MATH 595 - LECTURE 7

9) <u>Lie algebroid of v.b.</u> $E \to M$ a vector bundle

A **Derivation** of E is a pair (D, X) where

- $D : \Gamma(E) \to \Gamma(E)$ linear map
- $X_D \in \mathfrak{X}(M)$

$$D(fs) = f\,Ds + X_D(f), \quad \forall f \in C^\infty(M),\ s \in \Gamma(E)$$

If $D_1 = D_2$ then $X_{D_1} = X_{D_2}$, so one usually denotes a derivation simply by D. The vector field X_D is called the **symbol** of D.

In $\mathrm{Der}(E) = \{\text{derivations}\}$:

- Lie bracket:

$$[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$$

- The symbol map $\rho : \mathrm{Der}(E) \to \mathfrak{X}(M)$, a Lie algebra map

Q. Is $\mathrm{Der}(E)$ the space of sections of some vector bundle $A \to M$?

A. Yes! Apply <u>Serre-Swan Theorem:</u> Every finitely generated, projective, $C^\infty(M)$-module over a connected manifold M is the space of sections of a v.b. Can also use the following instead:

<u>Exercise.</u> Show that the Lie algebroid of $GL(E) \rightrightarrows M$, denoted $\mathfrak{gl}(E) \to M$, admits a natural linear isomorphism:

$$\Gamma(\mathfrak{gl}(E)) \simeq \mathrm{Der}(E)$$

which takes the Lie bracket and anchor to the commutator and the symbol in $\mathrm{Der}(E)$.

- ∇ connection on E $\Rightarrow$ $D := \nabla_X$ is derivation w/ symbol X

  $\Rightarrow$ $\mathfrak{gl}(E)$ has surjective anchor, so it is transitive algebroid

- $\{ D \in \mathrm{Der}(E) : X_D = 0 \} = \{ D : \Gamma(E) \to \Gamma(E) \mid C^\infty(M)\text{-linear} \} = \Gamma(\mathrm{End}(E))$

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathfrak{gl}(E) \xrightarrow{\rho} TM \longrightarrow 0$$

∇

<u>Conclusion:</u>

$\mathfrak{gl}(E) \cong \mathrm{End}(E) \oplus TM$ (non-canonical / depends on choice of ∇)

<u>Rmk.</u> From our previous discussion:

$$GL(E) \simeq \text{Gauge Groupoid of } \mathrm{Fr}(E) \to M \quad \circlearrowleft GL_r(\mathbb{R})$$

$$\Rightarrow \quad \mathfrak{gl}(E) \simeq \text{Atiyah algebroid of } \mathrm{Fr}(E) \to M \quad \circlearrowleft GL_r(\mathbb{R})$$

$$= T(\mathrm{Fr}(E)) / GL_m(\mathbb{R})$$

10) <u>Pullback of Lie algebroids.</u> $A \to M$ Lie algebroid, $\mu : N \to M$ smooth map

$$\mu^! A := A \times_{TM} TN = \{ (a, v) : \rho(a) = d\mu(v) \}$$

$\downarrow$

N

Assume this is a vector bundle (e.g., $\mu \pitchfork \rho$)

- Anchor: $\rho = pr_2 : \mu^! A \to TN$
- Lie bracket: $\Gamma(\mu^! A) \supset \{ (\alpha, X) \in \Gamma(A) \times \mathfrak{X}(N) : \rho(\alpha) = \mu_* X \}$

$$[(\alpha_1, X_1), (\alpha_2, X_2)] := ([\alpha_1, \alpha_2]_A, [X_1, X_2])$$

Extend to any section by requiring Leibniz

Rmks

- The Lie algebroid of $\mu^! \mathcal{G}$ is isomorphic to $\mu^! A(\mathcal{G})$
- Under appropriate conditions, one can restrict $A \to M$ to a submanifold $i: N \hookrightarrow M$: $A_N := i^* A$
- If $\mathcal{O} \subset M$ is an orbit of a Lie algebroid, one can always restrict A to $\mathcal{O}$, resulting in a transitive algebroid:

$$0 \to \mathfrak{g}_{\mathcal{O}} \to A_{\mathcal{O}} \to T\mathcal{O} \to 0 \qquad w/ \ \mathfrak{g}_{\mathcal{O}} = \bigcup_{x \in \mathcal{O}} \mathfrak{g}_x(A).$$

- An arbitrary Lie algebroid can be thought of as a collection of transitive algebroids parameterized by its leaves.

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Alternative Description of Lie algebroids:

- $A \to M$ any vector bundle: $\Omega^k(A) := \Gamma(\Lambda^k A^*)$ "A-forms"
  - $\Omega(A) = \left(\bigoplus_k \Omega^k(A), \wedge\right)$ is a (graded) algebra
  - $\Omega(A)$ is generated by $\Omega^0(A) = C^\infty(M)$ & $\Omega^1(A)$:

$$\omega = \sum_{i=1}^N f_i \, \theta_1 \wedge \dots \wedge \theta_k \qquad \theta_i \in \Omega^1(A), \quad f_i \in C^\infty(M)$$

- $$\begin{tikzcd} A_1 \arrow[r, "\Phi"] \arrow[d] & A_2 \arrow[d] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$ Any vector bundle map: $\Phi^*: \Omega^\bullet(A_2) \to \Omega^\bullet(A_1)$

$$(\Phi^* \omega)_x(\alpha_1, \dots, \alpha_k) := \omega_{\phi(x)}(\Phi(\alpha_1), \dots, \Phi(\alpha_k))$$

Proposition. Let $A \to M$ be a vector bundle. There is a 1:1 correspondence

$$\left\{ \begin{array}{c} \text{Lie algebroid structures} \\ \text{on } A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{linear operators } d_A: \Omega^\bullet(A) \to \Omega^{\bullet+1}(A) \text{ s.t.} \\ (i) \ d_A(\omega \wedge \eta) = d_A\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d_A \eta \\ (ii) \ d_A^2 = 0 \end{array} \right\}$$

PROOF

Given Lie algebroid $(A,[\cdot,\cdot],\rho)$ one defines the A-differential $d_A : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ by:

$$(d_A\omega)(\alpha_0,..,\alpha_k) := \sum_{i=0}^{k} (-1)^i \rho(\alpha_i)(\omega(\alpha_0,..,\hat{\alpha}_i,...,\alpha_k)) +$$

$$+ \sum_{0\le i<j\le k} (-1)^{i+j} \omega([\alpha_i,\alpha_j],\alpha_0,...,\hat{\alpha}_i,..,\hat{\alpha}_j,..,\alpha_k)$$

Just like the de Rham differential one checks that (i) & (ii) hold.

Conversely, given d_A satisfying (i) and (ii), we define:

$\rho : \Gamma(A) \to \mathfrak{X}(M)$, $\quad \rho(\alpha)(f) := d_A f(\alpha) \quad (f\in C^\infty(M))$

$[\,,\,] : \Gamma(A)\times\Gamma(A) \to \Gamma(A)$, $\quad \langle[\alpha,\beta],\theta\rangle := -d_A\theta(\alpha,\beta) - \rho(\alpha)(\theta(\beta)) + \rho(\beta)(\theta(\alpha)) \quad (\theta\in\Omega^1(A))$

Since $\rho(g\alpha) = g\rho(\alpha)$, $\forall g\in C^\infty(M)$, we see that ρ is induced by a bundle map $\rho : A \to TM$

Definition shows that $[\cdot,\cdot]$ is $\mathbb{R}$-linear, skew-symmetric.

To check Leibniz:

$$\begin{aligned}\langle[\alpha,f\beta],\theta\rangle &= -d_A\theta(\alpha,f\beta) - \rho(\alpha)(\theta(f\beta)) + \rho(f\beta)(\theta(\alpha))\\ &= -f\,d_A\theta(\alpha,\beta) - \rho(\alpha)(f)\,\theta(\beta) - f\rho(\alpha)(\theta(\beta)) - f\rho(\beta)(\theta(\alpha))\\ &= \langle f[\alpha,\beta] - \rho(\alpha)(f)\beta,\theta\rangle\end{aligned}$$

Finally, we need to check Jacobi identity. Since Leibniz holds, it is enough to check this on a local chart (U,x^i) over which $A\to M$ has a local basis of sections $\{e_a\}$. Then write:

$$\rho(e_a) = b_a^i \frac{\partial}{\partial x^i} \quad \text{w/ } b_a^i \in C^\infty(U)$$

$$[e_a,e_b] = C_{ab}^c e_c$$

Then:

$$\text{Jacobi id} \iff \underset{a,b,c}{\odot} [e_a, [e_b, e_c]] = 0$$

$$\iff \underset{a,b,c}{\odot} [e_a, C^d_{bc} e_d] = 0$$

$$\iff \underset{a,b,c}{\odot} \left(\rho(e_a)(C^d_{bc})\, e_d + C^d_{bc} [e_a, e_d]\right) = 0$$

$$\iff \boxed{\underset{a,b,c}{\odot} \left(B^i_a \frac{\partial C^d_{bc}}{\partial x^i} + C^d_{ae} C^e_{bc} \right) = 0} \quad (*)$$

Let $\{\theta^a\}$ be dual coframe: $\theta^a(e_b) = \delta^a_b$. Then:

- $d_A f(e_a) = \rho(e_a)(f) = B^i_a \, \partial f / \partial x^i \Rightarrow d_A f = \frac{\partial f}{\partial x^i} B^i_a \theta^a$

- $\langle \underbrace{[e_a, e_b]}_{= C^d_{ab} e_d}, \theta^c \rangle = - d_A \theta^c (e_a, e_b) \Rightarrow d_A \theta^c = -\frac{1}{2} C^c_{ab} \theta^a \wedge \theta^b$

- (i) + (ii) $\Rightarrow \quad 0 = d_A^2 \theta^c = -\frac{1}{2} d_A (C^c_{ab} \theta^a \wedge \theta^b)$

$$= -\frac{1}{2} \left(B^i_d \frac{\partial C^c_{ab}}{\partial x^i} \theta^d \wedge \theta^a \wedge \theta^b - C^c_{ab} d_A(\theta^a \wedge \theta^b) \right)$$

$$= -\frac{1}{2} \left(B^i_d \frac{C^c_{ab}}{\partial x^i} \theta^d \wedge \theta^a \wedge \theta^b - \frac{1}{2} C^e_{ab} C^a_{de} \theta^d \wedge \theta^e \wedge \theta^b + \frac{1}{2} C^c_{ab} C^b_{de} \theta^a \wedge \theta^d \wedge \theta^e \right)$$

$$\iff (*) \qquad \blacksquare$$

<u>Rmk.</u> The equation $\rho([\alpha, \beta]) = [\rho(\alpha), \rho(\beta)]$ in local coordinates and local sections as above is equivalent to:

$$\boxed{C^c_{ab} B^i_c = B^j_a \frac{\partial B^i_b}{\partial x^j} - B^j_b \frac{\partial B^i_a}{\partial x^j}} \quad (**)$$

Eqs $(*)$ & $(**)$ are the structure equations of a Lie algebroid. They caracterize locally a Lie algebroid. They appear in E. Cartan's work on the "Equivalence Problem".

If $A = A(G)$, then:

$$\cdot\,(\Omega^k(A), d_A) \simeq (\Omega^k_{inv}(G), d) = \text{left-invariant forms on } G$$

where a left-invariant form ω of degree k is

(i) a t-foliated form: $\omega \in \Gamma(\Lambda^k(\operatorname{Ker} dt)^*)$

(ii) left-invariant:

$$\omega_{gh}(d_h L_g v_1, \dots, d_h L_g v_k) = \omega_h(v_1, \dots, v_k), \quad v_1, \dots, v_k \in \operatorname{Ker} d_h t$$

Also d is the foliated de Rham differential:

$$(d\omega)(X_0, \dots, X_k) = \sum_i (-1)^i \omega(X_0, \dots, \hat{X}_i, \dots, X_k) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_k)$$

In particular, if $\Phi : G_1 \to G_2$ is a morphism, then:

$$\Phi^* : (\Omega^\bullet(G_2), d) \to (\Omega^\bullet(G_1), d), \quad \text{map of complexes}$$

$$\Rightarrow (\Phi_*)^* : (\Omega^\bullet(A_2), d_{A_2}) \to (\Omega^\bullet(A_1), d_{A_1}), \quad \text{map of complexes}$$

This suggests:

<u>Def.</u> A <u>Lie algebroid morphism</u> is a v.b. map

$$\begin{array}{ccc} A_1 & \xrightarrow{\Phi} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[\phi]{} & M_2 \end{array}$$

such that the induced pullback map is a map of complexes:

$$\Phi^* : (\Omega^\bullet(A_2), d_2) \to (\Omega^\bullet(A_1), d_1)$$

This allows to formally define subalgebroids:

<u>Def.</u> A <u>Lie subalgebroid</u> of $(A, [\cdot,\cdot]_A, \rho_A)$ is a Lie algebroid $(B, [\cdot,\cdot]_B, \rho_B)$ together with an algebroid morphism $\Phi : B \to A$ which is an injective immersion.

Exercise: Show that if $\phi : M_1 \to M_2$ is a diffeo, this definition is equivalent to the old one, i.e.,

$$\Phi^* d_{A_2} = d_{A_1} \Phi^* \qquad \text{iff} \qquad \begin{cases} \rho_2 \circ \Phi_*(\alpha) = d\phi \circ \rho_1(\alpha) \\ \Phi_*([\alpha,\beta]) = [\Phi(\alpha), \Phi(\beta)] \end{cases}$$

Hint. d_A is determined by its action on 0-forms and 1-forms.

Another by product of this discussion is:

Def. The cohomology of the complex $(\Omega^\bullet(A), d_A)$ is called the Lie algebroid cohomology of $(A, [\cdot,\cdot], d_A)$

Examples

1) $A = TM$: $H^\bullet(A) = H^\bullet_{dR}(M)$

2) $A = \mathfrak{g}$: $H^\bullet(A) = H^\bullet(\mathfrak{g})$ (Chevalley-Eilenberg coh)

3) $A = T\mathcal{F}$: $H^\bullet(A) = H^\bullet(\mathcal{F})$ (Foliated cohomology)

Rmk.

- Lie algebroid cohomology is often ∞-dim and hard to compute.
- One can show that $(\Omega^\bullet(A), d_A)$ is an elliptic complex iff A is a transitive algebroid. Hence, for transitive algebroids over compact M, $H^\bullet(A)$ is finite dimensional.

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A Lie algebroid can be thought of as a "geometric Lie algebra" or as "generalized tangent bundle". Here is another illustration of this mixed flavor.

Exponential Map (version 1)

$\mathcal{G} \rightrightarrows M$ Lie groupoid $\rightsquigarrow$ $\mathcal{B}(\mathcal{G})$ = group of bisections

$A \to M$ Lie algebroid $\rightsquigarrow$ $\Gamma(A)$ = Lie algebra of sections

Lemma.

If $A = A(\mathcal{G})$, for any $\alpha \in \Gamma(A)$ the left-invariant v.f. $\overleftarrow{\alpha} \in \mathfrak{X}(\mathcal{G})$ is complete iff $\rho(\alpha) \in \mathfrak{X}(M)$ is complete.

Proof

$\overleftarrow{\alpha}$ & $\rho(\alpha)$ are s-related:

$$d_g s(\overleftarrow{\alpha}_g) = d_g s\left(d_{1_{s(g)}} L_g(\alpha_{s(g)})\right) = d_{1_{s(g)}} s(\alpha_{s(g)}) = \rho(\alpha_{s(g)})$$

$$\Leftrightarrow \quad s_* \overleftarrow{\alpha} = \rho(\alpha)$$

- s is a surjective submersion: $\overleftarrow{\alpha}$ complete $\Rightarrow \rho(\alpha)$ complete

- Assume $\rho(\alpha)$ complete. Let $g : (a,b) \to \mathcal{G}$ be integral curve of $\overleftarrow{\alpha}$. Then $\gamma(t) = s(g(t))$ is integral curve of $\rho(\alpha)$ and can be extended to all $t \in \mathbb{R}$. If $b < +\infty$, let $h(t)$ be integral curve of $\overleftarrow{\alpha}$ with $h(b) = 1_{\gamma(b)}$. Then $h :]b-\varepsilon, b+\varepsilon[\to \mathcal{G}$, for some $\varepsilon > 0$ and we can define:

$$\tilde{g}(t) = \begin{cases} g(t), & t \in (a, b-\varepsilon] \\ g(b-\varepsilon)\, h(b-\varepsilon)^{-1}\, h(t), & t \in (b-\varepsilon, b+\varepsilon) \end{cases}$$

This is an integral curve of $\overleftarrow{\alpha}$ extending $g(t)$. Hence $b = +\infty$. Similar argument for a. $\blacksquare$

For any Lie algebroid A we call $\alpha \in \Gamma(A)$ complete if $\rho(\alpha)$ is complete. For example, compactly supported sections are complete, so there are many complete sections.

$\not{Z}$ The set $\Gamma_{cpl}(A) \subset \Gamma(A)$, in general, is not a subspace. The set $\Gamma_c(A) \subset \Gamma(A)$ is a Lie subalgebra

Def. The exponential map of a Lie groupoid $Exp : \Gamma_{cpl}(A) \to B(\mathcal{G})$

$$Exp(\alpha)(x) := \phi^1_{\overleftarrow{\alpha}}(1_x)$$

Rmk: By our conventions, $Exp(\alpha)$ is a t-parameterized bisection:

$$t \circ Exp(\alpha) = id \quad , \quad s \circ Exp(\alpha) : M \to M \text{ diffeo} .$$

Just like for a Lie group, the map $\mathbb{R} \to B(\mathcal{G}), t \mapsto exp(t\alpha)$ is a group homomorphism.