MATH 595 - Lecture 5

Theorem

$\Pi_1(M,\mathcal{F})$ and $Hol(M,\mathcal{F})$ have natural Lie groupoid structures and Φ is a Lie groupoid homomorphism which is a local diffeo.

Proof.

We will describe a basis for the topology, that are also the domains of charts.

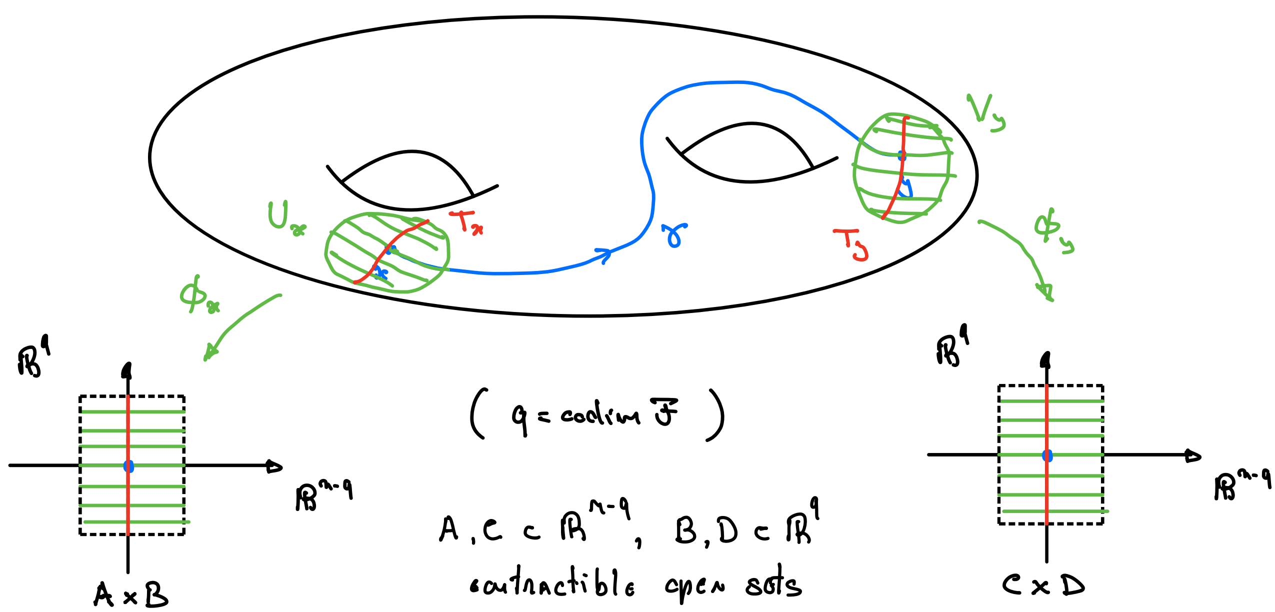
Fix path $\gamma : [0,1] \to M$, $\gamma(0) = x$, $\gamma(1) = y$, lying in a leaf L_0

Choose foliated charts centered at x & y, and transversals T_x & T_y:

We can assume that choices are small enough so that we have the holonomy transformation:

$$h(\gamma) : T_x \longrightarrow T_y \qquad h(\gamma)(z) := H(1,z)$$

$$H : [0,1] \times T_x \longrightarrow M \qquad \begin{cases} t \mapsto H(t,z) & \text{path in leaf} \\ H(0,z) = z \\ H(1,z) \in T_y \end{cases}$$



$(q = \operatorname{codim} \mathcal{F})$

$A, C \subset \mathbb{R}^{n-q}$, $B, D \subset \mathbb{R}^q$ contractible open sets charts: $\mathbb{R}^{2m\ q} \supset A \times B \times C \longrightarrow \Pi_1(M, \mathcal{F})$

$$\phi(a,b,c) = [\gamma_3 \circ \gamma_2 \circ \gamma_1] \quad w/ \quad \begin{cases} \gamma_1 = \phi_x^{-1}\left(\overline{(a,b)\,(0,b)}\right) \\ \gamma_2 = H(-, \phi_x^{-1}(0,b)) \\ \gamma_3 = \phi_x^{-1}\left(\overline{(0,\phi_y(\gamma_2(1)))\,(c,\phi_y(\gamma_2(1))}\right) \end{cases}$$

This defines a basis for the topology, as well as, local euclidean charts

Need to check that transition functions are smooth.

$\blacksquare$

Classes of Lie Groupoids

Recall X is called k-connected if $\pi_i(X)$ is trivial for $0 \leq i \leq k$

Def.

A Lie groupoid $\mathcal{G} \rightrightarrows M$ is <u>target k-connected</u> if $t^{-1}(x)$ is k-connected for all $x \in M$. Special cases:

- $k = 0$: $\mathcal{G}$ is t-connected
- $k = 1$: $\mathcal{G}$ is t-simply connected

Remark. Inversion is diffeo between $s^{-1}(x)$ and $t^{-1}(x)$ so

target k-connected $=$ source k-connected

- We do not assume M connected. So t-connected does not imply $\mathcal{G}$ is connected.

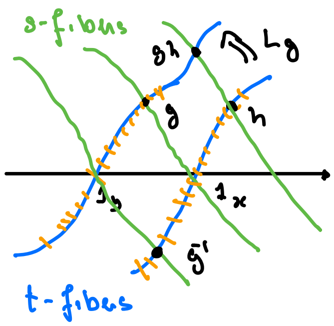
Proposition. Given any Lie groupoid $\mathcal{G}$:

$$\mathcal{G}^0 := \{ g \in \mathcal{G} : g \in \text{connected component of } t^{-1}(t(g)) \text{ containing } 1_{t(g)} \}$$

is an open, t-connected, subgroupoid of $\mathcal{G}$.

Proof.

- Let $y \xleftarrow{g} x \in \mathcal{G}^0 \iff 1_y$ & g belong to same connected component of $t^{-1}(y)$
- $L_g(1_x) = g$ & L_g maps connected components to connected comp.



$\boxed{\text{////}} = \mathcal{G}^0$

so $\left.\begin{array}{l} g,h \in \mathcal{G}^0 \Rightarrow gh \in \mathcal{G}^0 \\ g \in \mathcal{G}^0 \Rightarrow g^{-1} \in \mathcal{G}^0 \end{array}\right\}$

$\Rightarrow \mathcal{G}^0$ is a groupoid

To show that $\mathcal{G}^0$ is open it is enough $\exists\, M \subset U_{open} \subset \mathcal{G}^0$

By local normal form for submersions:

$1_x \in \mathcal{G}$ has open neighborhood $U_x \subset \mathcal{G}^0$

$\Rightarrow U = \bigcup_{x \in M} U_x$ is desired open set. $\blacksquare$

Proposition Let $\mathcal{G} \rightrightarrows M$ be t-connected Lie groupoid. Any open neighborhood $M \subset U \subset \mathcal{G}$ generates $\mathcal{G}$: given $g \in \mathcal{G}$ there exists $u_1, \dots, u_m \in U$ such that

$$g = u_1 \cdot \ldots \cdot u_m$$

Proof. It is enough to show that:

Claim. The set $S = \{ g \in t^{-1}(x) \mid g \text{ is product of elements } u_i \in U \}$ is both open & closed in $t^{-1}(x)$

- S is open: because left translations are diffeos between t-fibers
- S is closed: check that $t^{-1}(x) \setminus S$ is open in $t^{-1}(x)$. Choose

$g \notin S$, $t(g) = x$. Since inversion is also diffeo

$$gU^{-1} := \{ g u^{-1} : s(g) = s(u),\ u \in U \}$$

is neighborhood of g. But $gU^{-1} \cap S = \emptyset$: if $gu^{-1} \in S$ then

$gu^{-1} = u_1 \cdots u_k \Rightarrow g = u_1 \cdots u_k u \in S$, contradiction. ∎

Thm.

Let $\mathcal{G} \rightrightarrows M$ be a t-connected Lie groupoid. There exists a Lie groupoid $\tilde{\mathcal{G}} \rightrightarrows M$ and a Lie groupoid morphism $\Phi : \tilde{\mathcal{G}} \to \mathcal{G}$ such that:

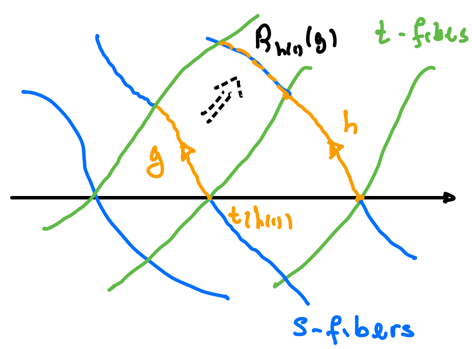
(i) $\tilde{\mathcal{G}}$ is t-simply connected

(ii) Φ is a surjective local diffeo.

Moreover, $\tilde{\mathcal{G}}$ is unique up to isomorphism.

Proof.

- $\tilde{\mathcal{G}} = \bigsqcup_{x \in M} \widetilde{s^{-1}(x)}$ with $\widetilde{s^{-1}(x)} = \{ [g] \mid g : I \to s^{-1}(x),\ g(0) = 1_x \}$
- $s([g]) = x$, $t([g]) = t(g(1))$
- $[g] \cdot [h] := [R_{h(1)}(g) \circ h]$



$\Rightarrow$ $\tilde{\mathcal{G}}$ is a groupoid

Why smooth?

$\mathcal{F}$ = Foliation by s-fibers

$\Pi_1(\mathcal{F}) \rightrightarrows \mathcal{G}$ - Lie groupoid so its source map $p : \Pi_1(\mathcal{G}) \to \mathcal{G}$ is submersion

$\Rightarrow$ $\tilde{\mathcal{G}} = p^{-1}(M) \subset \Pi_1(\mathcal{F})$ is submanifold. Check that operations make it into a Lie groupoid.

We have a surjective groupoid morphism:

$$\tilde{\mathcal{G}} \to \mathcal{G}, \quad [g] \mapsto g$$

which is smooth, being restriction of target of $\Pi_1(\mathcal{F}) \rightrightarrows \mathcal{G}$ to $\tilde{\mathcal{G}}$.

Exercise: Check its differential is isomorphism.

Let $\Phi' : \mathcal{G}' \to \mathcal{G}$ be another groupoid satisfying (i) and (ii).
Then looking at t-fibers:

$$\begin{tikzcd}
(t^{-1}_{\tilde{\mathcal{G}}}(x), 1_x) \arrow[rr, dashed, "\Psi_x"] \arrow[dr, "\Phi"'] & & (t^{-1}_{\mathcal{G}'}(x), 1_x) \arrow[dl, "\Phi'"] \\
& (t^{-1}(x), 1_x) &
\end{tikzcd}$$

These are covering maps $\Rightarrow \exists^1$ diffeo Ψ_x making diagram commute

$\Rightarrow \exists^1$ Lie groupoid isomorphism

$$\begin{tikzcd}
\tilde{\mathcal{G}} \arrow[rr, "\Psi"] \arrow[dr, "\Phi"'] & & \mathcal{G}' \arrow[dl, "\Phi'"] \\
& \mathcal{G} &
\end{tikzcd}$$

∎

The previous result is a groupoid version of the following

<u>Lie I</u>. Given a connected Lie group G, there exists a 1-connected Lie group $\tilde{G}$ w/ same Lie algebra. Moreover, $\tilde{G}$ is unique up to isomorphism.

There are two other results that form the foundations of the classical Lie group $\leftrightarrow$ Lie algebra correspondence.

Lie II. Given Lie groups G & H, w/ G 1-connected, and Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique Lie group homomorphism $\Phi : G \to H$ w/ $d_e\Phi = \phi$.

Lie III. Given a finite dimensional Lie algebra $\mathfrak{g}$ there exists a Lie group G w/ Lie algebra $\mathfrak{g}$

We would like to extend these to groupoids, so we now turn to the study of the infinitesimal versions of Lie groupoids.

II. Lie Algebroids

Lie Algebroid of a Lie Groupoid

Recall: G = Lie group, there isomorphism of vector spaces:

i) $\mathfrak{X}_{inv}(G) = \{ X \in \mathfrak{X}(G) : (L_g)_* X = X, \forall g \} \xleftrightarrow{\sim} T_e G$

$$X \longmapsto X|_e$$
$$\overleftarrow{\alpha} \longleftarrow \alpha, \quad \overleftarrow{\alpha}_g := d_e L_g(\alpha)$$

ii) $X_1, X_2 \in \mathfrak{X}_L(G) \Rightarrow [X_1, X_2] \in \mathfrak{X}_L(G)$

Def. Lie algebra of G is the vector space $\mathfrak{g} = T_e G$ w/ bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$[\alpha_1, \alpha_2] := [\overleftarrow{\alpha}_1, \overleftarrow{\alpha}_2]|_e$$

Rmk. Why left-invariant vector fields? Our convention for Lie bracket of vector fields is the usual one:

$$[X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f))$$

If $G = GL(V)$ then $T_e G = \mathfrak{gl}(V)$ (since $GL(V) \subset \mathfrak{gl}(V)$ is open)

<u>Exercise:</u> The bracket induced on $\mathfrak{gl}(V)$ is the usual commutator of linear transformations.

$$[A,B] = AB - BA$$

(If one uses right-invariant vector fields, then the bracket induced on $\mathfrak{gl}(V)$ is the anti-commutator)

<u>Def.</u> A <u>left-invariant vector field</u> on $\mathcal{G} \rightrightarrows M$ is a v.f. $X \in \mathfrak{X}(\mathcal{G})$:

(i) X is tangent to t-fibers: $d_g t(X_g) = 0$, $\forall g \in \mathcal{G}$

(ii) $d_h L_g (X_h) = X_{gh}$, $\forall (g,h) \in \mathcal{G}^{(2)}$.

<u>Lemma.</u> The vector space $\mathfrak{X}_{inv}(\mathcal{G})$ of left-inv. v.f. is closed under the Lie bracket of vector fields. It is isomorphic to the space of sections of the vector bundle over M:

$$A(\mathcal{G}) := \mathrm{Ker}(dt)_{u(M)} = u^*(\mathrm{Ker}\, dt)$$

<u>Proof:</u>

- X, Y tangent to t-fibers $\Rightarrow$ $[X,Y]$ is also tangent to t-fibers

- Fixing $g \in \mathcal{G}$, $L_g : \underbrace{t^{-1}(s(g))}_{x} \to t^{-1}(\underbrace{t(g)}_{y})$ is diffeo, so:

$$\left.\begin{array}{l} \cdot\ X|_{t^{-1}(x)},\ Y|_{t^{-1}(g)} \in \mathfrak{X}(t^{-1}(x)) \\ \cdot\ (L_g)_* X|_{t^{-1}(x)} = X|_{t^{-1}(y)},\ (L_g)_* Y|_{t^{-1}(x)} = Y|_{t^{-1}(y)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (L_g)_*[X,Y]|_{t^{-1}(x)} = (L_g)_*[X|_{t^{-1}(x)}, Y|_{t^{-1}(x)}] = [X|_{t^{-1}(y)}, Y|_{t^{-1}(y)}] = [X,Y]|_{t^{-1}(y)}$$

For second part observe that the map:

$$\mathfrak{X}_{inv}(\mathcal{G}) \ni X \longmapsto X|_{u(M)} \in \mathrm{Ker}(dt)_{u(M)}$$

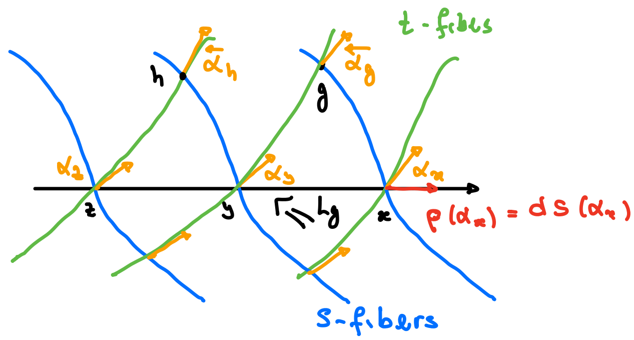
has inverse the map:

$$\mathrm{Ker}(dt)_{u(M)} \ni \alpha \longmapsto \overleftarrow{\alpha} \in \mathfrak{X}_{inv}(\mathcal{G}), \quad \overleftarrow{\alpha}|_g := d_e L_g(\alpha_{s(g)})$$

∎

We also define a vector bundle map $\rho : A(\mathcal{G}) \to TM$ by:

$$\rho(\alpha) = d_{1_x} s(\alpha) \quad \text{if } \alpha \in A_x = \mathrm{Ker}(d_{1_x} t)$$



Lemma

The bracket on sections $\Gamma(A(\mathcal{G}))$ and the map $\rho : \Gamma(A(\mathcal{G})) \to \mathfrak{X}(M)$ satisfy the Leibniz identity:

$$[\alpha_1, f\alpha_2] = f[\alpha_1, \alpha_2] + \rho(\alpha_1)(f)\,\alpha_2, \quad \alpha_1, \alpha_2 \in \Gamma(A(\mathcal{G})),\ f \in C^\infty(M)$$

Proof.

- $\overleftarrow{\alpha}$ and $\rho(\alpha)$ are s-related:

$$s_*(\overleftarrow{\alpha}) = \rho(\alpha) \implies \overleftarrow{\alpha}(s^* f) = s^*(\rho(\alpha)(f))$$

Also:

$$\overleftarrow{f\alpha} = s^*(f)\,\overleftarrow{\alpha}$$

Then:

$$\overleftarrow{[\alpha_1, f\alpha_2]} = [\overleftarrow{\alpha}_1, \overleftarrow{f\alpha_2}] = [\overleftarrow{\alpha}_1, s^*(f)\overleftarrow{\alpha}_2] =$$

(usual Leibniz for v.f.)

$$= s^*(f)\,[\overleftarrow{\alpha}_1, \overleftarrow{\alpha}_2] + \overleftarrow{\alpha}_1(s^*(f))\,\overleftarrow{\alpha}_2$$

$$= \overleftarrow{f[\alpha_1, \alpha_2]} + \overleftarrow{\rho(\alpha_1)(f)\,\alpha_2} = \overleftarrow{f[\alpha_1, \alpha_2] + \rho(\alpha_1)(f)\,\alpha_2}$$

∎