# MATH 595 - Lecture 3

## I.1. Definition & Examples (cont.)

__Last Time:__ After choosing base point:

Transitive Lie groupoids $\simeq$ Principal bundles

For __any__ Lie Groupoid $\mathcal{G} \rightrightarrows M$ we have isotropy bundle

$$\mathrm{Iso}(\mathcal{G}) = \coprod_{x \in M} \mathcal{G}_x$$

But this is not a (smooth) bundle of groups.

__Proposition__

If $\mathcal{G} = (P \times P)/K$ is the gauge groupoid of $\overset{\circlearrowleft K}{P} \to M$, then $\mathrm{Iso}(\mathcal{G}) \simeq$ associated bundle for $K \circlearrowleft K$ by conjugation. In particular, for any transitive groupoid $\mathcal{G} \rightrightarrows M$:

(i) Isotropy groups are all isomorphic

(ii) $\mathrm{Iso}(\mathcal{G}) \to M$ is a Lie groupoid

__Proof.__ If $\mathcal{G}$ is the gauge groupoid:

$$\mathcal{G} = (P \times P)/K \rightrightarrows M \qquad \mathrm{Iso}(\mathcal{G})_x = \{ [p_1, p_2] : \pi(p_1) = \pi(p_2) = x \}$$
$$= \{ [p, pg] : \pi(p) = x,\ g \in K \}$$

$$\begin{array}{ccc} P \times K & \overset{\Phi}{\hookrightarrow} & P \times P \\ \downarrow & & \downarrow \\ (P \times K)/K & \hookrightarrow & (P \times P)/K \end{array} \qquad (p, g) \overset{\Phi}{\longmapsto} (p, pg)$$

(here $K \circlearrowleft K$ by inner action $g \cdot k := k^{-1} g k$)

$\Phi$ is $K$-equivariant:

$$\Phi((p,g)\cdot k) = \Phi(pk, k^{-1}gk) = (pk, pkk^{-1}gk) = $$
$$= (pk, pgk) = \Phi(p,g)\cdot k$$

$$\Rightarrow \quad \tilde{\Phi} : (P\times K)/_K \longrightarrow (P\times P)/_K$$

This is an embedding on image $\mathrm{Iso}(\mathcal{G})$.

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10) <u>General Linear Groupoid.</u> $\pi : E \to M$ vector bundle

$$\begin{array}{l} GL(E) = \{ (y, A, x) \mid A : E_x \to E_y \text{ linear isomorphism} \} \\ \quad \downarrow\downarrow \\ \quad M \end{array}$$

- If $M = \{*\} \Rightarrow E = V$ is a vector space $\Rightarrow$ Lie group $GL(V)$
- $GL(E) \rightrightarrows M$ is transitive groupoid

<u>Exercise.</u> For a vector bundle $E \to M$ one has the bundle of frames ($r = \operatorname{rank} E$):

$$Fr(E) := \{ u : \mathbb{R}^r \to E_x \mid x \in M,\ u \text{ linear isomorphism} \}$$

This is a principal $GL_r(\mathbb{R})$-bundle. Show that $GL(E) \rightrightarrows M$ is canonically isomorphic to the gauge groupoid of $Fr(E) \to M$ with $GL_r(\mathbb{R})$ acting ($\circlearrowleft GL_r(\mathbb{R})$).

11) Restrictions. If $\mathcal{G} \rightrightarrows M$ is Lie and $N \subset M$ is submanifold

$$\begin{array}{c} \mathcal{G}|_N = s^{-1}(N) \cap t^{-1}(N) \\ \downarrow\downarrow \\ N \end{array}$$

is not Lie in general

Need conditions on $N$, e.g., $(t,s) : \mathcal{G} \to M \times N \pitchfork N \times N$. But other conditions work, e.g., $N$ is union of orbits of $\mathcal{G}$ (we say $N$ is "saturated"). In particular, for any orbit

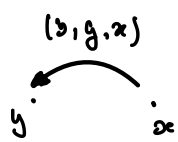
$$\mathcal{G}|_{\mathcal{O}} \rightrightarrows \mathcal{O} \quad \text{is a (transitive) Lie groupoid}$$

Conclusion:

A Lie groupoid can be thought of a collection of transitive Lie groupoids ($\simeq$ principal bundles) that are glued nicely.

12) Pullbacks. Restriction is special case of pullback under a map $\varphi : N \hookrightarrow M$:

$$\begin{array}{c} \varphi^!\mathcal{G} := N \times_M \mathcal{G} \times_M N = \{ (y,g,x) : \varphi(y) \xleftarrow{g} \varphi(x) \} \\ \downarrow\downarrow \\ N \end{array}$$




Not Lie in general

$\varphi^!\mathcal{G} \rightrightarrows N$ is Lie groupoid whenever $\varphi^!\mathcal{G} \subset N \times \mathcal{G} \times N$ is a submanifold

One has a morphism of Lie groupoids:

$$\begin{array}{ccc} \varphi^!\mathcal{G} & \xrightarrow{\Phi} & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ N & \xrightarrow[\varphi]{} & M \end{array} \qquad (y,g,x) \longmapsto g$$

Exercise: Show that $\varphi^!\mathcal{G}$ is a Lie groupoid whenever $\varphi$ is a submersion.

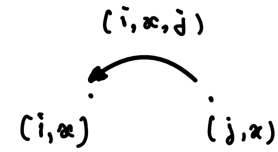
13) <u>Čech Groupoid.</u> $\mathcal{U} = \{U_i : i \in I\}$ open cover of $M$

$$N := \coprod_{i \in I} U_i \quad \text{(disjoint union)} \xhookrightarrow{\varphi} M$$

$\mathcal{G} = (M \rightrightarrows M)$ identity groupoid (one arrow for each object)

$\Rightarrow \mathcal{G}_{\mathcal{U}} := \varphi^! \mathcal{G} \rightrightarrows N$ Čech groupoid

$$\mathcal{G}_{\mathcal{U}} = \coprod_{i,j} U_i \cap U_j = \{(i,x,j) : x \in U_i \cap U_j\}$$
$$\downarrow\downarrow$$
$$\coprod_i U_i = \{(i,x) : x \in U_i\}$$




<u>Rmk:</u> If cover is not countable this violates our conventions

14) <u>Tangent Groupoid.</u> For any Lie groupoid $\mathcal{G} \rightrightarrows M$ apply tangent functor:

$$T\mathcal{G} \underset{ds}{\overset{dt}{\rightrightarrows}} TM$$

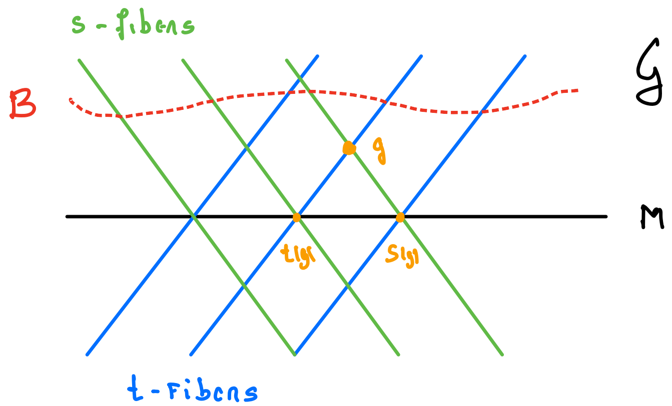
$$dm : T(\mathcal{G} {}_s\!\times_t \mathcal{G}) \longrightarrow T\mathcal{G}$$
$$T(\mathcal{G} {}_s\!\times_t \mathcal{G}) \cong T\mathcal{G} {}_{ds}\!\times_{dt} T\mathcal{G}$$

$$di : T\mathcal{G} \to T\mathcal{G} \qquad du : TM \to T\mathcal{G}$$

<u>Rmk:</u> There is also a $T^*\mathcal{G}$ groupoid as well as direct sums $\bigoplus_k T\mathcal{G}$ and $\bigoplus_k T^*\mathcal{G}$ (later in course). These will be relevant to understand geometric structures on groupoids

## Groupoids vs. Groups:

**Def:** A *bisection* of $\mathcal{G} \rightrightarrows M$ is a submanifold $B \subset \mathcal{G}$ such that $s|_B : B \to M$ & $t|_B : B \to M$ are diffeomorphisms




Equivalently, a bisection is a map $b: M \to \mathcal{G}$ such that $s \circ b = id_M$ and $t \circ b : M \to M$ is a diffeomorphism.

Bisections can be multiplied:

$$b_1 \circ b_2 (x) := b_1(t \circ b_2(x)) \cdot b_2(x)$$

This makes the space $\mathcal{B}(\mathcal{G})$ of bisections into a "Lie group". But this is $\infty$-dim and can be very wild.

**Rmk.** One can also define *local bisection*:

- A submanifold $B \subset \mathcal{G}$ such that $s|_B : B \to U$ and $t|_B : B \to V$ are diffeos onto open sets $U, V \subset M$

$\iff$ • a map $b: U \to \mathcal{G}$ such that $s \circ b = id_U$ and $t \circ b : U \to V$ is a diffeo.

**Proposition:**

Every $g \in \mathcal{G}$ belongs to the image of some local bisection.

**Proof.**

Choose a subspace $L \subset T_g\mathcal{G}$ complementary to both $\ker d_g s$ and $\ker d_g t$. Choose submanifolds $g \in B \subset \mathcal{G}$ with $T_g B = L$. If $B$ is small enough it is a bisection. ☒

**Proposition**

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

(i) $s^{-1}(x) \cap t^{-1}(y)$ are closed embedded submanifolds of $\mathcal{G}$

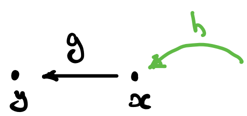
(ii) The isotropy groups $G_x$ are Lie groups

(iii) $t : s^{-1}(x) \longrightarrow \mathcal{O}_x$ is a principal $\mathcal{G}_x$-bundle

(iv) The orbits $\mathcal{O}_x$ are immersed submanifolds in $M$

To prepare for proof and for future use:

**Left translations:**

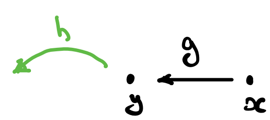



$$L_g: \quad h \longmapsto gh, \qquad h \in t^{-1}(x), \quad gh \in t^{-1}(y)$$

- $L_g : t^{-1}(s(g)) \longrightarrow t^{-1}(t(g))$

is a diffeomorphism with inverse $L_{g^{-1}}$

**Right translations:**




$$R_g: \quad h \longmapsto hg, \qquad h \in s^{-1}(y), \quad hg \in s^{-1}(x)$$

- $R_g : s^{-1}(t(g)) \longrightarrow s^{-1}(s(g))$

is a diffeomorphism with inverse $R_{g^{-1}}$

$$T_g t^{-1}(x) = \ker d_g t \quad , \quad T_g s^{-1}(y) = \ker d_g s \; ,$$

Proof of Proposition:

Fix s-Fiber $s^{-1}(x)$:

Claim: $D_g := \operatorname{Ker} d_g s \cap \operatorname{Ker} d_g t$ is a (constant rank) Distribution on $s^{-1}(x)$

Indeed, we have for any $g \in s^{-1}(x)$

- $d_{1_x} L_g : \operatorname{Ker} d_{1_x} t \to \operatorname{Ker} d_g t$ is isomorphism

- $s \circ L_g = s$ in $t^{-1}(x) \Rightarrow d_1 L_g (D_{1_x}) = D_g$

Hence, picking a basis $\{v_1, \dots, v_K\}$ for $D_{1_x}$ we obtain a basis of vector Fields $\{X_1, \dots, X_K\}$ on $s^{-1}(x)$ spanning $D$:

- $X_i : g \mapsto d_{1_x} L_g (v_i)$

This proves the claim.

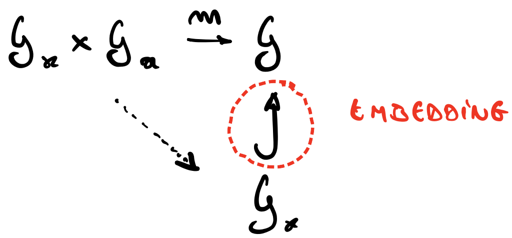
D is involutive distribution in $s^{-1}(x)$: it coincides with kernel of the differential of smooth map $t : s^{-1}(x) \to M$.

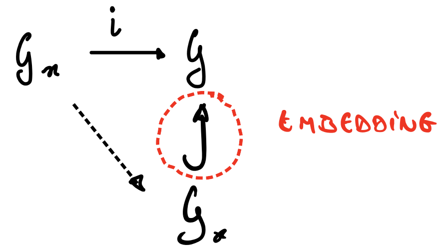
Frobenius Theorem $\Rightarrow$ $s^{-1}(x) \cap t^{-1}(y)$ are submanifolds (= connected components are leaves of D)

Note: Since source/target Fibers are Hausdorff, 2nd countable we can apply Frobenius. Also, $s^{-1}(x) \cap t^{-1}(y)$ are closed in $s^{-1}(x)$ $\Rightarrow$ leaves are closed, embedded $\Rightarrow$ $s^{-1}(x) \cap t^{-1}(x)$ are closed embedded submanifolds. $\Rightarrow$ (i) holds

(ii) $\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x) \subset \mathcal{G}$ are closed, embedded and are Groups

Need to check multiplication/inversion are smooth.




$\Rightarrow$ $m: G_x \times G_x \to G_x$ , $i: G_x \to G_x$ are smooth

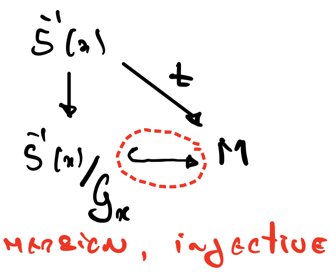
(iii) $s^{-1}(x) \times G_x \to s^{-1}(x)$ , $(g,h) \to gh$ (right action)

- Free action
- Orbits are fibers of $t: s^{-1}(x) \to M$ $\Rightarrow$ action is proper

$\Rightarrow$ principal right action $\Rightarrow$ $\Big\{$ $t: s^{-1}(x) \to s^{-1}(x)/G_x = O_x$ principal $G_x$-bundle

(iv) $O_x \cong s^{-1}(x)/G_x$ (bijection)

$\Rightarrow$ $O_x$ has smooth structure

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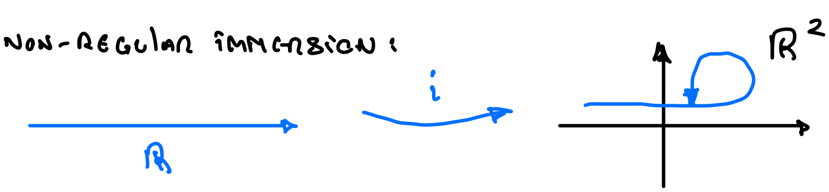
<u>Rmk:</u> An immersion $i: N \to M$ is called <u>regular</u> if for any map $f: P \to N$ the composition $f \circ i$ is smooth iff $f$ is smooth.

- Every embedding is a regular immersion
- The irrational line in torus

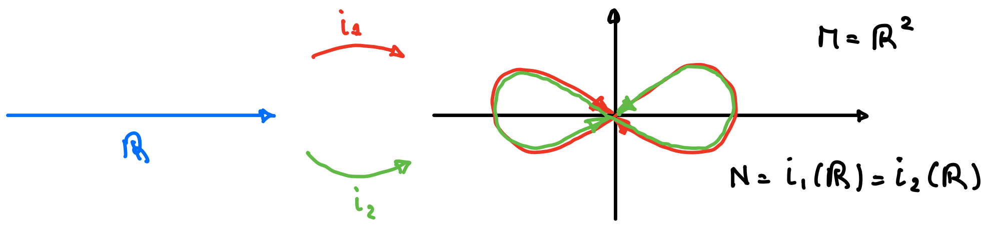
$$\mathbb{R} \to S^1 \times S^1, \quad t \mapsto (e^{it}, e^{i\lambda t}) \qquad (\lambda \notin \mathbb{Q})$$

is a regular immersion which is not an embedding

- A non-regular immersion:



: In general, a subset $N \subset M$ can have different smooth structures such that inclusion $N \hookrightarrow M$ is immersion.




However, a set $N \subset M$ can have at most one smooth str. such that $N \hookrightarrow M$ is a regular immersion, and that smooth structure is the unique one make the inclusion an immersion.

Alternative notations

Regular immersed = weakly embedded = initial submanifold

<u>Exercise (somewhat hard)</u>

The orbits of a Lie groupoid are regularly immersed

<u>Hint</u>: Look at the proof that the leaves of a foliation are regular immersed submanifolds (see, e.g., Warner "Foundations of Differentiable Manifolds and Lie Groups")