MATH 595 – Lecture 4

Groupoids and Foliation

Can associate Lie groupoids to foliations. These give important examples and play an important role in general theory.

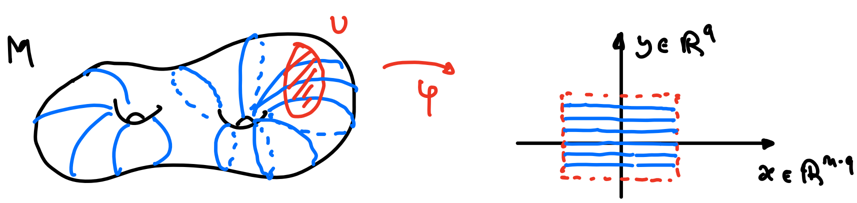
$(M, \mathcal{F})$ ≡ foliated manifold: this means that

(i) $\mathcal{F} = \{ L_\alpha : \alpha \in A \}$ partition of M into (regularly) immersed, connected, submanifolds $L_\alpha \hookrightarrow M$

(ii) ∃ foliated charts: every $p \in M$ belongs to a chart (U, φ)

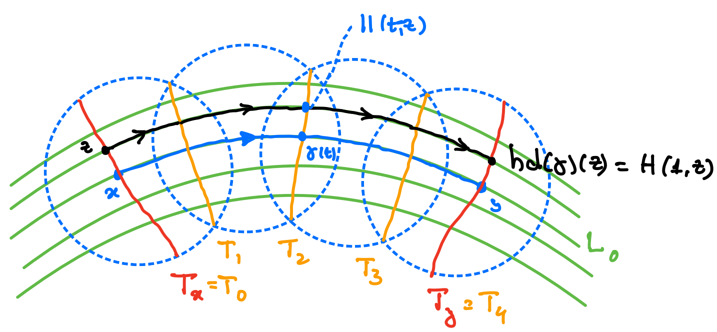
$\varphi : U \longrightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$

$\varphi(\text{connected components of } L_\alpha \cap U) = \{ (x, y) : y^1 = c_1, \dots, y^q = c_q \}$



$q \equiv \operatorname{codim} \mathcal{F}$

Holonomy of leafwise path $\gamma : [0,1] \to L_0$

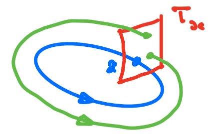


$$H : [0,1] \times T_x \longrightarrow M \qquad \begin{cases} t \longmapsto H(t,z) & \text{path in leaf} \\ H(0,z) = z \\ H(1,z) \in T_y \end{cases}$$

$$\mathrm{hol}^{T_y, T_x}(\gamma) := \mathrm{germ}_x \left(z \longmapsto H(1,z) \right)$$

Rmk. Holonomy measures the behaviour of nearby leaves wether they are expanding / contracting, etc.

For example, orbits of non-vanishing vector field $X \in \mathfrak{X}(M)$ form a foliation. For a periodic orbit: holonomy $\equiv$ Poincaré Return map



Properties:

- $\mathrm{hol}^{T_y,T_x}(\gamma)$ does not depend on choices of foliated charts
- If γ_1 and γ_2 are leafwise homotopic relative to end points
$$\mathrm{hol}^{T_y,T_x}(\gamma_1) = \mathrm{hol}^{T_y,T_x}(\gamma_2)$$
- If γ and η can be concatenated then
$$\mathrm{hol}^{T_z,T_x}(\eta\circ\gamma) = \mathrm{hol}^{T_z,T_y}(\eta)\circ\mathrm{hol}^{T_y,T_x}(\gamma)$$
- If T_x & S_x, T_y & S_y are transversals then:
$$\mathrm{hol}^{S_y,S_x}(\gamma) = \mathrm{hol}^{S_y,T_y}(\bar{y})\circ\mathrm{hol}^{T_y,T_x}(\gamma)\circ\mathrm{hol}(\bar{x})^{T_x,S_x}$$
- Two leafwise paths γ_1 & γ_2, with same end points, have <u>same holonomy</u> if $\mathrm{hol}^{T_y,T_x}(\gamma_1) = \mathrm{hol}^{T_y,T_x}(\gamma_2)$. This does not depend on choice of transversals. Denote by $[\gamma]_h$ the holonomy equivalence class of γ.

- For loops based at x, one obtains the <u>holonomy group</u> of the leaf L based at $x\in L$:
$$\mathrm{Hol}(L,x) := \{[\gamma]_h \mid \gamma:[0,1]\to L,\ \gamma(0)=\gamma(1)=x\}$$
It is a quotient of the fundamental group:
$$\pi_1(L,x)\longrightarrow \mathrm{Hol}(L,x),\quad [\gamma]\mapsto[\gamma]_h$$

Homotopy Groupoid of $(M, \mathcal{F})$

$$\Pi_1(M,\mathcal{F}) = \{ [\gamma] \mid \gamma : [0,1] \to L_\alpha \text{ continuous path} \}$$

$$\downarrow\downarrow$$
$$M$$

$[\;]$ = homotopy class in leaf, relative to end points

$[\gamma]\cdot[\eta] := [\gamma\circ\eta]$ concatenation of paths

- When $\operatorname{codim}\mathcal{F} = 0$ (so $\mathcal{F}$ = connected components of M) recover $\Pi_1(M)$.
- Orbit of x = Leaf L containing x
- Isotropy Group of $x \equiv \pi_1(L, x)$

Holonomy Groupoid of $(M, \mathcal{F})$

$$Hol(M,\mathcal{F}) = \{ [\gamma]_h \mid \gamma : [0,1] \to L_\alpha \text{ continuous path} \}$$

$$\downarrow\downarrow$$
$$M$$

$[\;]_h$ = holonomy class

- Orbits = Leaves
- Isotropy Group of $x \equiv Hol(L, x)$

We have a surjective groupoid homomorphism:

$$\Phi \quad \Pi_1(M,\mathcal{F}) \to Hol(M,\mathcal{F}), \quad [\gamma] \mapsto [\gamma]_h$$

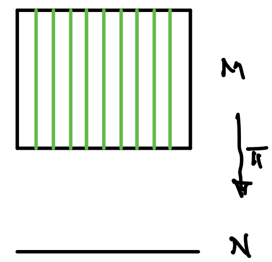
Theorem

$\Pi_1(M,\mathcal{F})$ and $Hol(M,\mathcal{F})$ have natural Lie groupoid structures and Φ is a Lie groupoid homomorphism which is a local diffeo.

Proof: see next class

Examples

1. **Simple Foliations.** $\pi : M \to N$ submersion w/ connected fibers



$\mathcal{F} = \{ \pi^{-1}(n) : n \in N \}$ = foliation by fibers

- Leaves have trivial holonomy, so:

$\mathrm{Hol}(M,\mathcal{F}) = M \times_N M \rightrightarrows M$ (submersion groupoid)

- If fibers are 1-connected then $\Pi_1(M,\mathcal{F}) = \mathrm{Hol}(M,\mathcal{F})$.

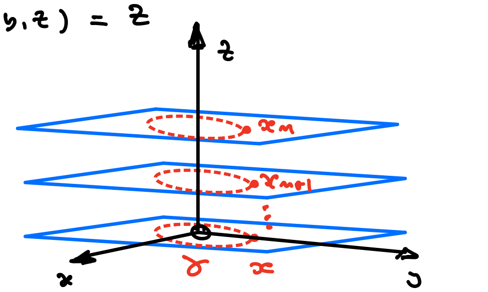
In general, $\Pi_1(M,\mathcal{F})$ will have source fibers which are the universal covering spaces of the fibers of π.

Rmk. In general, for any $(M,\mathcal{F})$, if L is leaf containing x:

- Homotopy groupoid: $s^{-1}(x) \circlearrowleft \pi_1(L,x)$, $t \downarrow L$ = universal covering space of L

- Holonomy groupoid: $s^{-1}(x) \circlearrowleft \mathrm{Hol}(L,x)$, $\downarrow L$ = covering space corresponding to $\mathrm{Ker}(\pi_1(L,x) \to \mathrm{Hol}(L,x))$

Let $M = \mathbb{R}^3 \setminus \{0\} \xrightarrow{\pi} \mathbb{R}$, $\pi(x,y,z) = z$



$\mathrm{Hol}(M,\mathcal{F})$ = submersion Grpd

$\Pi_1(M,\mathcal{F})$ is not Hausdorff!

Let $x_n \to x$, where

$\begin{cases} x_n \in \text{contractible leaves} \\ x \in \text{non-contractible leaf} \end{cases} \Rightarrow 1_{x_n} \longrightarrow \begin{cases} 1_x \\ [\gamma] \end{cases}$, γ any non-contractible loop based at x

Exercise. For a Lie groupoid $\mathcal{G} \rightrightarrows M$ the following are equivalent:

(i) $\mathcal{G}$ is Hausdorff

(ii) $u(M)$ is closed in $\mathcal{G}$

(iii) For each $x \in M$, any $g \in \mathcal{G}_x$ can be separated from 1_x.

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2. Linear Foliations.

- $$\begin{array}{c} \tilde{L} \circlearrowleft \Gamma \\ \downarrow \\ L \end{array}$$ covering space w/ group of deck transformations Γ

- $\Gamma \to GL(V)$ representation (linear action on vector space)

$\Rightarrow$ $M = (\tilde{L} \times V)/_{\Gamma}$ is foliated by

$\mathcal{F}^{lin}$ = linear foliation = projections of $\tilde{L} \times \{v\}$

Note: $L = \mathrm{proj}(\tilde{L} \times \{0\})$ is leaf of $\mathcal{F}^{lin}$

We also obtain a Lie groupoid:

- $$\begin{array}{c} \tilde{L} \times \tilde{L} \times V \\ \downdownarrows \\ \tilde{L} \times V \end{array}$$ = direct product of pair groupoid w/ identity groupoid

$$\Rightarrow \quad \begin{array}{c} \mathcal{G} = (\tilde{L} \times \tilde{L} \times V)/_{\Gamma} \\ \downdownarrows \\ M = (\tilde{L} \times V)/_{\Gamma} \end{array}$$

Orbits = leaves of $\mathcal{F}^{lin}$

Isotropy group of $L \cong \Gamma$

In general, this is neither the homotopy, nor holonomy groupoid of $\mathcal{F}^{lin}$.

Exercise: Show that:

(i) $\mathcal{G} = \Pi_1(M, \mathcal{F}^{lin})$ iff $\tilde{L}$ is 1-connected

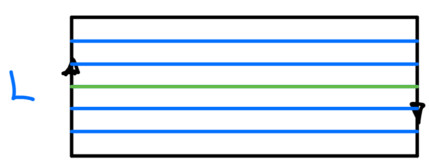
(ii) $\mathcal{G} = Hol(M, \mathcal{F}^{lin})$ iff $\Gamma \to GL(V)$ is faithfull rep.

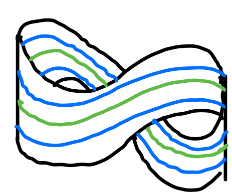
Note: Reeb Stability Thm states that if L is a compact leaf of $(M, \mathcal{F})$ w/ finite holonomy group $\Gamma = Hol(L, x)$, then a saturated neighborhood of L is isomorphic to such a linear foliation (take $\Gamma \curvearrowright \nu_x(L)$ obtained by linearizing holonomy).

For explicit example:

$$\begin{array}{c} \tilde{L} = \mathbb{R} \circlearrowleft \mathbb{Z} \\ \downarrow \\ L = S^1 \end{array} \quad , \quad \mathbb{Z} \to GL(\mathbb{R}), \quad m \cdot y = (-1)^m y$$

$\Rightarrow M = (\mathbb{R} \times \mathbb{R})/_{\mathbb{Z}} \cong$ Möbius Band foliated by "horizontal" circles





$\Rightarrow \tilde{L}$ is 1-connected $\Rightarrow \Pi_1(Mb, \mathcal{F}) = (\mathbb{R} \times \mathbb{R} \times \mathbb{R})/_{\mathbb{Z}} \rightrightarrows (\mathbb{R} \times \mathbb{R})/_{\mathbb{Z}}$

Note that: $\pi_1(L, x) = \mathbb{Z}$, $Hol(L, x) = \mathbb{Z}_2$

To obtain holonomy groupoid:

$$\begin{array}{c} \tilde{\tilde{L}} = S^1 \circlearrowleft \mathbb{Z}_2 \\ \downarrow \\ L = S^1 \end{array} \quad , \quad \mathbb{Z}_2 \to GL(\mathbb{R}), \quad (\pm 1) y := \pm y$$

$\Rightarrow Mb = (S^1 \times \mathbb{R})/_{\mathbb{Z}_2}$ $\quad Hol(Mb, \mathcal{F}) = (S^1 \times S^1 \times \mathbb{R})/_{\mathbb{Z}} \rightrightarrows (S^1 \times \mathbb{R})/_{\mathbb{Z}_2}$

<u>Rmk.</u> Given a foliation $(M, \mathcal{F})$ we have two canonical Lie groupoids associated w/ it. We will see that any "integration" fits into a diagram of surjective étale groupoid morphisms

$$\begin{tikzcd} \Pi_1(M,\mathcal{F}) \arrow[r] \arrow[rr, bend left, dashed] & \mathcal{G} \arrow[r] & \mathrm{Hol}(M,\mathcal{F}) \end{tikzcd}$$

So $\Pi_1(M,\mathcal{F})$ and $\mathrm{Hol}(M,\mathcal{F})$ are largest and smallest integrations of $(M,\mathcal{F})$.