MATH 595 - LECTURE 6

Def. A **Lie algebroid** is a vector bundle $A \to M$ together with:

- A Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$
- A bundle map $\rho : A \to TM$ covering id_M

satisfying the Leibniz identity:

$$[\alpha_1, f\alpha_2] = f[\alpha_1,\alpha_2] + \rho(\alpha_1)(f)\,\alpha_2, \quad \alpha_1,\alpha_2 \in \Gamma(A),\ f\in C^\infty(M)$$

Rmk.

- For $\mathcal{G} \rightrightarrows M$ we call $A(\mathcal{G})$, defined in previous lecture, the Lie algebroid of $\mathcal{G}$. When $A \simeq A(\mathcal{G})$ we say that A is integrable.
- ρ is called the anchor of A. Together w/ Leibniz, it is what makes a Lie algebroid a geometric object.

Exercise. Show that for any Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ the induced map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ preserves brackets: $\rho([\alpha,\beta]_A) = [\rho(\alpha),\rho(\beta)]$

What about morphisms?

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\Phi} & \mathcal{G}_2 \\ \downdownarrows & & \downdownarrows \\ M_1 & \xrightarrow[\phi]{} & M_2 \end{array} \quad \text{morphism} \quad \begin{cases} t\circ\Phi = \phi\circ t \\ \Phi(1_x) = 1_{\phi(x)} \end{cases} \Rightarrow \quad \begin{array}{ccc} A_1 & \xrightarrow{\Phi_*} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[\phi]{} & M_2 \end{array} \quad \text{v.b. map}$$

- $s\circ\Phi = \phi\circ s \Rightarrow \rho_2\circ\Phi_* = d\phi\circ\rho_1$

$\Phi(gh) = \Phi(g)\Phi(h) \Rightarrow \Phi_*$ preserves Lie brackets $(*)$

If ϕ is a diffeomorphism, then $\Phi_* : \Gamma(A_1) \to \Gamma(A_2)$, $\alpha \longmapsto \Phi_*\circ\alpha\circ\phi^{-1}$

$$(*) \iff \Phi_*([\alpha_1,\alpha_2]) = [\Phi_*(\alpha_1), \Phi_*(\alpha_2)]$$

Issue: In general, there is no map $\Phi_* : \Gamma(A_1) \to \Gamma(A_2)$.

We will deal w/ this later.

Examples

1) Tangent bundle. For any manifold M

- $A = TM$, w/ $\rho \equiv id : TM \to TM$

  $[\cdot,\cdot]$ = usual Lie bracket of v.f.

- It is easy to check that both $M\times M \rightrightarrows M$ & $\Pi_1(M) \rightrightarrows M$ have Lie algebroid $\simeq TM$ (we say integrate $A = TM$)

- For any Lie algebroid, the anchor

$$\rho : A \to TM$$

is a Lie algebroid morphism.

Exercise. If $\mathcal{G} \rightrightarrows M$ is Lie groupoid show that groupoid anchor $\Phi = (t,s) : \mathcal{G} \to M \times M$ is a groupoid morphism that differentiates to $\rho : A(\mathcal{G}) \to TM$ (we say that Φ integrates the morphism ρ)

2) Lie algebras $\iff$ Lie algebroids w/ $M = \{x\}$

Exercise: For any Lie algebroid $A \to M$, fixing $x \in M$:

$$\mathfrak{g}_x(A) := \operatorname{Ker} \rho_x \subset A_x$$

Show that for if $\alpha_x, \beta_x \in \mathfrak{g}_x$ then:

$$[\alpha_x, \beta_x] := [\tilde{\alpha}, \tilde{\beta}](x), \quad \text{w/} \begin{cases} \tilde{\alpha}, \tilde{\beta} \in \Gamma(A) \\ \tilde{\alpha}(x) = \alpha_x, \tilde{\beta}(x) = \beta_x \end{cases}$$

is well-defined, i.e., independent of choice of extensions.

- $\mathfrak{g}_x(A)$ is the isotropy Lie algebra of A at x:
  - $\mathfrak{g}_x(A) \subset A_x$ is a Lie subalgebroid
  - If $A = A(\mathcal{G})$, $\mathfrak{g}_x(A)$ is the Lie algebra of $\mathcal{G}_x$.

3) <u>Bundle of Lie algebras</u> $\Longleftrightarrow$ Lie algebroids w/ $\rho \equiv 0$

These do not need to be locally trivial:

- $A = \mathbb{R} \times \mathbb{R}^2$, $\downarrow pr$ $\mathbb{R}$

  - $\rho \equiv 0$
  - $e_1(t) = (t, (1,0))$ $\quad e_2(t) = (t, (0,1))$

$$[e_1, e_2] = t e_1$$

- $A = \mathbb{R} \times \mathbb{R}^3$, $\downarrow pr$ $\mathbb{R}$

  - $\rho \equiv 0$
  - $[e_1, e_2] = t e_3$, $[e_2, e_3] = e_1$, $[e_3, e_2] = e_1$

$$A_t \simeq \begin{cases} so(3), & t > 0 \\ e(2), & t = 0 \\ sl(2), & t < 0 \end{cases}$$

There is a bundle of Lie groups $\mathcal{G} \overset{t}{\underset{s}{\rightrightarrows}} M$ integrating this bundle of Lie algebroids. If we require $\mathcal{G}$ to have 1-connected fibers, $\mathcal{G}$ is non-Hausdorff.

4) <u>Involutive Distributions</u>. $D \subset TM$ a vector subbundle such that:

$$X, Y \in \mathfrak{X}(D) \Rightarrow [X,Y] \in \mathfrak{X}(D)$$

Then $A = D$ is Lie algebroid w/

- Anchor: $\rho : A \hookrightarrow TM$ (inclusion)
- bracket = usual Lie bracket of v.f.

Recall Frobenius Thm:

$$\left\{ \begin{matrix} \text{involutive distributions} \\ D \subset TM \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{Foliations } \mathcal{F} \\ \text{of } M \end{matrix} \right\}$$

$$D = T\mathcal{F}$$

<u>Exercise</u>. $\Pi_1(M, \mathcal{F})$ & $Hol(M, \mathcal{F})$ both have Lie algebroid isomorphic to $T\mathcal{F}$.

RMK. For a general Lie algebroid $A \to M$

$$\mathfrak{g}(A) = \operatorname{Ker}\rho = \bigcup_{x\in M} \operatorname{Ker}\rho_x$$

is a bundle of Lie algebras, but not smooth. We call A a regular Lie algebroid if ρ has constant rank so $\mathfrak{g}(A) \to M$ is bundle of Lie algebras. In this case we have a short exact sequence of Lie algebroids:

$$0 \to \operatorname{Ker}\rho \to A \to \operatorname{Im}\rho \to 0$$

where $\operatorname{Im}\rho \subset TM$ is an integrable distribution. The foliation $\mathcal{F}$ corresponding to $\operatorname{Im}\rho$ is called the orbit foliation of the regular algebroid A.

For general Lie algebroid $p: A \to M$ one still has an orbit foliation:

- It is the unique partition of M into connected, regular, immersed submanifolds $\mathcal{F}_A = \{ \mathcal{O}_i : i \in I \}$ with $T_x\mathcal{O}_i = \operatorname{Im}\rho_x, \forall x \in \mathcal{O}_i$
- $x, y \in M$ belong to same orbit iff $\exists$ smooth path $a: [0,1] \to A$ whose base path connects x & y:

$$\gamma : [0,1] \to M, \quad \gamma(t) = p(a(t)), \qquad \gamma(0) = x, \; \gamma(1) = y$$

and:

$$\rho(a(t)) = \frac{d}{dt}\gamma(t), \qquad \forall t \in [0,1].$$

- If $A = A(\mathcal{G})$ for some Lie groupoid $\mathcal{G} \rightrightarrows M$, then

Orbits of $A(\mathcal{G})$ = connected components of orbits of $\mathcal{G} \rightrightarrows M$

We will not prove this result. See references.

A Lie algebroid is called transitive if $\operatorname{Im}\rho = TM$; sequence becomes:

$$0 \to \operatorname{Ker}\rho \to A \xrightarrow{\rho} TM \to 0$$

5) __Atiyah algebroid.__ For any principal bundle $\underset{\circlearrowleft K}{P} \xrightarrow{\pi} M$.

$$\underset{\circlearrowleft K}{TP \longrightarrow P} \quad \rightsquigarrow \quad A := TP/_K \longrightarrow M = P/_K$$

$$\underset{\circlearrowleft K}{d\pi : TP \longrightarrow TM} \quad \rightsquigarrow \quad \rho : A \longrightarrow TM$$

$$\Gamma(A) \simeq \mathfrak{X}(P)^K \rightsquigarrow [\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

where:

$$\mathfrak{X}(P)^K := \{ X \in \mathfrak{X}(P) : k_* X = X, \ \forall k \in K \}$$

This is a transitive Lie algebroid. Also:

$$X \in \mathfrak{X}(P)^K, \ d\pi(X) = 0 \iff X : P \to \mathfrak{k}, \ K\text{-equivariant}$$

for adjoint action on $\mathfrak{k}$

$\iff$ Sections of adjoint bundle

$$P[\mathfrak{k}] := (P \times \mathfrak{k})/_K$$

So the Atiyah algebroid as associated short exact sequence:

$$0 \longrightarrow P[\mathfrak{k}] \longrightarrow TP/_K \longrightarrow TM \longrightarrow 0$$

__Exercise.__ Show that the Lie algebroid of the gauge groupoid

$$\mathcal{G} = (P \times P)/_K \rightrightarrows M$$

is the Atiyah algebroid.

Conclude that if a transitive Lie algebroid $A \to M$ over a connected base is integrable then $A \simeq$ Atiyah algebroid of some principal bundle.

6) Prequantization Algebroid. Fix $\omega \in \Omega^2_{cl}(M)$.

- $A_\omega := TM \oplus \mathbb{R}_M \qquad (\mathbb{R}_M := M \times \mathbb{R} \to M)$
- Anchor: $\rho := pr_1 : A_\omega \to TM$
- Lie bracket on $\Gamma(A_\omega) \simeq \mathfrak{X}(M) \times C^\infty(M)$

$$[(X,f),(Y,g)] := ([X,Y], X(g) - Y(f) + \omega(X,Y))$$

Transitive algebroid w/ Atiyah sequence:

$$0 \to \mathbb{R}_M \to TM \oplus \mathbb{R}_M \longrightarrow TM \to 0$$

Prequantization Problem. Given a closed 2-form ω is there a principal S^1-bundle $\pi : P \to M$ w/ connection $\theta \in \Omega^1(P)$ such that $d\theta = \pi^*\omega$?

Exercise. Show that if $\pi : P \to M$ is such prequantization bundle, then its gauge groupoid $(P \times P)/_{S^1} \rightrightarrows M$ has Lie algebroid A_ω:

$$\begin{array}{ccc}
\omega \text{ prequatizable} & \Rightarrow & A_\omega \text{ integrable} \\
\Updownarrow & & \Updownarrow \\
Per(\omega) \subset \mathbb{R} \text{ discrete} & & Per_S(\omega) \subset \mathbb{R} \text{ discrete} \\
\| & & \| \\
\left\{ \int_\sigma \omega : \sigma \in H_2(M,\mathbb{Z}) \right\} & \supset & \left\{ \int_\sigma \omega : \sigma \in \pi_2(M) \right\}
\end{array}$$

Examples:

- $M = \mathbb{T}^2 \times \mathbb{T}^2$, $\omega = pr_1^*\mu + \sqrt{2}\, pr_2^*\mu$ w/ $\mu \in \Omega^2(\mathbb{T}^2)$, $\int_{\mathbb{T}^2} \mu = 1$

  $Per(\omega) = \langle 1, \sqrt{2} \rangle \subset \mathbb{R} \Rightarrow$ not prequatizable

  $Per_S(\omega) = \{0\} \subset \mathbb{R} \Rightarrow A_\omega$ is integrable

- $M = \mathbb{S}^2 \times \mathbb{S}^2$, $\omega = pr_1^* \mu + \sqrt{2}\, pr_2^* \mu$ w/ $\mu \in \Omega^2(\mathbb{T}^2)$, $\int_{\mathbb{S}^2} \mu = 1$

  $Per(\omega) = Per_{\mathcal{S}}(\omega) = \langle 1, \sqrt{2} \rangle \subset \mathbb{R} \Rightarrow$ not prequantizable

  not integrable !!

7) Vector Fields. Given $X \in \mathfrak{X}(M)$

- $A = \mathbb{R}_M \to M$
- Anchor: $\rho(x, \lambda) := \lambda X_x$
- Lie Bracket: $\Gamma(\mathbb{R}_M) \simeq C^\infty(M)$

$$[f, g] := f X(g) - g X(f)$$

Any Lie algebroid structure on $\mathbb{R}_M \to M$ is of this form

$$X := \rho(e) \qquad e(x) = (x, 1)$$

The Flow groupoid $D(X) \rightrightarrows M$ has Lie algebroid $A = \mathbb{R}_M$.

8) Action algebroids. $a : \mathfrak{g} \to \mathfrak{X}(M)$ infinitesimal action

- $A = M \times \mathfrak{g} \to M$
- Anchor: $\rho : M \times \mathfrak{g} \to TM$, $(x, v) \mapsto a(v)_x$
- Lie Bracket: $\Gamma(A) = C^\infty(M, \mathfrak{g})$

$$[f, g](x) = [f(x), g(x)]_{\mathfrak{g}} + (\mathcal{L}_{a(f(x))} g)(x) - (\mathcal{L}_{a(g(x))} f)(x)$$

- The Lie algebroid of $X \in \mathfrak{X}(M)$ is a special case.
- The Lie groupoid associated w/ an action $G \circlearrowright M$ has Lie algebroid the one associated w/ the corresponding infinitesimal action.
- A Lie algebra action does not need to integrate to a Lie Group action. But the action algebroid is always integrable!