MATH 595 - Lecture 28

Last time:

- $\mathfrak{M}$ = category of smooth manifolds

- Fibered category over $\mathfrak{M}$: a functor $\pi: \mathcal{G} \to \mathfrak{M}$ satisfying:

  (i) For every $f: X' \to X$ & C in $\mathcal{G}$ over X, pullback exist:

$$\begin{tikzcd} C' \arrow[d, dashed, "\exists\, g"'] \\ C \end{tikzcd} \quad \overset{\pi}{\Rightarrow} \quad \begin{tikzcd} X' \arrow[d, "f"] \\ X = \pi(C) \end{tikzcd} \qquad C' = f^*C = C|_{X'} \text{ "pullback"}$$

  (ii) Existence of unique lifts:

$$\begin{tikzcd} C_1 \arrow[dd, dashed, "\exists!\, g"'] \arrow[dr, "g_1"] & \\ & C \\ C_2 \arrow[ur, "g_2"'] & \end{tikzcd} \quad \overset{\pi}{\Rightarrow} \quad \begin{tikzcd} X_1 \arrow[dd, "f"'] \arrow[dr, "f_1"] & \\ & X \\ X_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

- Fiber over X: $\mathcal{C}_X = \{ (c, g) : \pi(c) = X, \pi(g) = id_X \}$ (grpd)

Fibered categories generalize presheaves

In one direction, given a presheaf (i.e., a contravariant functor):

$$P: \mathfrak{M} \to \text{Sets}$$

one defines a fibered category:

$$\mathcal{G} := \begin{cases} \text{obj} = \{ (X, x) : X \in \mathfrak{M} \ \& \ x \in P(X) \} \\ \text{Arr} = \{ (X, x) \xrightarrow{f} (Y, y) \mid f: X \to Y, \ P(f)(y) = x \} \end{cases}$$

$\pi: \mathcal{G} \to \mathfrak{M}$:= forgetfull functor

Rmk: Note that this is a discretely fibered category, i.e. fibers $\mathcal{G}_X$ are identity grpds (as pullbacks are unique)

In the other direction, given fibered category $\pi : \mathcal{C} \to \mathfrak{M}$ make a choice of pullbacks:

- For each $C \in \mathcal{C}_X$ & map $f : X' \to X$ choose a lift $\tilde{f} : C' \to C$

With notation $C' = f^*C$, by property (ii), if $C_1, C_2 \in \mathcal{C}_X$ and $g : C_1 \to C_2$ then:

$$\begin{tikzcd} f^*C_1 \arrow[r, "f^*g"] \arrow[d] & f^*C_2 \arrow[d] \\ C_1 \arrow[r, "g"'] & C_2 \end{tikzcd}$$

We obtain a contravariant map

$$P : \mathfrak{M} \to \mathrm{Grpds}$$

where:

- $P(X) := \mathcal{C}_X$ (a groupoid)
- $P(f : X_1 \to X_2) := f^* : \mathcal{C}_{X_2} \to \mathcal{C}_{X_1}$ (a groupoid morphism)

But this is only a pseudo functor: by axiom 2, there is a

$$(f_1 \circ f_2)^* \underset{\tau_{f_1 f_2}}{\cong} f_2^* \circ f_1^*$$

where $\tau_{f_1 f_2}$ is a unique natural transformation satisfying some coherence conditions. One calls P a lax presheaf of grpds. One finds that:

- Different choices of pullbacks $\Rightarrow$ equivalent lax presheaves of grpds
- Lax presheaf of groupoids $\Rightarrow$ fibered category (similar to above)
- Equivalent lax presheaves of grpds $\Rightarrow$ equivalent fibered categories (we define equivalence later)

$$\boxed{\text{Lax presheaf of groupoids} \iff \text{Fibered category + choice of pullbacks}}$$

<u>Rmk</u>: A choice of pullbacks for $\pi : \mathcal{C} \to \mathfrak{M}$ is also called a cleavage.

Examples

1) The fibered category $\underline{M}$ has unique cleavage. The associated presheaf is the $\underline{\text{strict}}$ presheaf:

$$U \longmapsto C^\infty(U,M) \qquad (\text{identity groupoid})$$

More generally:

strict presheaves of groupoids $\longleftrightarrow$ Discretely fibered categories
(i.e., unique pullbacks)

2) $\pi: BG \to \mathcal{M}$ has a natural cleavage: pullbacks of principal G-bundles. The associated lax presheaf of groupoids is

- $X \longmapsto \{\text{Principal } G\text{-bundles over } M\}$
- $(f: X_1 \to X_2) \longmapsto f^* =$ Pullback of principal G-bundles

This is not a strict presheaf:

$$(f_1 \circ f_2)^* P \neq f_2^*(f_1^* P)$$

3) For any Lie grpd $G \rightrightarrows X$ we can define a strict presheaf of grpds:

$$U \longmapsto \text{Grpd w/} \begin{cases} \text{Objs}: & C^\infty(U,X) \\ \text{Arr}: & C^\infty(U,G) \end{cases}$$

$$(f: U \to V) \longmapsto \begin{array}{ccc} C^\infty(V,G) & \xrightarrow{f^*} & C^\infty(U,G) \\ \downarrow\downarrow & & \downarrow\downarrow \\ C^\infty(V,M) & \xrightarrow[f^*]{} & C^\infty(U,G) \end{array}$$

By correspondence above this defines a discrete fibered category. When G is the identity groupoid $X \rightrightarrows X$ we recover $\underline{X}$.

Note: In general, this fibered category is $\neq BG$

Prestacks & Stacks: Descent

So far we have not use covering families, i.e, the Grothendieck topology. These allows us to introduce "gluing axioms" similar to sheaves, and define stacks. Recall:

- A <u>covering family</u> of X is $\{U_i \xrightarrow{f_i} X\}$ w/ f_i étale and $\bigcup_i f(U_i) = X$.

Consider a presheaf over $\mathfrak{M}$, i.e., a contravariant functor

$$P : \mathfrak{M} \to \mathrm{Sets}$$

Given $f : U \to V$ & $x \in P(V)$, we use the usual notations;

$$x|_U := P(f)(x)$$

Also, for a covering family $\{f_i : U_i \to X\}$ we set:

$$U_{ij} := U_i \times_X U_j \;,\quad U_{ijn} := U_i \times_X U_j \times_X U_n \;,\quad \text{etc.}$$

Note that these are all good pullbacks. Recall:

<u>Def</u>:

(i) A presheaf P over $\mathfrak{M}$ is <u>separated</u> if

$$\left.\begin{array}{l} \{f_i : U_i \to X\} \text{ covering family} \\ x, x' \in P(X),\ \forall i : x|_{U_i} = x'|_{U_i} \end{array}\right\} \Rightarrow x = x'$$

(ii) A presheaf F over $\mathfrak{M}$ is a <u>sheaf</u> if

$$\left.\begin{array}{l} \{f_i : U_i \to X\} \text{ covering family} \\ x_i \in F(U_i),\ x_i|_{U_{ij}} = x_j|_{U_{ij}} \end{array}\right\} \Rightarrow \exists\, x \in F(X) \text{ w/ } x|_{U_i} = x_i$$

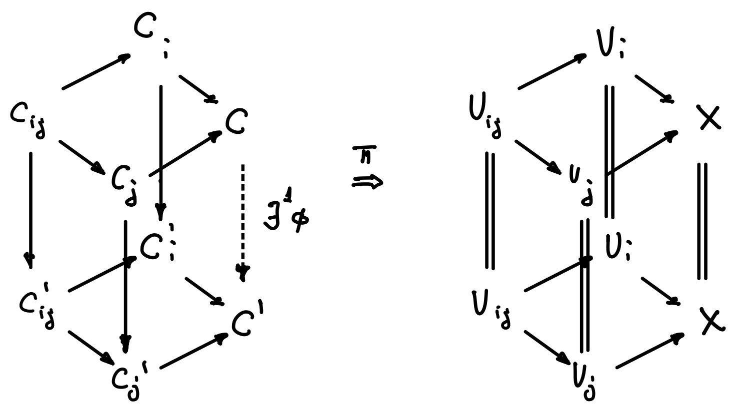
Since fibered categories can be thought as lax presheaves one can impose similar properties. These lead to prestacks & stacks.

First we define the analogue of separated:

<u>Def</u>: A fiber category $\pi: \mathcal{G} \to \mathfrak{M}$ is called a <u>prestack</u> if for some cleavage, for any $C, C' \in \mathcal{G}_X$, any covering family $\{f_i : U_i \to X\}$ and isomorphisms $\phi_i : C|_{U_i} \to C'|_{U_i}$

$$\phi_i = \phi_j \ (\text{on } C|_{U_{ij}}) \Rightarrow \exists^1 \text{ isomorphism } \phi : C \to C'.$$

<u>Rmk</u>: Without a choice of cleavage, the condition is that one can always fill the diagram:



<u>Examples</u>

1) For a Lie group G, $BG \to \mathfrak{M}$ is a prestack; the condition amounts to the gluing axiom for maps of spaces applied to principal bundles.

Similarly, principal G-bundles with connection form a prestack $BG^{\nabla} \to \mathfrak{M}$: principal bundle connections can also be glued.

More generally, for any Lie groupoid $B\mathcal{G}$ & $B\mathcal{G}^{\nabla}$ are prestacks

2) If $\pi: \mathcal{G} \to \mathfrak{M}$ is a discretely fibered category then it is a prestack iff the corresponding presheaf $P: \mathfrak{M} \to \mathrm{Grpds}$ is separated.

In particular, for any manifold, $\underline{M}$ is a prestack and for any Lie groupoid $\mathcal{G}$, $X \mapsto C^{\infty}(X, \mathcal{G})$ gives a prestack.

Def: A prestack $\pi : \mathcal{G} \to \mathcal{M}$ is called a <u>stack</u> if for some choice of cleavage, for any $X \in \mathcal{M}$, any covering family $\{f_i : U_i \to X\}$, any $C_i \in \mathcal{G}_{U_i}$ and $\phi_{ji} : C_i|_{U_{ij}} \to C_j|_{U_{ij}}$, satisfying the cocycle condition

$$\phi_{kj} \circ \phi_{ji} = \phi_{ki} \quad (\text{in } C_{U_{ijk}})$$

there exists $C \in \mathcal{G}_X$ and isomorphisms $\phi_i : C|_{U_i} \to C_i$ such that

$$\phi_{ji} \circ \phi_i = \phi_j \quad (\text{in } C|_{U_{ij}})$$

Rmk: Without a choice of cleavage, the condition says that given data:

$$\begin{tikzcd}[row sep=small] & C_i & & & U_i \\ C_{ij} \arrow[ur] \arrow[dr] & & \arrow[r, phantom, "\overset{\pi}{\Longrightarrow}"] & U_{ij} \arrow[ur] \arrow[dr] & \\ & C_j & & & U_j \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & C_{ij} \arrow[r] \arrow[dr] & C_i & & & U_{ij} \arrow[r] \arrow[dr] & U_i \\ C_{ijk} \arrow[ur] \arrow[r] \arrow[dr] & C_{ik} \arrow[ur] \arrow[dr] & C_j & \arrow[r, phantom, "\overset{\pi}{\Longrightarrow}"] & U_{ijk} \arrow[ur] \arrow[r] \arrow[dr] & U_{ik} \arrow[ur] \arrow[dr] & U_j \\ & C_{jk} \arrow[ur] \arrow[r] & C_k & & & U_{jk} \arrow[ur] \arrow[r] & U_k \end{tikzcd}$$

There is $C \in \mathcal{G}_M$ and arrows $C_i \to C$ filling diagram:

$$\begin{tikzcd}[row sep=small] & C_i \arrow[dr] & & & & U_i \arrow[dr] & \\ C_{ij} \arrow[ur] \arrow[dr] & & C & \arrow[r, phantom, "\overset{\pi}{\Longrightarrow}"] & U_{ij} \arrow[ur] \arrow[dr] & & X \\ & C_j \arrow[ur] & & & & U_j \arrow[ur] & \end{tikzcd}$$

Rmk: The data $\{C_i, \phi_{ij}\}$ satisfying cocycle condition is call **Descent Data**. The condition for a prestack to be a stack is called the **descent condition**.

Examples:

1) A discretely fibered category is a stack iff the corresponding presheaf is a sheaf. In particular, for any manifold $\underline{M}$ is a stack.

The fibered category associated w/ $X \mapsto C^\infty(X,G)$ is not, in general, a stack.

2) For $\pi : BG \to M$ descent data amounts to a collection of principal G-bundles $P_{U_i} \to U_i$ and transition functions giving a G-cocycle. Then there exists a principal G-bundle $P \to M$ and isomorphisms $f_i^* P \cong P_{U_i}$ compatible w/ transition maps. Hence, BG is a stack.

Rmk: Given a prestack there is a stackification. If one stackifyies the fibered category associated w/ $X \mapsto C^\infty(X,G)$ one obtains BG.

Maps of Stacks

Def: Let $\pi_1 : C_1 \to M$ & $\pi_2 : C_2 \to M$ be fibered categories:

i) A map of fibered categories is a functor $\Phi : C_1 \to C_2$ such that $\pi_2 \circ \Phi = \pi_1$

ii) A 2-isomorphism of fibered categories between $\Phi : C_1 \to C_2$ and $\Phi : C_1 \to C_2$ is a natural isomorphism $\tau : \Phi_1 \xrightarrow{\sim} \Phi_2$ w/ $\tau(x) \in C_x$.

iii) An equivalence of fibered categories is a map $\Phi : C_1 \to C_2$ admitting a quasi-inverse $\Psi : C_2 \to C_1$ (so $\Psi \circ \Phi \underset{\tau}{\cong} \mathrm{Id}_{C_1}$, $\Phi \circ \Psi \underset{\delta}{\cong} \mathrm{id}_{C_2}$)

A map, 2-isomorphism, or equivalence of stacks is just a map, 2-isomorphism or equivalence of the underlying fibered categories.

Hence, fibered categories and stacks are both 2-categories. The later is denoted $St(\mathfrak{M})$.

<u>Properties</u>:

(i) If $\Phi: C_1 \to C_2$ is a map of fibered categories then Φ is an equivalence iff for every $X \in \mathfrak{M}$ the restriction to the fibers $\Phi|_X: C_1|_X \to C_2|_X$ is an equivalence of Gpds.

(ii) If $\Phi: C_1 \to C_2$ is an equivalence of fibered categories then C_1 is a prestack (resp. stack) iff C_2 is a prestack (resp. stack).

<u>Examples</u>

1) Given $\phi: M \to N$ we obtain a map of stacks:

$$\underline{\phi}: \underline{M} \to \underline{N} \quad \left\{ \begin{array}{l} (f: X \to M) \mapsto \phi \circ f: X \to N \\ \begin{tikzcd} X_1 \arrow[d, "g"'] \arrow[dr, "f_1"] & \\ X_2 \arrow[r, "f_2"'] & M \end{tikzcd} \longmapsto \begin{tikzcd} X_1 \arrow[d, "g"'] \arrow[dr, "\phi\circ f_1"] & \\ X_2 \arrow[r, "\phi\circ f_2"'] & N \end{tikzcd} \end{array} \right.$$

Every map of stacks $\Phi: \underline{M} \to \underline{N}$ is of this form. Note that

$$\underline{M} \text{ is discretely fibered} \Rightarrow \left\{ \begin{array}{l} \text{No non-trivial 2-isomorphisms} \\ \text{between maps} \end{array} \right.$$

So we have a full embedding of 2-categories:

$$\mathfrak{M} \longrightarrow St(\mathfrak{M}) \quad \left\{ \begin{array}{l} M \mapsto \underline{M} \\ \phi \mapsto \underline{\phi} \end{array} \right.$$

<u>Remark</u>: One can also show that for any stack $\pi: G \to \mathfrak{M}$ there is a canonical equivalence of groupoids:

$$G_X \simeq \mathrm{Hom}(\underline{X}, G)$$

This example and remark formalize the idea that we can understand a (generalized) space by looking at all maps into the space

2) Let $\Phi : \mathcal{G} \to \mathcal{H}$ be a morphism of Lie groupoids. One obtains a map of stacks $\Phi_* : B\mathcal{G} \to B\mathcal{H}$ as follows:

- $$\begin{matrix} & & P \\ & G & \downarrow \\ \mathcal{G} & \rightrightarrows & M \end{matrix} \longmapsto \begin{matrix} & & \Phi_*(P) = (\mathcal{H} \times_N P)/\mathcal{G} \\ & G & \downarrow \\ \mathcal{H} & \rightrightarrows & N \end{matrix}$$

- $(f : P_1 \to P_2) \longmapsto \Phi_*(f)([h,p]) = [h, f(p)]$

Exercise: Show that $\Phi_* : B\mathcal{G} \to B\mathcal{H}$ is an equivalence of stacks iff $\Phi : \mathcal{G} \to \mathcal{H}$ is a Morita map.

Def. Let $\Phi : C_1 \to C_2$ be a map of fibered categories over $\mathfrak{M}$.

(i) Φ is a monomorphism if for every $X \in \mathfrak{M}$ the restriction to the fibers $\Phi : (C_1)_X \to (C_2)_X$ is fully faithful.

(ii) Φ is a epimorphism if for every $X \in \mathfrak{M}$ and $C_2 \in (C_2)_X$ there is a covering family $\{f_i : U_i \to X\}$ and $C_i \in (C_1)_{U_i}$ such that $\Phi(C_i) \simeq C_2|_{U_i}$.

RMK: If $\Phi : (C_1)_X \to (C_2)_X$ is ess. surjective for every $X \in \mathfrak{M}$, then (ii) holds for trivial covering family $\{id : X \to X\}$. So (ii) is weaker than this condition. Hence, an equivalence of fibered categories is both a monomorphism and a epimorphism but not the converse.

Exercise: Let $\phi : M \to N$ be a smooth map. Show that for the map of stacks $\underline{\phi} : \underline{M} \to \underline{N}$:

(i) $\underline{\phi}$ is always a monomorphism;

(ii) $\underline{\phi}$ is epimorphism iff ϕ is a surjective submersion.