MATH 595 – Lecture 27

Principal $\mathcal{G}$-bundles

Recall a principal $\mathcal{G}$-bundle is given by a $\mathcal{G}$-space P with a $\mathcal{G}$-invariant submersion $\pi: P \to M$ such that:

$$\begin{tikzcd} & P \arrow[r, "\pi"] \arrow[d] & X \\ \mathcal{G} \rightrightarrows & M & \end{tikzcd}$$

(*) $\quad \mathcal{G} \times_M P \longrightarrow P \times_M P, \quad (g,p) \mapsto (gp, p)$

is a diffeomorphism

RMK: (*) says that the action groupoid $\mathcal{G} \ltimes P \rightrightarrows P$ is isomorphic to the submersion groupoid $P \times_M P \rightrightarrows P$, which is a way of expressing that action is proper and free (since subm grpd is proper and free of isotropy).

Examples

1) For a Lie group $G \rightrightarrows \{*\}$ this recovers usual notion of principal G-bundle

2) Any Lie groupoid $\mathcal{G} \rightrightarrows M$, left action on itself gives a principal $\mathcal{G}$-bundle:

$$\begin{tikzcd} & \mathcal{G} \arrow[r, "s = \pi"] \arrow[d, "t"] & M = X \\ \mathcal{G} \rightrightarrows & M & \end{tikzcd}$$

3) For a general principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} X$, each fiber $\pi^{-1}(x)$ is isomorphic to a source fiber $s^{-1}(p)$ where $p = \mu(u)$, $u \in \pi^{-1}(x)$.

$$s^{-1}(p) \longrightarrow \pi^{-1}(x), \quad g \longmapsto gu$$

4) If $\mathcal{G} = G \ltimes M \rightrightarrows M$ a principal $\mathcal{G}$-bundle is just an ordinary principal G-bundle $\pi: P \to X$ together w/ a G-equivariant map $\mu: P \to M$.

Morphisms:

A morphism of principal G-bundles is a map between principal bundles

$$\Phi : P_1 \to P_2$$

which is G-equivariant:

$$\Phi(gu) = g\,\Phi(u)$$

In particular:

$$\begin{tikzcd} P_1 \arrow[rr, "\Phi"] \arrow[dr, "\mu_1"'] & & P_2 \arrow[dl, "\mu_2"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} P_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "\phi"'] & X_2 \end{tikzcd}$$

for a smooth map $\phi : X_1 \to X_2$.

Pullbacks:

- $P \xrightarrow{\pi} X$ principal G-bundle
- $\phi : Y \to X$

$\Rightarrow$ principal G-bundle: $\phi^* P \to Y$

$$\begin{tikzcd} \phi^* P := P \times_X Y \arrow[r, "\Phi"] \arrow[d] & P \arrow[d, "\pi"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

$\cdot\ \mu(u,y) := \mu(u) \quad g(u,y) := (gu, y)$

The map $\Phi : \phi^* P \to P$ is a morphism of principal G-bundles

Proposition:

Every morphism of principal G-bundles

$$\begin{tikzcd} P_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & P_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "\phi"'] & X_2 \end{tikzcd}$$

induces an isomorphism:

$$\begin{tikzcd} P_1 \arrow[r, "\sim"] \arrow[d, "\pi_1"'] & \phi^* P_2 \arrow[d, "\pi"] \\ X_1 \arrow[r, "\mathrm{id}"'] & X_1 \end{tikzcd} \qquad u \longmapsto (\Phi(u), \pi_1(u))$$

Proof: A morphism covering id is an isomorphism. $\blacksquare$

Local triviality

A principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} X$ is Trivial if it is iso to pullback of the unit principal $\mathcal{G}$-bundle $\mathcal{G} \xrightarrow{t} M$.

Lemma. For a principal $\mathcal{G}$-bundle the following are equivalent:

(i) $P \xrightarrow{\pi} X$ is trivial

(ii) There exists a morphism $\Phi : P \to \mathcal{G}$

(iii) $P \xrightarrow{\pi} X$ has a section

Proof:

(i) <=> (ii)

$$\begin{tikzcd}
P \arrow[r, "\simeq"] \arrow[d] \arrow[rr, bend left, dashed, "\Phi"] & \phi^*\mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\
X \arrow[r, "id"'] & X \arrow[r, "\phi"'] & M
\end{tikzcd}$$

(i) => (iii)

$$\cdot\ \phi^*\mathcal{G} = X \times_M \mathcal{G} \qquad (x, 1_{\phi(x)})$$

$$\begin{tikzcd}
X \times_M \mathcal{G} \arrow[d, "\pi"'] \\
X \arrow[u, bend right, dashed, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, 1_{\phi(x)}) \\
x \arrow[u, mapsto]
\end{tikzcd}$$

(iii) => (i)

$$\begin{tikzcd}
 & P \arrow[r, "\pi"] \arrow[d, "\mu"'] & X \arrow[l, bend left, "s"] \arrow[dl, dashed, "\phi := \mu \circ s"] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M &
\end{tikzcd}$$

$$P \simeq \phi^*\mathcal{G}$$

$$g\,s(x) \longmapsfrom (g, x)$$

$\blacksquare$

For any principal $\mathcal{G}$-bundle $\pi : P \to X$ is a surjective submersion, so it admits local sections.

=> principal $\mathcal{G}$-bundles are locally trivial

$$\begin{tikzcd}
 & P \arrow[r, "\pi"] \arrow[d, "\mu"'] & X \supset U_\alpha \arrow[l, dashed, "s_\alpha"'] \arrow[dl, "\phi_\alpha"] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M &
\end{tikzcd}
\qquad P|_{U_\alpha} \simeq \phi_\alpha^* \mathcal{G}$$

Cocycle description

Given principal $\mathcal{G}$-bundle $\pi : P \to X$ cover X by open sets $\{U_\alpha\}$ where there exist local section $s_\alpha : U_\alpha \to P$

- $\phi_\alpha := \mu \circ s_\alpha : U_\alpha \to M$, $P|_{U_\alpha} \simeq \phi_\alpha^* \mathcal{G}$
- On $U_{\alpha\beta} := U_\alpha \cap U_\beta$:

$$\phi_\alpha^* \mathcal{G}\,\big|_{U_{\alpha\beta}} \simeq P\big|_{U_{\alpha\beta}} \simeq \phi_\beta^* \mathcal{G}\,\big|_{U_{\alpha\beta}}$$

$$(x, g) \longmapsto (x, g_{\beta\alpha}(x)\, g)$$

where $g_{\beta\alpha} : U_{\alpha\beta} \to \mathcal{G}$ are arrows:

$$\phi_\beta(x) \xleftarrow{g_{\beta\alpha}(x)} \phi_\alpha(x)$$

- On triple intersections:

$$g_{\gamma\beta}(x)\, g_{\beta\alpha}(x) = g_{\gamma\alpha}(x) \qquad (x \in U_{\alpha\beta\gamma})$$

A $\mathcal{G}$-cocycle is a family $(\phi_\alpha, g_{\alpha\beta})$ w/ $\phi_\alpha : U_\alpha \to M$, $g_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{G}$:

$$s \circ g_{\beta\alpha} = \phi_\alpha \;,\; t \circ g_{\beta\alpha} = \phi_\beta \quad (\text{on } U_{\alpha\beta})$$

$$g_{\gamma\beta}\, g_{\beta\alpha} = g_{\gamma\alpha} \quad (\text{on } U_{\alpha\beta\gamma})$$

Two $\mathcal{G}$-cocycles $(\phi_\alpha, g_{\beta\alpha})$ & $(\tilde{\phi}_\alpha, \tilde{g}_{\alpha\beta})$ are equivalent if $\exists\, \lambda_\alpha : U_\alpha \to \mathcal{G}$

$$s \circ \lambda_\alpha = \phi_\alpha \;,\quad t \circ \lambda_\alpha = \tilde{\phi}_\alpha \quad (\text{on } U_\alpha)$$

$$\tilde{g}_{\beta\alpha} = \lambda_\beta \cdot g_{\beta\alpha}\, \lambda_\alpha^{-1} \quad (\text{on } U_{\alpha\beta})$$

After refinement, this gives equivalence relation and one finds:

$$\text{Principal } \mathcal{G}\text{-bundles}/_{\text{iso}} \;\longleftrightarrow\; \mathcal{G}\text{-cocycles}/_{\text{equiv.}}$$

Rmk: One can also describe generalized maps and Morita equivalences using principal $\mathcal{G}$-bundles (see Bibliography).

Differentiable Stacks

A differentiable stack is a (very general) notion of singular space, generalizing manifolds.

A differentiable stack "is" a Morita equivalence class of Lie groupoids. There is a more conceptual way of approaching them based on Grothendieck's philosophy of the "functor of points":

- A manifold M is completely determined, up to canonical isomorphism, by the set of all smooth maps $X \to M$, where X is a manifold. Equivalently, by the set of all smooth maps $\mathbb{R}^n \to M$ $(n = 0, 1, \dots)$

Formally, this means replacing M by the representable functor:

$$\underline{M} : \text{Manflds} \to \text{Sets} \qquad \begin{cases} \underline{M}(X) = \{ f : X \to M \} \\ \underline{M}(X \xrightarrow{g} X') = (f \mapsto f \circ g) \end{cases}$$

A singular space is a more general "functor" Manflds $\to$ Sets which is not necessarily representable (i.e., equivalent to some $\underline{M}$)

This philosophy is completed by observing that:

<u>Exercise:</u> Show that there is a bijection:

$$C^\infty(M_1, M_2) \longleftrightarrow \mathrm{Nat}(\underline{M_1}, \underline{M_2})$$

(This is a version of Yoneda's Lemma).

We are now going to formalize this and provide the connection with Lie groupoids.

Notation:

- $\mathcal{M}$ = category of C^∞-manifolds & C^∞-maps
- Given $X \in \mathcal{M}$ a covering family of X is any family $\{U_i \xrightarrow{f_i} X\}$ where f_i are étale & $\bigcup_{i \in I} f_i(U_i) = X$

RMK:

1) Covering families define a unique Grothendieck topology on $\mathcal{M}$, called the étale topology. A category equipped w/ a Grothendieck topology is called a site. In what follows $\mathcal{M}$ can be replaced by any site. One then obtains topological stacks, algebraic stacks, etc. by replacing $\mathcal{M}$ by Top or Sch.

2) $\mathrm{Obj}(\mathcal{M})$ is not a set. One can replace $\mathcal{M}$ by:

- $\mathcal{M}_{emb}$ = w/ objects embedded submanifolds in some $\mathbb{R}^N$
- $\mathcal{R}$ = w/ objects $= \{\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2, \ldots\}$
- Euc = w/ objects disjoint unions of open subsets in some $\mathbb{R}^N$

Defn: A category fibered in groupoids $\pi : \mathcal{G} \to \mathcal{M}$ is a functor from some category satisfying:

(i) For every $f: X' \to X$ & C in $\mathcal{G}$ over X, there exists $g: C' \to C$ in $\mathcal{G}$ with $\pi(g) = f$.

$$\begin{array}{ccc} C' & & X' \\ {\scriptstyle g}\downarrow & \overset{\pi}{\Rightarrow} & \downarrow{\scriptstyle f} \\ C & & X = \pi(C) \end{array}$$

(ii) Given a diagram:

$$\begin{tikzcd} C_1 \arrow[dr, "g_1"] \arrow[dd, dashed, "\exists ! g"'] & \\ & C \\ C_2 \arrow[ur, "g_2"'] & \end{tikzcd} \quad \overset{\pi}{\Rightarrow} \quad \begin{tikzcd} X_1 \arrow[dr, "f_1"] \arrow[dd, "f"'] & \\ & X \\ X_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

There is a unique lift g.

Rmks:

- We have not used yet covering families (i.e., the Grothendieck topology)
- By (ii) the object C' in (i) is unique up to a unique isomorphism

One calls C' a pullback of C via $f: X' \to X$ and one often writes

$$C' = C|_{X'} = f^*C$$

- Fixing $X \in \mathcal{M}$, we have the fiber over X, which is the subcategory $\mathcal{C}_X \subset \mathcal{C}$ with:

$$Obj(\mathcal{C}_X) = \{ C \in Obj(\mathcal{C}) : \pi(C) = X \}$$
$$Arr(\mathcal{C}_X) = \{ f \in Arr(\mathcal{C}) : \pi(f) = id_X \}$$

Exercise: Using (ii), show that fibers $\mathcal{C}_X$ are groupoids, i.e., every arrow in $\mathcal{C}_X$ has an inverse.

Abreviation:

Fibered category = category fibered in groupoids

Examples:

1) Fix $M \in \mathcal{M}$. Let $\mathcal{C} = \underline{M}$ be the category

$$Obj(\underline{M}) = \{ f : X \to M \}$$

$$Arr(\underline{M}) = \left\{ \begin{tikzcd} X_1 \arrow[d, "g"'] \arrow[dr, "f_1"] & \\ X_2 \arrow[r, "f_2"'] & M \end{tikzcd} \right\}$$

It is a fibered category for the forgetful functor:

$$\pi : \underline{M} \to \mathcal{M} \quad \begin{cases} (X \xrightarrow{f} M) \longmapsto X \\ \left(\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & M & \end{tikzcd} \right) \longmapsto (X_1 \xrightarrow{g} X_2) \end{cases}$$

Both axioms hold:

i) Given $(g : X' \to X) \in \mathrm{Arr}(\mathfrak{M})$ & $(f_2 : X \to M) \in \mathrm{Obj}(\underline{M})$ an object over M, we have the pullback:

$$\begin{tikzcd} X' \arrow[d, "g"'] \arrow[dr, "f_1 = f_2 \circ g"] & \\ X \arrow[r, "f_2"'] & M \end{tikzcd} \quad \overset{\pi}{\Longrightarrow} \quad \begin{tikzcd} X' \arrow[d, "g"] \\ X \end{tikzcd}$$

ii)

$$\begin{tikzcd} X_1 \arrow[dr, "g_1"] \arrow[drr] \arrow[dd, dashed, "g"'] & & \\ & X \arrow[r] & M \\ X_2 \arrow[ur, "g_2"'] \arrow[urr] & & \end{tikzcd} \quad \overset{\pi}{\Longrightarrow} \quad \begin{tikzcd} X_1 \arrow[d, "g"'] \arrow[dr, "g_1"] & \\ X_2 \arrow[r, "g_2"'] & X \end{tikzcd}$$

In this example:

pullbacks are unique $\Leftrightarrow$ Fibers are identity Groupoids

Def: A <u>discrete fiber category</u> over $\mathfrak{M}$ is a Fibered category $\pi : \mathcal{G} \to \mathfrak{M}$ such that $\mathcal{G}_X$ is an identity grpd for all $X \in \mathfrak{M}$.

2) Let G be a Lie group and $\mathcal{G} = BG$ be the category:

$\mathrm{Obj}(BG)$ = principal G-bundles; $p : P \to X$

$\mathrm{Arr}(BG)$ = Morphisms of principal G-bundles

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, "p_1"'] & P_2 \arrow[d, "p_2"] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

The forgetful functor $\pi : BG \to \mathfrak{M}$ is a Fibered category.

One checks that (i) and (ii) hold. Note that pullbacks are not unique, there are only unique up to a unique isomorphism.

<u>Exercise:</u> Show that principal G-bundles w/ connection also give a fibered category $\pi : BG^{\nabla} \to \mathfrak{M}$

3) More generally, any Lie groupoid $\mathcal{G} \rightrightarrows M$ defines a fibered category

$$\pi : B\mathcal{G} \to \mathfrak{M}$$

with:

$\mathrm{Obj}(B\mathcal{G}) = \left\{ \text{principal } \mathcal{G}\text{-bundles } \begin{array}{c} \mathcal{G} \\ \downdownarrows \\ M \end{array} \circlearrowleft P \xrightarrow{\pi} X \right\}$

$\mathrm{Arr}(B\mathcal{G}) = \{ \text{morphisms of principal } \mathcal{G}\text{-bundles} \}$

$\pi \equiv$ forgetful functor: $\pi(P) = X$

3) Let $\mathcal{C} = \mathcal{F}_g$ be the category:

$\mathrm{Obj}(\mathcal{F}_g) =$ Fiber bundles $p : E \to X$ w/ fiber a Riemann surface of genus g & complex structure smoothly varying on fibers

$\mathrm{Arr}(\mathcal{F}_g) =$ Commutative diagrams

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

with: $E_1 \to X_1 \times_{X_2} E_2$ a conformal isomorphism

The forgetful functor $\pi : \mathcal{F}_g \to \mathfrak{M}$ is a fibered category.

<u>Rem:</u> Often fibered categories arise as in previous example from moduli problems. Then one thinks of the fibered category $\pi : \mathcal{C} \to \mathfrak{M}$ as:

- An object in $\mathcal{C}$ over M is a $\mathcal{C}$-family parametrized by M
- Aim is to classify all objects over $pt \in \mathfrak{M}$.