MATH 595 - Lecture 26

Back to orbifolds

$X \equiv$ orbifold

To each orbifold atlas: $\mathcal{U} = \{(U_i, G_i, \phi_i) : i \in I\}$

$$U = \bigsqcup_{i \in I} U_i, \quad \phi = \{\phi_i\} : U \to X$$

$$\Psi(\mathcal{U}) = \{\psi \in \mathrm{Diff}_{loc}(U) : \phi \circ \psi = \phi|_{\mathrm{Dom}(\psi)}\}$$

This is a pseudogroup over U. So we have effective étale groupoid:

$$\Gamma(\mathcal{U}) \equiv \Gamma(\Psi(\mathcal{U})) \rightrightarrows U$$

Exercise: If M is a manifold and $\mathcal{U}$ is an atlas, show that $\Gamma(\mathcal{U})$ is the cover groupoid.

Proposition: For any orbifold atlas $\mathcal{U}$ of X:

(i) $\Gamma(\mathcal{U})$ is a proper, effective, étale groupoid.

(ii) If $\mathcal{U}'$ is another orbifold atlas of X, then $\Gamma(\mathcal{U})$ and $\Gamma(\mathcal{U}')$ are Morita equivalent.

Proof:

(i) Let $(p,q) \in U_i \times U_j \subset U \times U$. We claim that:

- $\exists\, K \ni (p,q)$ compact neighborhood w/ $(t \times s)^{-1}(K) \subset \Gamma(\mathcal{U})$ compact

This implies that $t \times s : \Gamma(\mathcal{U}) \to U \times U$ is proper.

- If $\phi_i(p) \neq \phi_j(q)$ this is clear since X is locally compact & Hausdorff, and G_i are finite.

- If $\phi_i(p) = \phi_j(q) = x$: By compatibility of charts, & their properties, one can find open $p \in V \subset U_i$ & embedding of orbifold charts $\lambda : (V, (G_i)_V, \phi_i|_V) \to (U_j, G_j, \phi_j)$ so that $\phi \circ \lambda = \phi|_V$, $\lambda(p) = q$. Note that then $\lambda \in \Phi(\mathcal{U})$.

We may assume that $(G_i)_V = (G_i)_p$ by eventually shrinking V. Then $\lambda(V)$ is $(G_j)_{\lambda(V)} = (G_j)_q$-stable, and it follows from properties of charts:

$$(t \times s)^{-1}(\lambda(V) \times V) = \{ \operatorname{germ}_z(\lambda_g \circ \lambda) : g \in (G_j)_q,\ z \in V \} \simeq (G_j)_q \times V$$

Since G_j are finite, claim follows.

(ii) If $\mathcal{U}$ & $\mathcal{V}$ are equivalent atlas, so $\mathcal{U} \sqcup \mathcal{V}$ is also an atlas. Then there are groupoid morphisms:

$$\begin{tikzcd} \Gamma(\mathcal{U}) \arrow[dr] & \\ & \Gamma(\mathcal{U} \sqcup \mathcal{V}) \\ \Gamma(\mathcal{V}) \arrow[ur] & \end{tikzcd}$$

These are Morita maps since they preserve $\perp$-data. ▨

On the other hand:

<u>Proposition</u> For any proper, effective, groupoid $\mathcal{G} \rightrightarrows M$ there is a canonical orbifold structure on $X = M/\mathcal{G}$ such that for any orbifold atlas $\mathcal{U}$, $\Gamma(\mathcal{U})$ & $\mathcal{G}$ are Morita equivalent.

<u>Proof:</u>

For each $p \in M$ one can find a saturated neighborhood $U_p \subset M$ and a $\mathcal{G}_p$-invariant neighborhood $0 \in V_p \subset T_pM$ such that:

$$\mathcal{G}|_{U_p} \simeq \mathcal{G}_p \ltimes V_p$$

Then the collection $(V_p, \mathcal{G}_p, \phi_p)$ w/ $\phi_p : V_p \xrightarrow{\sim} U_p \to M/\mathcal{G}$ is an orbifold atlas $\mathcal{U}$ for $M/\mathcal{G}$.

Notice that:

$$U = \coprod_{p \in M} U_p \ , \quad i : U \to M \ , \quad \mathcal{G}_U := i^* \mathcal{G} \to \mathcal{G}$$

is a Morita map. On the other hand:

$$\mathcal{G}_U \to \Gamma(\mathcal{U}) \ , \quad g \longmapsto \operatorname{germ}_{s(g)} b \ ,$$ b local bisection w/ $b(s(g)) = g$

is also a Morita map, so:

$$\mathcal{G} \underset{M}{\simeq} \Gamma(\mathcal{U}).$$

∎

Thm. For any Lie groupoid $\mathcal{G}$ the following are equivalent:

(i) $\mathcal{G}$ is Morita equiv to a proper effective étale gpd

(ii) $\mathcal{G}$ is Morita equiv to a groupoid associated w/ an orbifold atlas

(iii) $\mathcal{G}$ is Morita equiv to the holonomy gpd of a foliation w/ compact leaves & finite holonomy

(iv) $\mathcal{G}$ is Morita equiv to the action gpd of an proper effective Lie group action w/ finite isotropy.

Proof.

- (i) <=> (ii): This follows from previous props.
- (iii) => (i): $\operatorname{Hol}(M,\mathcal{F}) \rightrightarrows M \underset{M}{\simeq} \operatorname{Hol}(M,\mathcal{F})|_T \rightrightarrows T$ for a complete, transversal T. Later gpd satisfies (i)
- (iv) => (i): If $G \curvearrowright M$ is proper, effective, w/ finite isotropy then connected components of orbits form a foliation. If T is complete transversal, then $G \ltimes M \rightrightarrows M \underset{M}{\simeq} (G \ltimes M)|_T \rightrightarrows T$. Later gpd satisfies (i).

· (ii) $\Rightarrow$ (iii), (iv): We saw that we can find compact, connected Lie group action $K \curvearrowright M$, which is effective and has finite isotropy, so that $X \simeq M/K$. The groupoid $K \ltimes M \rightrightarrows M$ is Morita Equiv. to $\Gamma(\mathcal{U})$ for an orbifold atlas $\mathcal{U}$ and satisfies (iv). The orbits of K form a foliation $(M, \mathcal{F})$ and $\mathrm{Hol}(M, \mathcal{F}) \simeq K \ltimes M$, so (iii) also holds. $\blacksquare$

This all discussion sugests:

Def: Let X be a top. space.

(i) An <u>orbifold atlas</u> for X is a pair $(\mathcal{G}, \phi)$ where $\mathcal{G}$ is a proper Lie groupoid w/ finite isotropy & $\phi: M \to X$ is a map inducing a homeomorphism $M/\mathcal{G} \xrightarrow{\sim} X$.

(ii) Two orbifold atlas $(\mathcal{G}_1, \phi_1)$ and $(\mathcal{G}_2, \phi_2)$ are <u>equivalent</u> if there exists a Morita equivalence giving comm. diagram

$$\begin{tikzcd} & \mathcal{H} \arrow[dl] \arrow[dr] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} M_1/\mathcal{G}_1 \arrow[r, "\sim"] \arrow[d, "\phi_1"'] & M_2/\mathcal{G}_2 \arrow[d, "\phi_2"] \\ X \arrow[r, equal, "\mathrm{id}"'] & X \end{tikzcd}$$

(iii) An <u>orbifold structure</u> on X is an equivalence class of orbifold atlas.

Recall that:

$\mathcal{G}$ proper & finite isotropy $\Leftrightarrow$ $\mathcal{G}$ Morita equivalent to proper, étale grpd

Def: An <u>effective orbifold</u> X is an orbifold structure on X admitting an atlas $(\mathcal{G}, \phi)$ where $\mathcal{G}$ is proper, effective, étale grpd.

Note that:

- Effective orbifolds = classical orbifolds with classical atlas
- Orbifolds = cannot be defined using classical atlas (properties of atlas collapse without effective assumption)

Prop: Any orbifold structure on X has an underlying classical (effective) structure.

Proof:

$$\mathcal{G} \text{ proper étale} \Rightarrow \mathrm{Eff}(\mathcal{G}) \text{ proper, effective, étale}$$

$\blacksquare$

Example:

Let $G \times M \to M$ be an action of a finite group (possibly ineffective). Then $\mathcal{G} = G \ltimes M \rightrightarrows M$ defines an orbifold structure on M/G. By factoring the kernel K of the action:

$$K = \{ g \in G : gp = p, \forall p \in M \}$$

we obtain an eff. action $G/_K \curvearrowright M$, and $G/_K \ltimes M \rightrightarrows M$ gives the classical orbifold structure on:

$$M/G = M/(G/K).$$

An extreme case is when $K = G$, i.e. a trivial action: the underlying classical orbifold is a manifold!

- What do we gain with the groupoid approach to orbifolds?

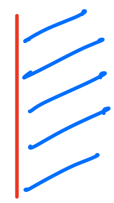
1) It solves some issues
2) It is conceptually simpler
3) It extends to even more singular spaces

1) Let X be an orbifold. What is a suborbifold $Y \subset X$?

- Classically, there are problems. Take $\mathbb{Z}_2 \circlearrowright \mathbb{R}^2$, $(x,y) \to (-x,y)$

$$\Rightarrow \quad X = \mathbb{R}^2/\mathbb{Z}_2$$

Is the set $x=0$ a suborbifold?



$x=0$

$\mathbb{R}^2/\mathbb{Z}_2$

- We can think of it as a 1-dim manifold (= orbifold w/ no isotropy). But what happen to isotropy groups?

- We can think of it as a 1-dim orbifold with isotropy $\mathbb{Z}_2$ at every point. Not an effective orbifold

- Non-classically:
  - $\mathcal{G} \rightrightarrows M$ a orbifold atlas for X
  - $N \subset M$ closed submanifold such that $\mathcal{G}|_N \rightrightarrows N$ is subgroupoid of $\mathcal{G} \rightrightarrows M$

$$\Rightarrow \quad Y = N/\mathcal{G}_N \hookrightarrow X = M/\mathcal{G}_M \text{ is suborbifold}$$

In example:

$$\mathcal{G} = \mathbb{Z}_2 \ltimes \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$$
$$\cup \qquad\qquad \cup$$
$$\mathcal{G}|_{\{0\}\times\mathbb{R}} = \mathbb{Z}_2 \ltimes (\{0\}\times\mathbb{R}) \rightrightarrows \{0\}\times\mathbb{R}$$

2) Homotopy groups of an orbifold:

- $\mathcal{G} \rightrightarrows M$ a groupoid representing X

$\rightsquigarrow$ nerve of $\mathcal{G}$: is the simplicial manifold

$$\underset{\mathcal{G}^{(0)}}{\underset{\parallel}{M}} \underset{s}{\overset{t}{\leftleftarrows}} \underset{\mathcal{G}^{(1)}}{\underset{\parallel}{\mathcal{G}}} \overset{pr}{\Lleftarrow} \underset{\mathcal{G}^{(2)}}{\underset{\parallel}{\mathcal{G} {}_t\!\times_s \mathcal{G}}} \cdots \qquad \mathcal{G}^{(n)} = \underbrace{\mathcal{G} {}_t\!\times_s \mathcal{G} \times \cdots {}_t\!\times_s \mathcal{G}}_{n\text{-times}}$$

Face Maps: $d_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)} \; (i = 0, \dots, n)$

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n), & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n), & 1 \leq i \leq n-1 \\ (g_1, \dots, g_{n-1}), & i = n \end{cases}$$

Degeneracies: $s_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n+1)} \; (i = 1, \dots, n+1)$

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, 1_x, g_i, \dots, g_{n-1})$$

As for any simplicial set we have its (FAT) geometric realization:

$$\|\mathcal{G}^\bullet\| := \left(\coprod_n \mathcal{G}^{(n)} \times \Delta_n \right) / \sim \qquad \text{(w/ quotient topology)}$$

where:

- $\Delta_n = \{ (t_0, \dots, t_n) : t_i \geq 0, \; \sum_{i=0}^{n} t_i = 1 \}$
- $\partial_i : \Delta_{n-1} \to \Delta_n \; (i = 0, \dots, n)$ face maps:

$$\partial_i (t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

- $\sim$ is equiv. relation generated by:

$$(d_i(g), t) \sim (g, \partial_i(t))$$

If $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a Morita map one gets a simplicial map $\phi^\bullet : \mathcal{G}_1^\bullet \to \mathcal{G}_2^\bullet$ and hence a continuous map

$$\|\phi\| : \|\mathcal{G}_1^\bullet\| \to \|\mathcal{G}_2^\bullet\|$$

One can show that this gives an isomorphism of homotopy groups:

Thm: If $\mathcal{G}_1$ & $\mathcal{G}_2$ are Morita equivalent then $\|\mathcal{G}_1^\bullet\|$ and $\|\mathcal{G}_2^\bullet\|$ are weak homotopy equivalent.

One can use $\mathcal{G}^\bullet$ & $\|\mathcal{G}^\bullet\|$ to attach geom. & topological invariants to the orbifold X represented by the atlas $(\mathcal{G}, \phi)$:

1) The orbifold homotopy groups:

$$\pi_n^{orb}(X,x_0) := \pi_n(\|\mathcal{G}^\bullet\|, [1_{x_0}])$$

2) For any ring R, the singular cohomology of X:

$$H^n_{orb}(X,R) := H^n(\|\mathcal{G}^\bullet\|, R)$$

3) The de Rham cohomology of X:

- $\Omega^n(X) := \{\omega \in \Omega^n(M) : s^*\omega - t^*\omega = 0\}$
- $d : \Omega^n(X) \to \Omega^{n+1}(X)$

<u>De Rham Theorem:</u>

$$H^\bullet(\Omega(X), d) \cong H^\bullet_{orb}(X, \mathbb{R})$$

4) Riemann. Metric on X:

- η I-metric on $\mathcal{G}$: metric on $\mathcal{G}$ making s & t Riem. sub and $i : \mathcal{G} \to \mathcal{G}$ an isometry.

There is a Gauss-Bonnet Theorem, etc.

<u>Example:</u>

Can use π_n^{orb} to find obst. to be a global quotient:

<u>Prop:</u> If $\pi_1^{orb}(X) \neq 1$, then X is not a global quotient.

<u>Sketch of proof:</u>

$G \curvearrowright M$ is effective, proper action w/ finite isotropy, $X = M/G$. Then $\exists$ long exact seq in homotopy:

$$\cdots \to \pi_n(G) \to \pi_n(M) \to \pi_n^{orb}(X) \to \pi_{n-1}(G) \to \cdots \quad (*)$$

When G is finite this gives:

$$\begin{cases} \pi_m^{orb}(X) \simeq \pi_m(M), & m \geq 2 \\ 1 \to \pi_1(M) \to \pi_1^{orb}(X) \to G \to 1 \end{cases}$$

But if X is not smooth, then $G \neq 1$, so $\pi_1^{orb}(X) \neq 1$

$\blacksquare$

Corollary

If $K \curvearrowright N$ is a proper effective action with finite isotropy on a 1-connected manifold N, then $X = N/K$ is not a global quotient.

Proof:

The long exact sequence $(*)$ gives $\pi_1^{orb}(X) = 1$

$\blacksquare$

Tear Drop:

$$\left.\begin{aligned} &S^1 \curvearrowright S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \\ &\theta \cdot (z,w) = (e^{i m \theta} z, e^{i n \theta} w) \quad (n \neq m) \end{aligned}\right\} \Rightarrow X = S^3/S^1 \text{ not global quotient}$$