MATH 595 - Lecture 25

If $\mathcal{O}$ is an orbit of $\mathcal{G} \rightrightarrows M$ then

$$\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O} \quad \circlearrowleft \quad \nu(\mathcal{O})$$

The linear local model for $\mathcal{G}$ around $\mathcal{O}$ is:

$$\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O})$$

Linearization Thm (Zung & Weinstein)

Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid and fix $\mathcal{O} \subset M$. There exist open sets $\mathcal{O} \subset U \subset M$, $\mathcal{O} \subset V \subset \nu(\mathcal{O})$ and a groupoid isomorphism

$$\mathcal{G}|_U \simeq (\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}))|_V$$

The open sets U and V, in general, will not be saturated (= union of orbits). For that:

Corollary (Invariant Linearization Thm)

Let $\mathcal{G} \rightrightarrows M$ be a s-proper groupoid and fix $\mathcal{O} \subset M$. There exist open saturated neighborhoods $\mathcal{O} \subset U \subset M$ and $\mathcal{O} \subset V \subset \nu(\mathcal{O})$ and a groupoid isomorphism

$$\mathcal{G}|_U \simeq (\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}))|_V$$

Proof of Corollary:

Every s-proper groupoid is proper

Orbits of s-proper groupoids are stable: every neighborhood of an orbit contains a saturated neighborhood (exercise!).

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By considering various classes of groupoids, we obtain several well-known theorems.

- If $(M,\mathcal{F})$ is a foliation w/ compact leaves and finite holonomy then $\mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ is an s-proper groupoid:

$$\text{Inv. Linearization thm} \equiv \text{Reeb stability Thm}$$

- If $G \ltimes M \rightrightarrows M$ is an action groupoid associated w/ an action of a compact Lie group, it is s-proper and:

$$\text{Inv. Linearization thm} \equiv \text{Linearization of action}$$

- If $M \times_N M \rightrightarrows M$ is submersion grpd of a proper submersion $\phi: M \to N$, then it is s-proper and:

$$\text{Inv. Linearization thm} \equiv \text{Ehresmann Theorem}$$

(proper submersions are locally trivial)

<u>Corollary</u>: Let $\mathcal{G} \rightrightarrows M$ be a proper étale groupoid. Every $x \in M$ has a neighborhood U such that:

$$\mathcal{G}|_U \cong \mathcal{G}_x \ltimes V_x$$

where $0 \in V_x \subset T_x M$ is a $\mathcal{G}_x$-invariant neighborhood

<u>Proof</u>: $\mathcal{G}$ étale $\Rightarrow$ discrete orbits $\Rightarrow$ every $x \in M$, has neighborhood $\tilde{U}$ with $\tilde{U} \cap \mathcal{O}_x = \{x\}$. Apply linearization Thm. to $\mathcal{G}|_{\tilde{U}}$: There is smaller neighborhood $x \in U \subset \tilde{U}$ and $0 \in V_x \subset T_x M$:

$$\mathcal{G}|_U \cong (\mathcal{G}_x \ltimes T_x M)|_{V_x}$$

Choosing a $\mathcal{G}_x$-inv metric, we see we can choose V_x to be $\mathcal{G}_x$-invariant.

$\blacksquare$

<u>Note</u>: One can prove this corollary directly (see Moerdijk-Mrcun)

Corollary: Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid. Every orbit $\mathcal{O} \subset M$ has saturated neighborhood U such that

$$(\mathcal{G}|_U \rightrightarrows U) \underset{M}{\cong} (\mathcal{G}_x \ltimes V_x \rightrightarrows V_x)$$

where $0 \in V_x \subset \nu_x(\mathcal{O})$ is $\mathcal{G}_x$-invariant neighborhood.

Proof of Corollary

Fix $x \in \mathcal{O}$ and choose $T \subset M$ a transverse submanifold to $\mathcal{O}$ through x:

$$T_x M = T_x \mathcal{O} \oplus T_x T$$

If T is small enough, then $T \pitchfork$ to every orbit it meets and $T \cap \mathcal{O} = \{x\}$ (since $\mathcal{O}$ is embedded). It follows that:

- $\mathcal{G}|_T \rightrightarrows T$ is a proper Lie groupoid with orbit $\{x\}$

Apply linearization Thm to $\mathcal{G}|_T$ around $\{x\}$: eventually after shrinking T:

$$\mathcal{G}|_T \cong (\mathcal{G}_x \ltimes T_x(T))|_{V_x}$$

As in previous corollary, eventually after shrinking T, we can write

$$\mathcal{G}|_T \cong \mathcal{G}_x \ltimes V_x$$

for some $\mathcal{G}_x$-invariant open $0 \in V_x \subset T_x(T)$

Finally, observe that:

- $\mathcal{G}|_T \underset{M}{\cong} \mathcal{G}_U$, where $U = \pi^{-1}(\pi(T))$ is open saturated $(\pi : M \to M/\mathcal{G})$

$\blacksquare$

Sketch of Proof of Linearization Thm

The main idea is to use a "Groupoid Metric", i.e., a Riem. metric adapted to transverse data.

Recall that given submersion $\phi : M \to N$ & Riem metric η_M on M, one calls ϕ a Riemannian Submersion if for any $p_1, p_2 \in M$ with $\phi(p_1) = \phi(p_2) = q$ the linear isomorphism:

$$(\mathrm{Ker}\, d_{p_1}\phi)^{\perp} \xrightarrow[\sim]{d_{p_1}\phi} T_q N \xleftarrow[\sim]{d_{p_2}\phi} (\mathrm{Ker}\, d_{p_2}\phi)^{\perp}$$

is an isometry. Then $\phi(M)$ inherits a unique Riem metric η_N s.t.:

$$(d_p\phi) : (\mathrm{Ker}\, d_p\phi)^{\perp} \longrightarrow T_p N$$

is an isometry for all $p \in M$. We write $\eta_N = \phi_* \eta_M$.

For a Riemannian submersion:

(i) Geodesic $\perp$ to a fiber $\Rightarrow$ $\perp$ to every fiber

(ii) Fibers are equidistant

Def: A 2-metric on a Groupoid $\mathcal{G} \rightrightarrows M$ is a Riemann metric $\eta^{(2)}$ on the composable arrows:

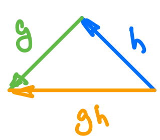
$$\mathcal{G}^{(2)} = \mathcal{G}\, {}_s\!\times_t \mathcal{G}$$

For which:

(i) The 3 maps:
$$\mathcal{G}^{(2)} \begin{array}{l} \xrightarrow{pr_2} \mathcal{G} \\ \xrightarrow{m} \mathcal{G} \\ \xrightarrow{pr_1} \mathcal{G} \end{array}, \quad (g,h) \begin{array}{l} \to h \\ \to gh \\ \to g \end{array}, \text{ are}$$

Riem. submersions.

(ii) The natural action $S_3 \circlearrowleft \mathcal{G}^{(2)}$ is by isometries

$$\mathcal{G}^{(2)} \simeq \{$$ 
$$\} \circlearrowright S_3$$

E.g. $(g,h) \mapsto (h^{-1}, g^{-1})$

$(g,h) \mapsto (h, h^{-1}g^{-1})$

For such a 2-metric $\eta^{(2)}$, one obtains

(i) The metrics induced on $\mathcal{G}$ by pr_1, m, pr_2 coincide. We denote it by $\eta^{(1)}$.

(ii) The metric $\eta^{(1)}$ makes s & t Riem submersions and is invariant under inversion $i : \mathcal{G} \to \mathcal{G}$.

(iii) The metric induced by s & t on M coincide. We denote it by $\eta^{(0)}$

$$\begin{tikzcd}
(\mathcal{G}^{(2)}, \eta^{(2)}) \arrow[r, shift left=2, "pr_2"] \arrow[r, "m"] \arrow[r, shift right=2, "pr_1"'] & (\mathcal{G}, \eta^{(1)}) \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & (M, \eta^{(0)}) \\
\circlearrowleft S_3 & \circlearrowleft S_2 &
\end{tikzcd}$$

Now fix orbit $\mathcal{O}$. Then we have the exponential maps of the metrices, which because of properties above give commutative diagram:

$$\begin{tikzcd}
\nu(\mathcal{G}_{\mathcal{O}}^{(2)}) \cong \nu(\mathcal{G}_{\mathcal{O}})^{(2)} \arrow[r, "\exp^{\eta^{(2)}}"] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & \mathcal{G}^{(2)} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\nu(\mathcal{G}_{\mathcal{O}}) \arrow[r, "\exp^{\eta^{(1)}}"'] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\
\nu(\mathcal{O}) \arrow[r, "\exp^{\eta^{(0)}}"'] & M
\end{tikzcd}$$

(Defined only on neighborhoods of zero sections of ν)

By restricting to domains of injectivity, we obtain groupoid isom with $\mathcal{G}|_U$, for $\mathcal{O} \subset U = \exp^{\eta^{(0)}}(V)$, $\mathcal{O} \subset V \subset \nu(\mathcal{O})$. So this shows:

<u>Thm</u> If a groupoid $\mathcal{G} \rightrightarrows M$ admits a 2-metric, then it can be linearized around an orbit.

The linearization of proper groupoids now follows from:

Thm

Every proper groupoid admits a 2-metric

The proof is by averaging, using the facts:

- Proper groupoids admit invariant volume forms/densities
- Proper groupoids admit invariant partitions of unity.

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Historical Remarks:

- Linearization of proper groupoids was conjectured by A. Weinstein (2002). He also showed it was enough to prove the case where $G = \{x\}$
- N.T. Zung (2006) gave a proof of the case $G = \{e\}$ (but proof has a gap) using analytic methods (iteration scheme on Banach Spaces)
- M. Crainic & I. Struchiner (2013) gave a complete proof using vanishing of deformation cohomology for proper groupoids.
- Groupoid metrics were introduced by del Hoyo & RLF (2018) and gave the geometric proof sketched above.

Effective Groupoids vs Pseudo Groups

For a manifold M let

$$\mathrm{Diff}_{loc}(M) = \{ \phi : U \to V \mid U, V \subset M \text{ open}, \ \phi \text{ diffeo} \}$$

Rmk: These are diffeos that are locally defined, not smooth maps that are local diffeos!

Def: A <u>pseudogroup</u> on a manifold M is a collection $\Psi \subset \mathrm{Diff}_{loc}(M)$ satisfying:

(i) $\phi, \varphi \in \Psi$, $\mathrm{Im}\,\varphi \subset \mathrm{Dom}(\phi) \Rightarrow \phi \circ \varphi \in \Psi$

(ii) $id_M \in \Psi$

(iii) $\phi \in \Psi \Rightarrow \phi^{-1} \in \Psi$

(iv) $\phi \in \Psi$, $U \subset \mathrm{Dom}(\phi)$ open $\Rightarrow \phi|_U \in \Psi$

(v) $\phi \in \mathrm{Diff}_{loc}(M)$, $\{U_i\}_{i\in I}$ is open cover of $\mathrm{Dom}(\phi)$ and $\phi|_{U_i} \in \Psi \Rightarrow \phi \in \Psi$

<u>Rmk.</u> If a subset $P \subset \mathrm{Diff}_{loc}(M)$ satisfies (i)-(iii), one obtains a pseudogroup generated by P by imposing (iv) & (v).

Cartan introduced pseudogroups as ∞-dim generalizations of Lie groups. We will see that they are essentially equivalent to effective étale groupoids.

<u>Examples:</u>

1) $\mathrm{Diff}_{loc}(M)$ is a pseudo-group.

2) Given a Riemannian manifold (M, η), the set of all local isometries is pseudo-group. Similarly for any geometric structure (symplectic structure, cplx str., etc.)

3) Given an effective étale groupoid $\mathcal{G} \rightrightarrows M$ the set

$$\Psi(\mathcal{G}) := \{ t \circ b \mid b : U \to \mathcal{G} \text{ local bisection} \}$$

is a pseudogroup. Note that we need the effective condition to be able to "glue" bisections: if $\phi_i = t \circ b_i : U_i \to V_i$ and $U_1 \cap U_2 \neq \emptyset$ then $b_1 = b_2$ on $U_1 \cap U_2$, by effectiveness.

Hence,

$$b(p) = \begin{cases} b_1(p), & p \in U_1 \\ b_2(p), & p \in U_2 \end{cases}$$

is a smooth bisection, and $\phi = t \circ b \in \Psi(\mathcal{G})$ satisfies $\phi|_{U_i} = \phi_i$.

4) If $\mathcal{G} \rightrightarrows M$ is any groupoid

$$P := \{ t \circ b \mid b : U \to \mathcal{G} \text{ local bisection} \}$$

satisfies (i)-(iv) but not (v). It generates a pseudogroup which will be denoted by $\Psi(\mathcal{G})$.

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Given a pseudogroup $\Psi \subset \mathrm{Diff}_{loc}(M)$ we can associate to it the groupoid of germs of Ψ:

$$\Gamma(\Psi) = \{ \mathrm{germ}_p(\phi) : \phi \in \Psi, \ p \in \mathrm{Dom}(\phi) \}$$

With the sheaf topology this is an effective étale groupoid (perhaps not 2nd countable). One checks easily:

- For any pseudogroup Ψ: $\Psi(\Gamma(\Psi)) = \Psi$
- F any étale grpd $\mathcal{G}$: $\Gamma(\Psi(\mathcal{G})) = \mathrm{Eff}(\mathcal{G})$

In particular:

Proposition.

There is a 1:1 correspondence between effective étale groupoids (not ness 2nd countable) and pseudogroups:

$$\mathcal{G} \longmapsto \Psi(\mathcal{G})$$

whose inverse is:

$$\Gamma(\Psi) \longmapsfrom \Psi$$

Given manifolds M & N and pseudogroups $\Psi \subset \mathrm{Diff}_{loc}(M)$ $\Phi \subset \mathrm{Diff}_{loc}(N)$ an equivalence from Ψ to Φ is a collection

$$E = \{ h : U \to V \mid U \subset M, V \subset N \text{ open}, h \text{ diffeo} \}$$

satisfying:

(i) $\bigcup_{h \in E} \mathrm{Dom}(h) = M$, $\bigcup_{h \in E} \mathrm{Im}\, h = N$

(ii) $\psi \in \Psi, \phi \in \Phi, h_1, h_2 \in E \Rightarrow \begin{cases} h_1 \circ \psi \circ h_2^{-1} \in \Phi \\ h_1^{-1} \circ \phi \circ h_2 \in \Psi \end{cases}$

(iii) E is maximal among collections satisfying (i) & (ii)

Proposition

Two effective étale groupoids are Morita equivalent iff the corresponding pseudogroups are equivalent.

Proof: Exercise. $\blacksquare$