MATH 595 - Lecture 29

Differentiable Stacks

What are the stacks that can be represented by Lie groupoids? In other words, what are the stacks equivalent to $B\mathcal{G}$ for for some Lie groupoid $\mathcal{G} \rightrightarrows M$?

Def.: A stack $\pi: \mathcal{C} \to \mathfrak{M}$ is called <u>representable</u> if it is equivalent to $\underline{M}$ for some $M \in \mathfrak{M}$

We also need the notion of representable map between stacks. For that we use fiber products of fibered categories:

(*)
$$\begin{tikzcd}
\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathcal{C}_2 \arrow[d, "\Phi_2"] \\
\mathcal{C}_1 \arrow[r, "\Phi_2"'] & \mathcal{D}
\end{tikzcd}$$

It is defined similarly to fiber product of groupoids:

$$\mathrm{Obj}(\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2) = \left\{ (c_1, \phi, c_2) : \begin{array}{l} \pi_1(c_1) = \pi_2(c_2) = X \quad \& \\ \Phi_1(c_1) \xleftarrow{\phi} \Phi_2(c_2) \in \mathcal{D}_X \end{array} \right\}$$

$$\begin{tikzcd} (c_1, \phi, c_2) \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$$\mathrm{Arr}(\mathcal{C}_1 \times_{\mathfrak{M}} \mathcal{C}_2) = \left\{ \begin{array}{c} (c_1', \phi', c_2') \xleftarrow{(\alpha, \beta)} (c_1, \phi, c_2) \\ \Downarrow \pi \\ X' \xleftarrow{f} X \end{array} : \begin{array}{c} \pi_1(\alpha) = f = \pi_2(\beta) \\ \begin{tikzcd} \Phi_1(c_1) \arrow[r, "\phi"] \arrow[d, "\Phi_1(\alpha)"'] & \Phi_2(c_2) \arrow[d, "\Phi_2(\beta)"] \\ \Phi_1(c_1') \arrow[r, "\phi'"'] & \Phi_2(c_2') \end{tikzcd} \end{array} \right\}$$

obvious functor $\pi: \mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \to \mathfrak{M} \implies$ a fibered category.

Rmks

1) Fibers are the weak fiber products of the groupoid fibers
2) (*) is only 2-commutative.
3) Fiber products of prestacks (resp. stacks) are prestacks (resp. stacks).
4) If M, N, Q are manifolds w/ smooth maps $f: M \to Q$, $g: N \to Q$ then:

$$\underline{M} \times_{\underline{Q}} \underline{N} = \underline{M \times_Q N}$$

Def: A map of stacks $\Phi: \mathcal{C} \to \mathcal{D}$ is called representable if for any manifold M and any map $\underline{M} \to \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \underline{M}$ is representable.

In this case, we can represent Φ by "honest" maps:

$$\begin{tikzcd}
\underline{N} \simeq \mathcal{C} \times_{\mathcal{D}} \underline{M} \arrow[r] \arrow[d] \arrow[d, dashed, "\underline{\phi}"'] & \mathcal{C} \arrow[d, "\Phi"] \\
\underline{M} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

Pullbacks of manifolds only exist under strong assumptions, like transversality. In fact, one finds:

Lemma: For a smooth map $\phi: M \to N$ the map of stacks $\underline{\phi}: \underline{M} \to \underline{N}$ is representable iff ϕ is a submersion.

Proof: Exercise. ∎

<u>Corollary</u>: If $\underline{\Phi} : \mathcal{C} \to \mathcal{D}$ is a representable map of stacks then for any manifold M and map $p : \underline{M} \to \mathcal{D}$ the resulting map $p^*\underline{\Phi}$:

$$\begin{tikzcd} \underline{M} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d, "p^*\underline{\Phi}"'] & \mathcal{C} \arrow[d, "\underline{\Phi}"] \\ \underline{M} \arrow[r, "p"'] & \mathcal{D} \end{tikzcd}$$

is a submersion.

This is also useful to define properties of representable maps of stacks.

$$P = \begin{cases} \text{property of smooth maps stable under pullback by} \\ \quad \text{submersions} \end{cases}$$

<u>Examples</u>: injective, surjective, immersion, submersion, embedding, open/closed embedding, étale

<u>Def</u>: A representable map $\underline{\Phi} : \mathcal{C} \to \mathcal{D}$ of stacks <u>has property P</u> if for every representable map $p : \underline{M} \to \mathcal{D}$ the pullback $p^*\underline{\Phi} : \underline{M} \times_{\mathcal{D}} \mathcal{C} \to \underline{M}$ has property P.

This coincides with property P for smooth maps, since representable maps $\underline{\Phi} : \underline{M} \to \underline{N}$ are submersions.

<u>Def</u>: A stack $\pi : \mathcal{G} \to \mathcal{M}$ is <u>locally representable</u> if there is $M \in \mathcal{M}$ and a representable epimorphism $q : \underline{M} \to \mathcal{G}$. One calls $q : \underline{M} \to \mathcal{G}$ a <u>presentation</u> or an <u>atlas</u> for $\mathcal{G}$,

A stack $\pi : \mathcal{G} \to \mathcal{M}$ which admits a presentation is called a <u>differentiable stack</u>. The reason for this name is that they are precisely the stacks equivalent to Lie groupoids:

Thm

For any Lie groupoid $\mathcal{G} \rightrightarrows M$, $B\mathcal{G}$ is a differentiable stack with presentation:

$$q: \underline{M} \to B\mathcal{G} \qquad (f: X \to M) \longmapsto f^*\mathcal{G}$$

Moreover, there is an equivalence:

$$\underline{M} \times_{B\mathcal{G}} \underline{M} \simeq \underline{\mathcal{G}}$$

Proof:

To check that $q: \underline{M} \to B\mathcal{G}$ is an epimorphism, just observe that given $(f: X \to M) \in \underline{M}(X)$ and $(P \to X) \in B\mathcal{G}(X)$, we can find a covering family $\{U_i \to X\}$ where both $P|_{U_i}$ & $f^*\mathcal{G}|_{U_i}$ are trivial principal $\mathcal{G}$-bundles, hence isomorphic.

Consider a map $p: \underline{X} \to B\mathcal{G}$. The map p amounts to:

- $f: U \to X \overset{p}{\Longrightarrow}$ principal $\mathcal{G}$-bundle P_U
- $$\begin{tikzcd} U \arrow[d, "g"'] \arrow[dr] & \\ V \arrow[r] & X \end{tikzcd} \overset{p}{\Longrightarrow} \text{map of principal } \mathcal{G}\text{-bundles} \quad \begin{tikzcd} P_U \arrow[r] \arrow[d] & U \arrow[d, "g"] \\ P_V \arrow[r] & V \end{tikzcd}$$

In particular, consider a covering family $\{U_\alpha \to X\}$ for which P_{U_α} & $P_{U_{\alpha\beta}}$ are trivial principal $\mathcal{G}$-bundles. We obtain a $\mathcal{G}$-cocycle:

- $g_\alpha: U_\alpha \to X$
- $g_{\alpha\beta}: U_{\alpha\beta} \to \mathcal{G}$

So they yield a principal $\mathcal{G}$-bundle $P \to X$. We leave as an exercise to check that:

$$\underline{P} = \underline{X} \times_{B\mathcal{G}} \underline{M}$$

So $q: \underline{M} \to B\mathcal{G}$ is representable.

Finally, to check $\underline{X} \times_{B\mathcal{G}} \underline{X} \simeq \underline{\mathcal{G}}$, just apply the previous argument to $M = X$ and $f = s : \mathcal{G} \to X$. This gives a 2-commutative diagram:

$$\begin{tikzcd} \underline{\mathcal{G}} \arrow[r, "\underline{t}"] \arrow[d, "\underline{s}"'] \arrow[dr, phantom, "\Nearrow"] & \underline{X} \arrow[d, "p"] \\ \underline{X} \arrow[r] & B\mathcal{G} \end{tikzcd}$$

which satisfies the pullback universal property. $\square$

Thm

Let $\pi : \mathcal{C} \to \mathcal{M}$ be a differentiable stack w/ a presentation:

$$q : \underline{M} \to \mathcal{C}$$

Then $\underline{M} \times_{\mathcal{C}} \underline{M}$ is representable by a smooth manifold $\mathcal{G}$ which has a canonical Lie groupoid structure over M. The stack $B\mathcal{G}$ is canonically equivalent to $\mathcal{C}$, and any two groupoids $\mathcal{G}_1$ & $\mathcal{G}_2$ presenting $\mathcal{C}$ are Morita equivalent.

Sketch of Proof:

Since $q : \underline{M} \to \mathcal{C}$ is representable, so is $\underline{M} \times_{\mathcal{C}} \underline{M}$. Denote by $\mathcal{G}$ a manifold representing it. Then:

- source/target maps are the projections $\underline{M} \times_{\mathcal{C}} \underline{M} \rightrightarrows \underline{M}$ (being pullbacks of rep. maps, are submersions)
- unit map is the diagonal $\underline{M} \to \underline{M} \times_{\mathcal{C}} \underline{M}$
- inverse map is the map switching factors $\underline{M} \times_{\mathcal{C}} \underline{M} \to \underline{M} \times_{\mathcal{C}} \underline{M}$
- product map:

$$\mathcal{G} \times_{M} \mathcal{G} \longrightarrow (\underline{M} \times_{\mathcal{C}} \underline{M}) \times_{\underline{M}} (\underline{M} \times_{\mathcal{C}} \underline{M}) \simeq \underline{M} \times_{\mathcal{C}} \underline{M} \times_{\mathcal{C}} \underline{M} \xrightarrow{\pi_{1,3}} \underline{M} \times_{\mathcal{C}} \underline{M} \simeq \mathcal{G}$$

Having the groupoid $\mathcal{G} \rightrightarrows M$, we have the prestack associated with the separated pre-sheaf of grpds:

$$U \longmapsto C^\infty(U, \mathcal{G})$$

Denote this prestack by $\pi : \mathcal{D} \to \mathcal{M}$, so:

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "t"', shift right] \arrow[d, "s", shift left] \\ X \arrow[ur, "f"] \arrow[r] & M \end{tikzcd}$$

$$Obj(\mathcal{D}) = \{ (X, f) : X \in \mathcal{M},\ f \in C^\infty(X, \mathcal{G}) \}$$

$$Arr(\mathcal{D}) = \{ (X, f) \xrightarrow{F} (X', f') : F : X \to X' \ \& \ f = f' \circ F \}$$

We define a map of prestacks $\Phi : \mathcal{D} \to \mathcal{G}$ by:

- $$\underline{X} \xrightarrow{f} \underline{\mathcal{G}} \simeq \underline{M} \times_C \underline{M} \longrightarrow C$$

$$\Phi(X, f) := q(f) \in Hom(\underline{X}, C_X) \simeq (C)_X$$

- $$\left(\begin{tikzcd} \underline{X} \arrow[r] \arrow[dr, "f"'] & \underline{X}' \arrow[d, "f'"] \\ & \underline{M} \times_C \underline{M} \end{tikzcd} \right) \longleftrightarrow \begin{matrix} q(f) \longrightarrow q(f') \\ \Phi(F) = q(F) \end{matrix}$$ (using definition of pullback)

Using that $q : \underline{M} \to \underline{\mathcal{G}}$ is epimorphism, one checks that this map induces an isomorphism on the stackification:

$$\hat{\Phi} : B\mathcal{G} \longrightarrow G$$

$\blacksquare$

<u>Epilogue:</u> This is just the beginning of the story: one can now define geometric / topological structures on a differentiable stack, by defining them on a Lie groupoid as long as they are Morita invariant.

That's all Folks!