$$\boxed{\text{MATH 595 - LECTURE 9}}$$

Actions and Representations of Lie Algebroids

Def. An *action of a Lie algebroid* $A \to M$ along a map $\mu : S \to M$ is given by a linear map $\sigma : \Gamma(A) \to \mathfrak{X}(S)$ such that:

(i) $\sigma(f\alpha) = (f\circ\mu)\,\sigma(\alpha)$ $\qquad \forall\, \alpha,\beta \in \Gamma(A),\ f \in C^\infty(M)$

(ii) $\sigma([\alpha,\beta]) = [\sigma(\alpha),\sigma(\beta)]$

Notice that (i) is equivalent to say an action is given by a v.b. map

$$\begin{tikzcd} \mu^* A \arrow[r, "\tilde\sigma"] \arrow[d] & TS \arrow[d] \\ S \arrow[r, equal] & S \end{tikzcd} \qquad , \qquad \tilde\sigma(\mu^*\alpha) = \sigma(\alpha)$$

By the next result, this map satisfies:

$$\begin{tikzcd} \mu^* A \arrow[r, "\tilde\sigma"] \arrow[d] & TS \arrow[d, "d\mu"] \\ A \arrow[r, "\rho"'] & TM \end{tikzcd} \qquad d\mu \circ \tilde\sigma(p,a) = \rho_{\mu(p)}(a)$$

Lemma For an action $\sigma : \Gamma(A) \to \mathfrak{X}(S)$ the vector fields $\sigma(\alpha)$ & $\rho(\alpha)$ are μ-related:

$$\mu_*\sigma(\alpha) = \rho(\alpha)$$

Proof. For any $\alpha,\beta \in \Gamma(A)$, $f \in C^\infty(M)$

$$\sigma([\alpha, f\beta]) = \sigma(f[\alpha,\beta] + \rho(\alpha)(f)\beta)$$

$$\| \qquad\qquad = (f\circ\mu)\sigma([\alpha,\beta]) + \rho(\alpha)(f)\circ\mu\, \sigma(\beta)$$

$$[\sigma(\alpha), \sigma(f\beta)] = [\sigma(\alpha), f\circ\mu\, \sigma(\beta)]$$

$$= (f\circ\mu)\sigma([\alpha,\beta]) + \sigma(\alpha)(f\circ\mu)\,\sigma(\beta)$$

$$\Rightarrow \quad \sigma(\alpha)(f\circ\mu) = \rho(\alpha)(f)\circ\mu,\ \forall f \iff \mu_*\sigma(\alpha) = \rho(\alpha) \qquad \boxtimes$$

For an action $\sigma : \Gamma(A) \to \mathfrak{X}(S)$ on a map $\mu : S \to M$ one has an action Lie algebroid

$$\mu^* A \to S$$

where:

- Anchor: $\rho_{\mu^* A} := \sigma : \mu^* A \to TS$
- Lie bracket: $[\mu^*\alpha, \mu^*\beta]_{\mu^* A} := \mu^*[\alpha, \beta]_A$ and extend to any section by Leibniz.

It follows that for a Lie algebroid action:

- Orbits := orbits of $\mu^* A \to S$ = Regularly immersed submanifolds of S
- Isotropy Lie algebras := $\operatorname{Ker} \sigma_p \simeq$ Lie subalgebras of $\operatorname{Ker} \rho_{\mu(p)}$

(left) Groupoid action $\mathcal{G} \rightrightarrows M$ on $\mu : S \to M$ $\Rightarrow$ Algebroid action of $\sigma : \mu^* A(\mathcal{G}) \to TS$

$$\sigma_p : A_{\mu(p)} \to T_p S, \quad \begin{cases} R_p : t^{-1}(\mu(p)) \to S, \; g \mapsto g^{-1} \cdot p \\ \sigma_p := d_{\mu(p)} R_p \end{cases}$$

This can also be expressed by:

$$\sigma(\alpha)_p := \frac{d}{dt} \exp(-t\alpha) \cdot p \Big|_{t=0}$$

where a bisection $b \in \mathcal{B}(\mathcal{G})$ acts by:

$$p \mapsto b\left((s \circ b)^{-1}(\underline{\mu(p)})\right) \cdot p$$

If instead right groupoid action, then

$$\sigma_p : A_{\mu(p)} \to T_p S, \quad \begin{cases} \mathcal{L}_p : t^{-1}(\mu(p)) \to S, \; g \mapsto p \cdot g \\ \sigma_p := d_{\mu(p)} \mathcal{L}_p \end{cases}$$

Now one has: $\sigma(\alpha)_p := \frac{d}{dt} p \cdot \exp(t\alpha) \Big|_{t=0}$

where a bisection acts by:

$$p \mapsto p \cdot b(\underline{\mu(p)})$$

Examples

1) For a Lie algebra $\mathfrak{g} \to d \times \mathfrak{g}$ this notion reduces to the usual notion of infinitesimal $\mathfrak{g}$-action on S

2) Any Lie algebroid $A \to M$ acts on its base $\mu = id : M \to M$

If $A = A(\mathcal{G}) \longrightarrow M$ then

- A acts on $t : \mathcal{G} \to M$: $\quad \sigma(\alpha) = -\overrightarrow{\alpha}$
- A acts on $s : \mathcal{G} \to M$: $\quad \sigma(\alpha) = \overleftarrow{\alpha}$

These are of course the differentials of the actions of $\mathcal{G}$ on M and on itself by left/right translations.

3) Given a principal G-bundle $\pi : P \to M$ the Atiyah algebroid $A = TP/_G \to M$ acts on $\pi : P \to M$:

$$\sigma : A_{\pi(p)} \to T_p P \; : \; \sigma_p([v]) := v$$

Again this is the differential of the action $(P \times P)/_G \rightrightarrows M$ on $\pi : P \to M$.

4) Given any v.b. $\mu : E \to M$ we have an action of $\mathfrak{gl}(E) \to M$ on $\mu : E \to M$ by setting:

$$\sigma : \mathrm{Der}(E) \longrightarrow \mathfrak{X}(E) \, , \quad \sigma(D) = \tilde{X}^*_D$$

where $\tilde{X}^*_D$ is the vector field whose flow is φ^t_D.

——— / ———

In last example, $\tilde{X}^*_D$ is a fiberwise linear vector field on $E \to M$:

- $f : E \to M$ is fiberwise linear if:

$$f(\lambda v) = \lambda f(v) \, , \quad \forall \lambda \in \mathbb{R}$$

$$\Leftrightarrow \quad m_\lambda^* f = \lambda f \qquad (m_\lambda(v) = \lambda v)$$

- $X \in \mathfrak{X}(E)$ is <u>fiberwise linear</u> if

$$f \in C^\infty_{lin}(E) \Rightarrow X(f) \in C^\infty_{lin}(E)$$

$$\Leftrightarrow \quad m_\lambda^* X = X$$

Fix contractible open chart (U, x^i) for M so $E|_U \to U$ has local basis of sections $\{e_a\} \Rightarrow (x^i, \xi^a)$ coordinates on E

$$X \in \mathfrak{X}^{lin}(E|_U) \Leftrightarrow X = X^i(x) \frac{\partial}{\partial x^i} + X^a_b(x) \xi^b \frac{\partial}{\partial \xi^a}$$

<u>Lemma 1.</u> There is a Lie algebra isomorphism:

$$\mathrm{Der}(E) \simeq \mathfrak{X}^{lin}(E)$$

$$D \longmapsto \widetilde{X}^*_D \quad \left(= \frac{d}{dt} \varphi^t_D \Big|_{t=0}\right)$$

In local coordinates:

$$\begin{cases} D(e_a) = D^b_a(x)\, e_b \\ X_D = X^i_D(x) \frac{\partial}{\partial x^i} \end{cases} \longleftrightarrow \quad \widetilde{X}^*_D = X^i_D(x) \frac{\partial}{\partial x^i} - D^b_a(x) \xi^a \frac{\partial}{\partial \xi^b}$$

Check the minus sign! (e.g., $[D_1, D_2] \to [\widetilde{X}^*_{D_1}, \widetilde{X}^*_{D_2}]$ only holds because of the minus sign!)

<u>Lemma 2.</u> The following are equivalent:

(i) An action $\sigma: \Gamma(A) \to \mathfrak{X}^{lin}(E)$

(ii) A map $\nabla: \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, s) \mapsto \nabla_\alpha s$, satisfying

(a) $s \mapsto \nabla_\alpha s$ is $\mathbb{R}$-linear

(b) $\alpha \mapsto \nabla_\alpha s$ is C^∞-linear

(c) $\nabla_\alpha(fs) = f \nabla_\alpha s + \rho(\alpha)(f)\, \alpha$

(d) $\nabla_{[\alpha,\beta]} s = \nabla_\alpha \nabla_\beta s - \nabla_\beta \nabla_\alpha s$

<u>Proof:</u>

- $\mathfrak{X}^{lin}(E) \ni \sigma(\alpha) \longleftrightarrow \nabla_\alpha \in \mathrm{Der}(E)$ (a), (b) hold ✓

- (c) $\Leftrightarrow$ $\sigma(\alpha)((f\circ\pi)s) = \sigma(\alpha)(f\circ\pi)s + (f\circ\pi)\sigma(s) = \rho(\alpha)(f)\circ\pi + (f\circ\pi)\sigma(\alpha)s$
- (d) $\Leftrightarrow$ $\sigma([\alpha,\beta]) = [\sigma(\alpha),\sigma(\beta)]$

$\blacksquare$

Def. A map $\nabla : \Gamma(A)\times\Gamma(E) \to \Gamma(E)$ satisfying (a), (b), (c) is called an <u>A-connection</u> on the vector bundle $E\to M$. If additionally, it satisfies (d) it is called a flat <u>A-connection</u>.

Note that if ∇ is a flat A-connection we have a map

$$\Gamma(A) \longrightarrow \mathrm{Der}(E) = \Gamma(\mathfrak{gl}(E)), \quad \alpha \longmapsto \nabla_\alpha$$

which is $C^\infty(M)$-linear (by (b)). Hence, it is induced by a v.b. map

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathfrak{gl}(E) \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd}$$

This map satisfies:

- $\rho_{\mathfrak{gl}(E)} \circ \nabla_\alpha = X_{\nabla_\alpha} = \rho_A(\alpha)$ (by (c))
- preserves Lie brackets (by (d))

Hence, it is a Lie algebroid morphism $A \to \mathfrak{gl}(E)$ covering id.
Conversely, every such morphism determines a flat connection.

Def. A <u>representation of a Lie algebroid</u> $A\to M$ on a v.b. $E\to M$ is given by any of the following equivalent data:

(a) A linear algebroid action $\sigma : \Gamma(A) \to \mathfrak{X}^{lin}(E)$;

(b) A flat A-connection ∇;

(c) A Lie algebroid morphism $\phi : A \to \mathfrak{gl}(E)$ covering id_M.

We say that a rep is **faithful** if ϕ is injective.

Given two reps (E_1,∇^1), (E_2,∇^2) of A:

- Direct sum of Reps: $E_1 \oplus E_2$ w/ $\nabla_\alpha(s_1 \oplus s_2) = \nabla^1_\alpha s_1 \oplus \nabla^1_\alpha s_2$
- Tensor product of Reps: $E_1 \otimes E_2$ w/ $\nabla_\alpha(s_1 \otimes s_2) = \nabla^1_\alpha s_1 \otimes s_2 + s_1 \otimes \nabla_\alpha s_2$

Trivial Rep: $\mathbb{R}_M \to M$: $\nabla_\alpha(f) = \rho(\alpha)(f)$

$\Rightarrow$ $\mathrm{Rep}(A)$ is a semi-ring.

Examples.

0. Any Rep of $\mathcal{G} \rightrightarrows M$ on v.b. $E \to M$ induces a Rep of $A = A(\mathcal{G})$ on E.

1. $A = TM$: $\mathrm{Rep}(TM) = \{$flat vector bundles$\}$

2. $A = \mathfrak{g} \ltimes M \to M$ action algebroid
$\mathrm{Rep}(A) = \{\mathfrak{g}$-equivariant vector bundles$\}$

3. $(M,\mathcal{F})$ foliated manifold $A = T\mathcal{F}$ has a natural Rep on $\nu(\mathcal{F}) \to M$:

$X \in \Gamma(A) = \mathfrak{X}(\mathcal{F})$, $\overline{Y} \in \Gamma(\nu(\mathcal{F})) \Leftrightarrow Y \in \mathfrak{X}(M)$, $\overline{Y} = Y \bmod \mathfrak{X}(\mathcal{F})$

$$\nabla_X \overline{Y} := \overline{[X,Y]}$$

This is known as the Bott connection. In foliation theory is called a "partial connection" $\equiv$ $T\mathcal{F}$-connection

4. Every Lie algebroid $A \to M$ has a canonical Rep on the line bundle $\Lambda^{top} A \otimes \Lambda^{top} T^*M \to M$. It is defined by:

$$\nabla_\alpha(\alpha_1 \wedge \dots \wedge \alpha_r \otimes \mu) = \sum_{j=1}^{r} \alpha_1 \wedge \dots \wedge [\alpha,\alpha_j] \wedge \dots \wedge \alpha^r \otimes \mu + \alpha_1 \wedge \dots \wedge \alpha_r \otimes \mathcal{L}_{\rho(\alpha)} \mu$$

(Check this!)

5. Every <u>regular</u> Lie algebroid $A \to M$ has canonical reps on the isotropy bundle $\mathfrak{g}(A) \to M$ and on the normal bundle to the orbit foliation:

$$\nabla_\alpha \beta := [\alpha, \beta] \qquad \beta \in \Gamma(\mathfrak{g}(A)) \qquad (\mathfrak{g}(A) = \ker\rho)$$

$$\nabla_\alpha \bar{X} := \overline{[\rho(\alpha), X]} \qquad \bar{X} \in \nu(\mathcal{F}_A) \qquad (\mathcal{F}_A = \operatorname{Im}\rho)$$

One can put them together into a single Rep: $\ker\rho \oplus \operatorname{Im}\rho$ which can be thought of as the Adjoint Rep of A.
For non-regular algebroids this only exists as a "Representation up to homotopy"

6. Let $A \to M$ be <u>any</u> Lie algebroid. Fix orbit $\mathcal{O} \subset M$
Then we have restriction $A_{\mathcal{O}} \to \mathcal{O}$. This as a canonical representations:

i) $\mathfrak{g}_{\mathcal{O}} = (\ker\rho)|_{\mathcal{O}} = \ker\rho_{\mathcal{O}}$ (special case of 5)!)

ii) $\nu(\mathcal{O}) \to \mathcal{O}$ (not special case of 5!)

$$\nabla_\alpha \bar{X} = \overline{[\rho(\tilde{\alpha}), \tilde{X}]}|_{\mathcal{O}}, \qquad \tilde{\alpha} \in \Gamma(A),\ \tilde{X} \in \mathfrak{X}(M)$$

local extensions of α & X

The later can be thought as the Bott Connection of orbit $\mathcal{O}$