MATH 595 - Lecture 8

Flow of a section $\alpha \in \Gamma(A)$

Proposition

Let $\alpha \in \Gamma_{cpl}(A)$. There exists a unique 1-parameter group of automorphisms $\phi_\alpha^t \in \mathrm{Aut}(A)$ covering the flow of $\rho(\alpha)$:

$$\begin{tikzcd} A \arrow[r, "\phi_\alpha^t"] \arrow[d] & A \arrow[d] \\ M \arrow[r, "\phi_{\rho(\alpha)}^t"'] & M \end{tikzcd}$$

which are uniquely characterized by:

- $\phi_\alpha^0 = id \,, \quad \frac{d}{dt}(\phi_\alpha^t)^*(\beta)\Big|_{t=0} = [\alpha, \beta] \,, \quad \forall \beta \in \Gamma(A)$

Defn One calls ϕ_α^t the (infinitesimal) flow of a section $\alpha \in \Gamma(A)$

(It can also be defined for non-complete sections in some domain $D = D(\alpha)$)

Rmk. When $A = TM$, this diagram becomes:

$$\begin{tikzcd} TM \arrow[r, "d\phi_X^t"] \arrow[d] & TM \arrow[d] \\ M \arrow[r, "\phi_X^t"'] & M \end{tikzcd}$$

and the equations define the flow the vector field $\alpha = X \in \Gamma(A) = \mathfrak{X}(M)$ by defining its differential

Proof.

Let $D_\alpha : \Gamma(A) \to \Gamma(A)$ be the map

$$D_\alpha(\beta) := [\alpha, \beta]$$

By Leibniz, This is a derivation of the v.b. $A \to M$ with symbol $X_D = \rho(\alpha)$:

$$D_\alpha(f\beta) = [\alpha, f\beta] = f[\alpha,\beta] + \rho(\alpha)(f)\beta = fD(\alpha) + \rho(\alpha)(f)\beta$$

The result follows from a general result about derivations of a vector bundle $E \to M$: For any $D \in \mathrm{Der}(E)$ whose symbol X_D is complete there is a 1-parameter group ϕ_D^t of vector bundle automorphisms:

$$\begin{tikzcd} E \arrow[r, "\phi_D^t"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "\phi_{X_D}^t"'] & M \end{tikzcd}$$

characterized by:

$$\phi_D^0 = \mathrm{id}, \quad \frac{d}{dt}(\phi_D^t)^*(\beta)\Big|_{t=0} = D(\beta), \ \forall \beta \in \Gamma(E)$$

To construct ϕ_D^t we define a vector field $\tilde{X}_D \in \mathfrak{X}(E^*)$. Note that $C^\infty(E^*)$ is generated by these two types of functions:

- evaluation on a section $\beta \in \Gamma(E)$:

$$ev_\beta : E^* \to \mathbb{R}, \quad \xi_x \mapsto \langle \beta(x), \xi_x \rangle$$

- pullbacks of $f \in C^\infty(M)$:

$$\pi^* f : E^* \to \mathbb{R}, \quad \text{where } \pi : E^* \to M$$

So we can define $\tilde{X}_D \in \mathfrak{X}(E^*)$ by setting:

$$\tilde{X}_D(ev_\beta) := ev_{D(\beta)}, \quad \tilde{X}_D(f \circ \pi) := X_D(f) \circ \pi$$

Then $\pi_*(\tilde{X}_D) = X_D$ and the flow of $\tilde{X}_D$ gives 1-parameter group:

$$\begin{tikzcd} E^* \arrow[r, "\phi_{\tilde{X}_D}^t"] \arrow[d] & E^* \arrow[d] \\ M \arrow[r, "\phi_{X_D}^t"'] & M \end{tikzcd}$$

w/ $\phi_{\tilde{X}_D}^0 = \mathrm{id}$

$$\frac{d}{dt}(\phi_{\tilde{X}_D}^t)^* F\Big|_{t=0} = \tilde{X}_D(F), \ \forall F \in C^\infty(E^*)$$

Transposing:
$$\begin{tikzcd} E \arrow[r, "\phi^t_D = (\phi^{-t}_{X_D})^*"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "\phi^t_{X_D}"'] & M \end{tikzcd} \qquad \text{w/} \quad \phi^0_D = id$$
$$\frac{d}{dt}(\phi^t_D)^*\beta\Big|_{t=0} = D(\beta), \ \forall \beta \in \Gamma(E)$$
$\blacksquare$

Exercise.

Assume $A = A(\mathcal{G})$ for some $\mathcal{G} \rightrightarrows M$. Show that:
$$\phi^t_\alpha = (\Phi_{\exp(t\alpha)})_*$$
where $\Phi_b : \mathcal{G} \to \mathcal{G}$ is conjugation by $b \in B(\mathcal{G})$:
$$\Phi_b(g) := b(t(g))^{-1} \cdot g \cdot b(s(g))$$

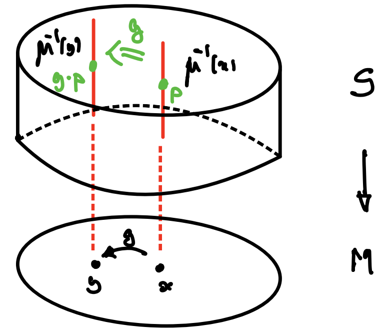
Actions and Representations of Lie Groupoids

Groupoids act along maps:

<u>Def</u> A <u>(left) action</u> of $\mathcal{G} \rightrightarrows M$ on a map $\mu : S \to M$ is a map $A : \mathcal{G}\,{}_s\!\times_\mu S \to S$, $(g, p) \mapsto g \cdot p$, satisfying:

- $\mu(g \cdot p) = t(g)$
- $1_{\mu(p)} \cdot p = p$
- $g(h \cdot p) = (gh) \cdot p$

We call μ the <u>moment map</u>



Notice that:

- When $G \rightrightarrows \{*\}$ is a group, this recovers the usual notion of a G-action
- Each isotropy group $\mathcal{G}_x$ acts on fiber $\mu^{-1}(x)$. But note that $\mu^{-1}(x)$ may not be submanifolds (often they are).

- For an action of $\mathcal{G} \rightrightarrows M$ on $\mu : S \to M$ we have **action groupoid**:

$$\mathcal{G} \ltimes S \rightrightarrows S \quad \text{where} \quad \mathcal{G} \ltimes S := \mathcal{G} \,{}_s\!\times_\mu S \qquad g\cdot p \xleftarrow{(g,p)} p$$

$$\begin{cases} s(g,p) = p \\ t(g,p) = g\cdot p \end{cases} \qquad (g,p)(h,q) = (gh, q) \ \text{ if } \ p = g\cdot q$$

- **Orbits** := orbits of $\mathcal{G} \ltimes S \rightrightarrows S$ = Regularly immersed submanifolds of S

- **Isotropy groups** := $(\mathcal{G} \ltimes S)_p \simeq$ closed subgroups of $\mathcal{G}_{\mu(p)}$

- **space of orbits** := $\mathcal{G} \backslash S$ (not smooth manifold, in general)

Def. An right action of $\mathcal{G} \rightrightarrows M$ on $\mu : P \to M$ is called <u>principal</u> if there exists a surjective submersion $\pi : P \to B$ s.t.

(i) action is fiberwise: $\pi(pg) = \pi(p)$

(ii) $P \times_M \mathcal{G} \to P \times_B P$, $(p,g) \mapsto (p, pg)$ is a diffeomorphism.

One calls $\pi : P \to B$ a <u>principal $\mathcal{G}$-bundle</u>.

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & G & P \arrow[dll, "\mu"'] \arrow[dr, "\pi"] & \\ M & & & B \end{tikzcd}$$

(In this case, the orbit space $\mathcal{G} \backslash P \simeq B$ is smooth)

<u>Rmk.</u> For actions we allow S to be non-Hausdorff but we assume that for every orbit $\mathcal{O}$ of $\mathcal{G} \rightrightarrows M$ the preimage $\mu^{-1}(\mathcal{O})$ is Hausdorff space. In particular, the fibers $\mu^{-1}(x)$ are also Hausdorff spaces. For now one can assume that S is Hausdorff.

Examples

1. Given a Lie group action $G \times M \to M$:

   - $G \rightrightarrows \{*\} \curvearrowright \mu : M \to \{*\}$ and action is principal iff it is principal in the usual sense.

   - $G \ltimes M \rightrightarrows M \curvearrowright \mu : P \to M$ iff $G \curvearrowright P$ and $\mu : P \to M$ is a G-equivariant map.

2. Every Lie groupoid acts on itself (on left & right) by left/right translations:

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, shift left, "s"] \arrow[d, shift right, "t"'] & \curvearrowright & \mathcal{G} \arrow[dll, "\mu = t"] \arrow[r, dashed, "\pi = s"] & M \\
M & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
M & \mathcal{G} \arrow[l, dashed, "\pi = t"'] \arrow[drr, "\mu = s"'] & \curvearrowleft & \mathcal{G} \arrow[d, shift left, "s"] \arrow[d, shift right, "t"'] \\
& & & M
\end{tikzcd}
$$

Both these actions are principal

3. Given a G-principal bundle $\pi : P \to M$ the gauge groupoid $\mathcal{G} = (P \times P)/_G \rightrightarrows M$ acts on $\pi : P \to M$

$[p_1, p_2] \cdot p := [p \cdot g, p]$

where $p_2 g = p$

$$
\begin{tikzcd}
P \times_M P \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll, "\pi"] \arrow[r, dashed] & \{*\} & \curvearrowleft\ G \arrow[l] \\
M & & & &
\end{tikzcd}
$$

This action is principal.

4. Given a vector bundle $\mu : E \to M$ the groupoid $GL(E) \rightrightarrows M$ acts on $\mu : E \to M$

$$
\begin{tikzcd}
GL(E) \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & E \arrow[dll, "\mu"] \\
M & &
\end{tikzcd}
\qquad \phi \cdot e = \phi(e) \quad \text{if} \quad x \xrightarrow{\phi} y
$$

This action is not principal (why?).

We will comeback to $\mathcal{G}$-principal bundle. The last action is "linear":

Defn. A <u>representation</u> of a Lie Groupoid $\mathcal{G} \rightrightarrows M$ is a $\mathcal{G}$-action on a vector bundle $\mu: E \to M$ which is linear:

$$\forall g \in \mathcal{G}, \quad E_{s(g)} \to E_{t(g)}, \; e \mapsto g \cdot e, \text{ is linear map.}$$

- For each $x \in M$, we obtain a Rep of $\mathcal{G}_x$ on E_x.

- Clearly, a Rep of $\mathcal{G} \rightrightarrows M$ on $E \to M$ is the same thing as a Lie Groupoid Morphism covering id_M:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & GL(E) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "id"'] & M \end{tikzcd}$$

We say that the representation is faithful if Φ is 1:1.

- Given two Reps of $\mathcal{G} \rightrightarrows M$:
  - Direct sum $E_1 \oplus E_2 \to M$: $\quad g \cdot (e_1 + e_2) = g e_1 + g e_2$
  - Tensor product $E_1 \otimes E_2 \to M$: $\quad g \cdot (e_1 \otimes e_2) = (g e_1) \otimes (g e_2)$
  - Trivial Rep: $\mathbb{R}_M \to M$: $\quad g \cdot (x, \lambda) = (y, \lambda)$ if $x \xrightarrow{g} y$

$\rightsquigarrow (Rep(\mathcal{G}), \oplus, \otimes)$ is a semi-ring

<u>Examples</u>

1. A Rep of pair Groupoid $M \times M \rightrightarrows M$ on $\mu: E \to M$ amounts to a trivialization of $\mu: E \to M$.

   A Rep of $\Pi_1(M)$ on $\mu: E \to M$ amounts to give a //-transport map $\Leftrightarrow$ Flat connection on ∇

2. A Rep of action Groupoid $G \ltimes M \rightrightarrows M$ on $\mu: E \to M$ is the same thing as a G-equivariant v.b.: $Rep(G \ltimes M) \simeq Vect_G(M)$

3. Given a principal G-bundle $\pi : P \to M$ and Rep of gauge groupoid $\mathcal{G} = (P \times P)/G \rightrightarrows M$ on $\mu : E \to M$, then fixing $x \in M$:

- $G = \mathcal{G}_x \curvearrowright V \equiv E_x$
- V.b. isomorphism:

$$\begin{tikzcd} (P \times V)/G \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd} \qquad \begin{array}{l} [p, e_x] \mapsto [p, p'] \cdot e_x \\ \text{w/ } \pi(p') = x \end{array}$$

$$\mathrm{Rep}\left((P \times P)/G\right) \simeq \mathrm{Rep}(G)$$

4. Let $(M, \mathcal{F})$ be a foliated manifold. We obtain a linear action by linearizing holonomy transformations:

$$\begin{tikzcd} \mathrm{Hol}(M, \mathcal{F}) \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & \nu(\mathcal{F}) \arrow[dll] \\ M & & \end{tikzcd}$$

$x = \gamma(0)$, $v \in \nu_x(\mathcal{F}) = T_xM / T_x\mathcal{F}$

$$[\gamma]_h \cdot v := \underbrace{d_x \,\mathrm{hol}(\gamma)}_{\mathrm{hol}_{Lin}(\gamma)}(v)$$

Note that $\mathrm{hol}_{Lin}(\gamma)$ can be trivial and $\mathrm{hol}(\gamma)$ non-trivial.

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