MATH 595 - Lecture 24

Recall our aim:

- Which groupoids represent orbifolds?

For example, we saw above that manifolds are represented by surjective submersions.

The groupoids representing orbifolds that we studied all had discrete (in fact, finite) isotropy. We will show that groupoids with finite isotropy are Morita equivalent to:

<u>Def:</u> A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called <u>étale</u> if $\dim \mathcal{G} = \dim M$.

Note that:

$\mathcal{G}$ is étale $\iff$ s local diffeo

$\iff$ t local diffeo

$\iff$ u local diffeo

$\mathcal{G}$ étale $\Rightarrow$ $\left\{\begin{array}{l}\text{Fibers of } s \ \& \ t \\ \text{isotropy groups} \\ \text{orbits}\end{array}\right.$ are all discrete

<u>Examples:</u>

1) Any manifold $M \rightrightarrows M$. Note a submersion groupoid is not étale if fibers of submersion are not discrete. So "étale" is not Morita invariant

2) Any discrete group $G \rightrightarrows \{*\}$ or any action groupoid of a discrete group is étale

3) For any foliation $(M, \mathcal{F})$ and any transversal T to $\mathcal{F}$, the restrictions $\mathrm{Hol}(M,\mathcal{F})|_T \rightrightarrows T$ and $\Pi_1(M,\mathcal{F})|_T \rightrightarrows T$ are étale groupoids.

4) For any manifold M, one can form the groupoid $\Gamma(M) \rightrightarrows M$ of germs of local diffeos: the topology on $\Gamma(M)$ is the sheaf topology, i.e., the topology that makes source map discrete. Hence, $\Gamma(M)$ is étale groupoid (but space of arrows is not 2^{nd} countable!)

<u>Rmk:</u> $\Gamma(\mathbb{R}^q)$ is often called the <u>Haefliger groupoid</u>.

If $\dim M = q$, $\Gamma(M)$ & $\Gamma(\mathbb{R}^q)$ are Morita equivalent.

"Étale" is not preserved under Morita equivalence. But:

<u>Thm</u>

A groupoid is Morita equivalent to an étale groupoid iff all its isotropy groups are discrete.

<u>Proof.</u>

Since Morita equivalent gpds have isomorphic isotropy Lie groups, discreteness is preserved under Morita equivalence. Hence

$$\mathcal{G} \underset{M}{\simeq} \mathcal{E} \text{ étale} \Rightarrow \mathcal{G} \text{ has discrete isotropy}$$

For the converse, observe that if $\mathcal{G}$ has discrete isotropy then $\dim \mathcal{O}_m = \dim \mathfrak{s}^{-1}(m)$, so connected components of orbits form a foliation. Restricting to a complete transversal T to orbit foliation, we have $\mathcal{G} \underset{M}{\simeq} \mathcal{G}_T$, $\mathcal{G}_T$ étale. ∎

Assume $\mathcal{G} \rightrightarrows M$ is étale:

$$\mathrm{Eff}: \mathcal{G} \longrightarrow \Gamma(M), \quad g \longmapsto \mathrm{germ}_{s(g)}(t \circ (s|_U)^{-1})$$

w/ $g \in U \subset \mathcal{G}$ open such that $s|_U : U \to s(U)$ is diffeo

Note that Eff is a Lie groupoid morphism

Def. An effective étale groupoid is an étale groupoid for which Eff is injective.

Rmk: For any étale groupoid $\mathrm{Eff}(\mathcal{G}) \subset \Gamma(\mathcal{G})$ is an open subgroupoid which is effective. It is called the effect of $\mathcal{G}$

Proposition If $\mathcal{G}_1$ & $\mathcal{G}_2$ are étale groupoids which are Morita equivalent then $\mathcal{G}_1$ is effective iff $\mathcal{G}_2$ is effective

Proof: Exercise.

∎

Examples

1) If G is a discrete group an action $G \circlearrowright M$ is effective iff the action groupoid $G \ltimes M \rightrightarrows M$ is effective. More generally, given an action $G \circlearrowright M$ with discrete isotropy, it is effective iff for any complete transversal T to the orbit foliation $(G \ltimes M)|_T \rightrightarrows T$ is effective.

2) If $(M, \mathcal{F})$ is a foliation and $T \subset M$ is a complete transversal the effect of $\Pi_1(M, \mathcal{F})|_T \rightrightarrows T$ is $\mathrm{Hol}(M, \mathcal{F})|_T \rightrightarrows T$. In fact

$$\mathrm{Eff}([\gamma]) = \mathrm{hol}^{T,T}(\gamma)$$

In particular, $\mathrm{Hol}(M, \mathcal{F})|_T \rightrightarrows T$ is an effective, étale groupoid.

Def: A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called <u>proper</u> if it is Hausdorff and $t \times s : \mathcal{G} \to M \times M$ is proper.

<u>Convention:</u> When we say "$f : M \to N$" is proper, then it is assumed that both M & N are Hausdorff.

<u>Rmk.</u> If $f : M \to N$ is map between Hausdorff spaces then

$$f \text{ proper} \iff \begin{cases} \text{Any sequence } \{x_n\} \subset M \text{ such that } f(x_n) \text{ converges} \\ \text{has a convergent subsequence} \end{cases}$$

In particular, f proper $\Rightarrow$ f closed

<u>Exercise:</u> Show that for a (Hausdorff) groupoid $\mathcal{G}$:

$$\mathcal{G} \text{ compact} \iff \mathcal{G} \text{ proper } \& \ M \text{ is compact}$$

<u>Exercise:</u> A groupoid $\mathcal{G} \rightrightarrows M$ is called s-proper if $s : \mathcal{G} \to M$ is a proper map. Show that:

$$\mathcal{G} \text{ s-proper} \Rightarrow \mathcal{G} \text{ proper } \& \text{ all orbits are compact}$$

The converse holds for a s-connected groupoid $\mathcal{G}$

<u>Hint:</u> A submersion w/ compact, connected fibers is proper.

$$\mathcal{G} \text{ compact} \underset{\not\Leftarrow}{\Rightarrow} \mathcal{G} \text{ s-proper} \underset{\not\Leftarrow}{\Rightarrow} \mathcal{G} \text{ proper}$$

<u>Examples</u>

1. An action $G \curvearrowright M$ is proper iff the action groupoid $G \ltimes M \rightrightarrows M$ is proper. $G \ltimes M \rightrightarrows M$ is s-proper iff G is compact

2. Given submersion $\phi : M \to N$, the groupoid $M \times_N M \rightrightarrows M$ is always proper. It is s-proper iff ϕ is proper.

3. Given foliation $(M,\mathcal{F})$, the groupoid $\mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ is S-proper iff $(M,\mathcal{F})$ has compact leaves with finite holonomy.

Basic Properties. If $\mathcal{G} \rightrightarrows M$ is a proper groupoid then:

(i) All isotropy groups are compact

(ii) All orbits are closed, embedded, submanifolds

(iii) Orbit space is Hausdorff

Proof:

(i) $\mathcal{G}_x = (s\times t)^{-1}(x,x)$ and $\{(x,x)\} \subset M\times M$ is compact.

(ii) Recall that $\mathcal{O}_x$ is the immersed submanifold

$$i : s^{-1}(x)/\mathcal{G}_x \hookrightarrow M \ , \quad g\mathcal{G}_x \mapsto t(g)$$

We claim this immersion is proper, so result follows. Let $i(g_n \mathcal{G}_n) = x_n \to x_\infty$ be a convergent sequence. Since $K = \{(x_n, x) : n\in \mathbb{N}\} \cup \{(x_\infty, x)\} \subset M\times M$ is compact, $\{g_n\} \subset (t\times s)^{-1}(K)$ has convergent subsequence: $g_{n_i} \to g_\infty$. But $s(g_\infty) = \lim s(g_n) = x$, so $g_{n_i}\mathcal{G}_x \in s^{-1}(x)/\mathcal{G}_x$ is convergent subsequence, showing that i is proper.

(iii) The projection $\pi : M \to M/\mathcal{G}$ is an open map (for any Lie grpd). We show that any distinct $\mathcal{O}_{x_1}, \mathcal{O}_{x_2} \in M/\mathcal{G}$ can be separated.

Assume not. Then for any m, letting $U_{1,m} := B_{1/m}(x_1)$, $U_{2,m} := B_{1/m}(x_2)$ we have

$$\pi(U_{1,m}) \cap \pi(U_{2,m}) \neq \phi$$

In other words, there exists $g_m \in \mathcal{G}$ w/ $s(g_m) \in U_{1,m}$ & $t(g_m) \in U_{2,m}$. Then $(t\times s)(g_m) \to (x_2, x_1)$. By properness, there exists a convergent subsequence $g_{m_i} \to g_\infty$. But $s(g_\infty) = x_1$, $t(g_\infty) = x_2$ contradicting $\mathcal{O}_{x_1} \cap \mathcal{O}_{x_2} = \phi$. ∎

Proposition: Hausdorff and proper are Morita invariants

We use the following lemma whose proof is left as an exercise:

Lemma. Consider a (good) pull back
$$\begin{tikzcd} M' \arrow[r] \arrow[d, "g"'] & N' \arrow[d, "f"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$
and assume that M & N are Hausdorff. Then:

(i) N' Hausdorff $\Rightarrow$ M' Hausdorff

(ii) f proper $\Rightarrow$ g proper

When h is surjective submersion, the reverse implications hold.

Proof: If $\Phi: \mathcal{G}_1 \to \mathcal{G}_2$ is Morita map w/ $\phi: M_1 \to M_2$ surjective submersion, apply lemma to pullback diagram:
$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "\Phi"] \arrow[d, "t\times s"'] & \mathcal{G}_2 \arrow[d, "t\times s"] \\ M_1\times M_1 \arrow[r, "\phi\times\phi"'] & M_2\times M_2 \end{tikzcd}$$
To conclude that $\mathcal{G}_1$ is Hausdorff/proper iff $\mathcal{G}_2$ is Hausdorff/proper.

Given any Morita equivalence

$$\begin{tikzcd}[column sep=small] & \mathcal{H} \arrow[dl, "s"'] \arrow[dr, "\sim"] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \Rightarrow \begin{tikzcd}[column sep=small] & \mathcal{G}_1 \times_{\mathcal{G}_1} \mathcal{H} \arrow[dl, "pr_1"', "s"] \arrow[dr, "\Phi\circ pr_2", "\sim"'] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd} \Rightarrow \begin{tikzcd}[column sep=small] & \mathcal{G}_1 \times_{\mathcal{G}_1} \mathcal{H} \times_{\mathcal{G}_2} \mathcal{G}_2 \arrow[dl, "pr_1"', "s"] \arrow[dr, "pr_2", "\sim"'] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd}$$

surjective left leg — Both legs surjective

So apply first part of proof. ∎

RMK: "s-proper" is not Morita invariant: if $G \curvearrowright M$ is proper & free action of a non-compact Lie group w/ quotient $N = M/G$ then $(G \ltimes M \rightrightarrows M) \simeq (N \rightrightarrows N)$.

Recall that if $\mathcal{O}$ is an orbit of $\mathcal{G} \rightrightarrows M$ then

$$\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O} \curvearrowright \nu(\mathcal{O})$$

The action groupoid $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O})$ is called the linear local model for $\mathcal{G}$ around $\mathcal{O}$.

More explicit description:

- $s^{-1}(x) \xrightarrow{t} \mathcal{O}$ is principal $\mathcal{G}_x$-bundle
- $\mathcal{G}_x \curvearrowright \nu_x(\mathcal{O})$

$\Rightarrow \nu(\mathcal{O}) \simeq (s^{-1}(x) \times \nu_x(\mathcal{O}))/\mathcal{G}_x$

- $\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \simeq (s^{-1}(x) \times s^{-1}(x) \times \nu_x(\mathcal{O}))/\mathcal{G}_x$

(Quotient of (pair groupoid $\times$ vector space))

Examples:

1) For a Lie group action $G \ltimes M \rightrightarrows M$, fixing an orbit $\mathcal{O} = G \cdot x$

$$\nu(\mathcal{O}) \simeq (G \times \nu_x(\mathcal{O}))/G_x = M^{\text{lin}}$$

This has a G-action, and we find:

$$\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}) \rightrightarrows \nu(\mathcal{O}) \simeq G \ltimes M^{\text{lin}} \rightrightarrows M^{\text{lin}}$$

2) For a foliation $(M, \mathcal{F})$, fixing a leaf L, we find:

$$\nu(L) \simeq (L^h \times \nu_x(L))/\mathrm{Hol}_x(L) = M^{\text{lin}}$$

This has a linear foliation $\mathcal{F}^{\text{lin}}$ and we find

$$\mathrm{Hol}(M, \mathcal{F})|_L \ltimes \nu(L) \rightrightarrows \nu(L) \simeq \mathrm{Hol}(M^{\text{lin}}, \mathcal{F}^{\text{lin}}) \rightrightarrows M^{\text{lin}}$$

3) For a submersion $\phi : M \to N$, the orbits of $M \times_N M \rightrightarrows M$ are the fibers of ϕ. Fix a fiber $\mathcal{O} = \phi^{-1}(y)$. Then

- $\nu(\mathcal{O}) \simeq \phi^*(T_y N) = \phi^{-1}(y) \times T_y N \to \phi^{-1}(y)$
- $\mathcal{G}|_{\mathcal{O}} = \phi^{-1}(y) \times \phi^{-1}(y) \rightrightarrows \phi^{-1}(y)$

The local model is just the direct product groupoid:

$$\phi^{-1}(y) \times \phi^{-1}(y) \times T_y N \rightrightarrows \phi^{-1}(y)$$

Linearization Thm (Zung & Weinstein)

Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid and fix $\mathcal{O} \subset M$. There exist open sets $\mathcal{O} \subset U \subset M$, $\mathcal{O} \subset V \subset \nu(\mathcal{O})$ and a groupoid isomorphism

$$\mathcal{G}|_U \simeq (\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}))|_V$$

Corollary (Invariant Linearization Thm)

Let $\mathcal{G} \rightrightarrows M$ be a s-proper groupoid and fix $\mathcal{O} \subset M$. There exist open saturated neighborhoods $\mathcal{O} \subset U \subset M$ and $\mathcal{O} \subset V \subset \nu(\mathcal{O})$ and a groupoid isomorphism

$$\mathcal{G}|_U \simeq (\mathcal{G}_{\mathcal{O}} \ltimes \nu(\mathcal{O}))|_V$$

Proof of Corollary:

Every s-proper groupoid is proper

Orbits of s-proper groupoids are stable: every neighborhood of an orbit contains a saturated neighborhood.

$\blacksquare$

These results generalize several well-known theorems:

- Reeb stability theor.
- Linearization of proper actions
- Ehresmann Theorem (proper submersions are locally trivial)

Corollary: Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid. Every orbit $\mathcal{O} \subset M$ has saturated neighborhood U such that

$$(\mathcal{G}|_U \rightrightarrows U) \underset{M}{\simeq} (\mathcal{G}_x \ltimes V_x \rightrightarrows V_x)$$

where $0 \in V_x \subset \nu_x(\mathcal{O})$ is $\mathcal{G}_x$-invariant neighborhood.

Proof of Corollary

Fix $x \in \mathcal{O}$ and choose $T \subset M$ a transverse submanifold to $\mathcal{O}$ through x:

$$T_x M = T_x \mathcal{O} \oplus T_x T$$

If T is small enough, then $T \pitchfork$ to every orbit it meets and $T \cap \mathcal{O} = \{x\}$ (since $\mathcal{O}$ is embedded). It follows that:

- $\mathcal{G}|_T \rightrightarrows T$ is a proper Lie Groupoid with orbit $\{x\}$

By linearization Thm, eventually after shrinking T, we can assume:

$$\mathcal{G}|_T \simeq (\mathcal{G}_x \ltimes T_x(T))|_V$$

Choose $\mathcal{G}_x$-inv metric, we see we can choose $V = V_x$ a $\mathcal{G}_x$-invariant neighborhood, so:

$$\mathcal{G}|_T \simeq \mathcal{G}_x \ltimes V_x$$

Finally, observe that:

- $\mathcal{G}|_T \underset{M}{\simeq} \mathcal{G}_U$, where $U = \pi^{-1}(\pi(T))$ is open saturated $(\pi: M \to M/\mathcal{G})$

∎

Corollary: Let $\mathcal{G} \rightrightarrows M$ be a proper étale groupoid. Every $x \in M$ has a neighborhood U such that:

$$\mathcal{G}|_U \simeq \mathcal{G}_x \ltimes V_x$$

where $0 \in V_x \subset T_x M$ is a $\mathcal{G}_x$-invariant neighborhood

Proof: $\mathcal{G}$ étale $\Rightarrow$ discrete orbits $\Rightarrow$ every $x \in M$, has neighborhood $\tilde{U}$ with $\tilde{U} \cap \mathcal{O}_x = \{x\}$. Apply linearization Thm. to $\mathcal{G}|_{\tilde{U}}$

∎

Note: One can prove this corollary directly. Not hard.