to M$ a vector bundle with a contravariant connection. We say that $c : I \to E$ is a **parallel curve** along a cotangent path $a : I \to T^*M$ if c lies above a and:

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Proposition

Given (M,π), vector bundle (E,∇), a cotangent path $a : [0,1] \to T^*M$ and a point $u_0 \in E_{\gamma_a(0)}$ there is a unique parallel curve $c_{u_0} : I \to E$ along a starting at u_0. The end point of this curve $c_{u_0}(1)$ depends linearly on u_0.

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$\Rightarrow$ **parallel transport along the cotangent path a for (E,∇):**

$$\tau_a : E_{\gamma_a(0)} \to E_{\gamma_a(1)}, \quad u_0 \mapsto c_{u_0}(1).$$

Parallel transport - properties

(i) If $\bar{a}$ is the reverse cotangent path: $\tau_{\bar{a}} \circ \tau_a =$id;

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Example (Linear Poisson structure $M = \mathfrak{g}^*$)

Contravariant connection on $E = T^*\mathfrak{g}^*$:

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Since π^{lin} vanishes at the origin, any $v \in T_0^*\mathfrak{g}^* \simeq \mathfrak{g}$ defines the constant cotangent path $a_v(t) = v$:

$$\tau_{a_v} = \mathrm{Ad}_{\exp(v)} : \mathfrak{g} \to \mathfrak{g}.$$

3) Flat contravariant connections

When ∇ is a flat contravariant connection on $E \to (M, \pi)$:

- cotangent homotopic paths $a_0, a_1 : I \to T^*M$ induce the same parallel transport: $\tau_{a_0} = \tau_{a_1}$;

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For flat line bundles:

Proposition

If (L, ∇) is a flat line bundle and $a : [0,1] \to T^*M$ is a cotangent path, for any section μ of $L \to M$ which does not vanish along γ_a:

$$\tau_a(\mu_{\gamma_a(0)}) = \exp\left(-\int_a c(L, \nabla)\right)\ \mu_{\gamma_a(1)}.$$

Linear Poisson holomomy

Definition

Given a cotangent path $a : [0,1] \to T^*M$ on (M, π) lying in a symplectic leaf S, the parallel transport map for the contravariant Bott connection

$$\mathrm{Hol}_a := \tau_a : \nu^*_{\gamma_a(0)}(S) \to \nu^*_{\gamma_a(1)}(S),$$

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One can relate linear Poisson holonomy to the modular class:

Theorem

Let (M,π) be a Poisson manifold. For a cotangent path $a:[0,1]\to T^*M$ whose base path is a loop:

$$\det(\mathrm{Hol}_a) = \exp\left(-\int_a \mathrm{mod}(M,\pi)\right).$$

4) Geodesics for contravariant connections

Contravariant connections on $E = T^*M$ play a special role.

Definition

Let (M, π) be a Poisson manifold. A **contravariant connection on** (M, π) is a contravariant connection ∇ on the bundle T^*M. Its **torsion** is the T^*M-valued bivector field:

$$T_\nabla(\alpha, \beta) := \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta]_\pi.$$

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- On constant 1-forms: $\nabla_v w := \frac{1}{2}[v, w]_\mathfrak{g}$,
- extend to any forms, by imposing the properties of a connection.

This connection is torsionless and, in general, non-flat:

$$T = 0, \quad R(v, w)z = \frac{1}{4}[v, [w, z]].$$

Torsion and connections

Geometric interpretation of torsion:

Proposition

Given a contravariant connection ∇ and a cotangent surface $\Phi : T([0,1] \times [0,1]) \to T^*M$,

$$\Phi(t,\varepsilon) = \Phi_t(t,\varepsilon)\,\mathrm{d}t + \Phi_\varepsilon(t,\varepsilon)\,\mathrm{d}\varepsilon.$$

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$$T(\Phi_t, \Phi_\varepsilon) = D_{\Phi_t}\Phi_\varepsilon - D_{\Phi_\varepsilon}\Phi_t.$$

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Local coordinate expressions: In a local chart (U, x^i) for (M, π):

$$\nabla_{\mathrm{d}x^i}\mathrm{d}x^j = \sum_k \Gamma^{ij}_k \mathrm{d}x^k.$$

The $\Gamma^{ij}_k \in C^\infty(U)$ are called the **Christoffel symbols**.

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Geodesics

Definition

Let ∇ is a contravariant connection on (M, π). A cotangent path $a : I \to T^*M$ is called a **geodesic** if it is parallel along itself:

$$D_a a = 0.$$

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In local coordinates: $a(t) = \sum_i a_i(t) \mathrm{d}x^i$ with base path $\gamma_a(t) = (\gamma_a^i(t))$ is a geodesic iff:

$$\begin{cases} \dot{a}_k(t) = - \sum_{1 \leq i,j \leq n} \Gamma_k^{ij}(\gamma_a(t))\, a_i(t)\, a_j(t), \\ \dot{\gamma_a}^k(t) = \sum_{1 \leq i \leq n} \pi^{ik}(\gamma_a(t))\, a_i(t). \end{cases} \qquad (k = 1, \ldots n)$$

Geodesic spray and Geodesic flow

Geodesics are the integral curves of $X \in \mathfrak{X}(T^*M)$, given in local coordinates (x^i, p_i) by:

$$X = \sum_{1 \leq i,k \leq n} \pi^{ik}(x) p_i \frac{\partial}{\partial x^k} - \sum_{1 \leq i,j,k \leq n} \Gamma_k^{ij}(x) p_i p_j \frac{\partial}{\partial p_k}.$$

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Proposition

Given a contravariant connection ∇ on (M, π), there is a unique torsion free contravariant connection $\tilde{\nabla}$ with the same geodesics as ∇.

5) Existence of symplectic realizations

Theorem

Let X be the geodesic spray of a contravariant connection on (M,π). There is an open neighborhood $U \subset T^*M$ of the zero-section on which the 2-form

$$\omega := -\int_0^1 (\phi_X^{-t})^* \omega_{\mathrm{can}}\, \mathrm{d}t$$

is symplectic and $\mathrm{pr}|_U : (U, \omega^{-1}) \to (M,\pi)$ is a symplectic realization.

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$$\omega := -\int_0^1 (\phi_X^{-t})^* \omega_{\mathrm{can}} \, \mathrm{d}t$$

is symplectic and $\mathrm{pr}|_U : (U, \omega^{-1}) \to (M,\pi)$ is a symplectic realization.

When $(M,\pi) = (\mathbb{R}^n, \pi)$ and we let X be the geodesic spray of the contravariant connection ∇ defined by:

$$\nabla_{\mathrm{d}x^i} \mathrm{d}x^j = 0,$$

we recover the result we saw before.