

## MATH 595 - LECTURE 23

Last time: Different notions of equivalence

A Lie groupoid morphism  $\begin{array}{ccc} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$  is called a:

- Isomorphism:  $\exists \Phi: \mathcal{H} \rightarrow \mathcal{G}$  w/  $\Phi \circ \Phi = \text{id}_{\mathcal{G}}$ ,  $\Phi \circ \Phi = \text{id}_{\mathcal{H}}$
- Strong equivalence:  $\exists \Phi: \mathcal{H} \rightarrow \mathcal{G}$  w/  $\Phi \circ \Phi \stackrel{\simeq}{\sim} \text{id}_{\mathcal{G}}$ ,  $\Phi \circ \Phi \stackrel{\simeq}{\sim} \text{id}_{\mathcal{H}}$
- Morita map:

(i) Ess. surjective:  $\text{top}_{\mathbb{R}^2}: \mathcal{H} \times_N M \rightarrow N$  is surjective submersion

(ii) Fully Faithful:  $\begin{array}{ccc} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\ \text{txs} \downarrow & & \downarrow \text{txs} \\ M \times M & \xrightarrow{\phi \times \phi} & N \times N \end{array}$  is (good) pullback

We saw that:

Isomorphism  $\Rightarrow$  strong equivalence  $\Rightarrow$  Morita map

Morita is the RIGHT notion of equivalence.

### Theorem (del Hoyo)

A groupoid morphism  $\Phi: \mathcal{G} \rightarrow \mathcal{H}$  is a Morita map iff it preserves transversal data

Proof:

$$(i) \quad \Phi \text{ fully faithful} \Rightarrow \begin{cases} \mathcal{G}_\alpha \simeq \mathcal{H}_{\Phi(\alpha)} \\ M/\mathcal{G} \rightarrow N/\mathcal{H} \text{ injective} \\ d_\alpha^\sharp: \nu_\alpha(\mathcal{O}) \rightarrow \nu_{\Phi(\alpha)}^\sharp(\Phi(\mathcal{O})) \text{ injective} \end{cases}$$

Since  $\Phi$  is categorical equivalence (set theoretically) first two follow

For last one, use the pullback diagram:

$$\begin{array}{ccc}
 T_{u(\alpha)} \mathcal{G} & \xrightarrow{d\bar{\Phi}} & T_{u(\phi(\alpha))} \mathcal{H} \\
 dt \times ds \downarrow & & \downarrow dt \times de \\
 T_x M \times T_x M & \xrightarrow{d\phi \times d\phi} & T_{\phi(x)} N \times T_{\phi(x)} N
 \end{array}$$

$$\left. \begin{array}{l} v \in T_x M \\ d\phi(v) \in T_{\phi(x)} \phi(0) \end{array} \right\} \Rightarrow (0, d\phi(v)) \in \text{Im}(dt \times ds) \Rightarrow (0, v) \in \text{Im}(dt \times ds) \Rightarrow v \in T_x 0$$

$$(ii) \quad \bar{\Phi} \text{ is surjective} \Leftrightarrow \begin{cases} M/\mathcal{G} \rightarrow N/\mathcal{H} \text{ surjective \& open} \\ d_x \bar{\Phi} : \nu_x(0) \rightarrow \nu_{\phi(x)}(\phi(0)) \text{ surjective} \end{cases}$$

For first item note that we have diagram

$$\begin{array}{ccc}
 \mathcal{H} \times M & \xrightarrow{\text{top}_\alpha} & N \\
 \downarrow \text{open, surjective} & & \downarrow \text{open, surjective} \\
 M/\mathcal{G} & \longrightarrow & N/\mathcal{H}
 \end{array}$$

Hence:

$$\text{top}_\alpha \text{ surjective/open} \Leftrightarrow M/\mathcal{G} \rightarrow N/\mathcal{H} \text{ surjective/open}$$

On the other hand, the diagram:

$$\begin{array}{ccccc}
 T(\mathcal{H} \times M) & \xrightarrow{d\text{top}_\alpha} & T_h \mathcal{H} & \xrightarrow{dt} & T N \\
 \downarrow d\text{top}_\alpha & & \downarrow ds & & \downarrow d\lambda_{\phi(x)} \\
 T_x M & \xrightarrow{d\phi} & T_{\phi(x)} N & \longrightarrow & \nu_{\phi(x)}(\phi(0))
 \end{array}$$

Shows that

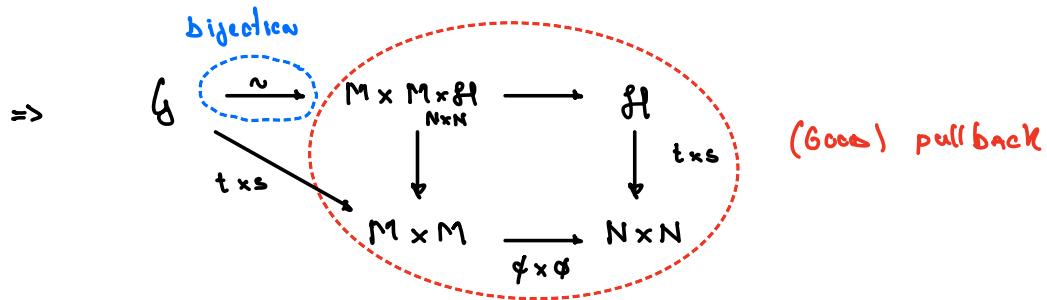
$$\begin{aligned}
 d_x(\text{top}_\alpha) \text{ surjective} &\Leftrightarrow d\phi : T_x M \rightarrow \nu_{\phi(x)}(\phi(0)) \text{ surjective} \\
 &\Leftrightarrow d\phi : \nu_x(0) \rightarrow \nu_{\phi(x)}(\phi(0)) \text{ surjective}
 \end{aligned}$$

(iii)  $\bar{\Phi}$  preserves transversal data  $\Rightarrow \bar{\Phi}$  fully faithful

$$\left. \begin{array}{l} \mathcal{G}_\alpha \simeq \mathcal{H}_{\phi(\alpha)} \\ M/\mathcal{G} \simeq N/\mathcal{H} \end{array} \right\} \Rightarrow \bar{\Phi} : \mathcal{S}'(\alpha) \cap \mathcal{E}'(\alpha) \rightarrow \mathcal{S}'(\phi(\alpha)) \cap \mathcal{E}'(\phi(\alpha)) \text{ is diffeo}$$

Groupsoids and their formal set theoretical pullback

$$d_*\phi: U_x(\mathcal{O}) \rightarrow U_{\phi(x)}(\phi(\mathcal{O})) \text{ surjective} \Rightarrow \begin{cases} \phi \times \phi: M \times M \rightarrow N \times N \\ t \times s: \mathcal{F} \rightarrow N \times N \end{cases} \text{ transverse maps}$$



It remains to check that diff. of  $G \rightarrow M \times M \times \mathcal{F}$  is bijective

This follows from diagram chasing:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_{\delta}(\tilde{t}^{-1}(s) \cap \tilde{s}^{-1}(x)) & \longrightarrow & T_{\delta} G & \xrightarrow{d t \times d s} & T_x M \times T_y M \longrightarrow V_x(G) \longrightarrow 0 \\ & & d_{\delta} \tilde{\Phi} \downarrow s_1 & & d_{\delta} \tilde{\Phi} \downarrow & & \phi \times \phi \downarrow & & s_1 \downarrow d\phi \\ 0 & \longrightarrow & T_{\phi(y)}(\tilde{t}^{-1}(\phi(s)) \cap \tilde{s}^{-1}(\phi(x))) & \longrightarrow & T_{\phi(y)} \mathcal{F} & \xrightarrow{d t \times d s} & T_{\phi(x)} N \times T_{\phi(y)} N \longrightarrow V_{\phi(x)}(\phi(\mathcal{O})) \longrightarrow 0 \end{array}$$

□

### Examples

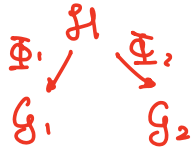
1) If  $G \rightrightarrows M$  is a transitive Lie groupoid, then for any  $p \in M$ ,  $G_p \hookrightarrow G$  is a Morita map:  $M/G = \{*\}$  &  $U_p(M) = 0$

2) If  $G \rightrightarrows M$  is any groupoid and  $T \subset M$  is a submanifold that intersects transversally every orbit of  $G$ , then  $G|_T \rightrightarrows T$  is a Lie groupoid and  $G|_T \hookrightarrow G$  is a Morita map:

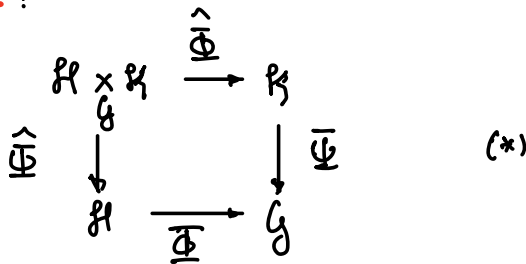
- $(\pi, \tilde{\Phi})$  foliation:  $\text{Hol}(\pi, \tilde{\Phi})|_T \rightrightarrows T \hookrightarrow \text{Hol}(\pi, \tilde{\Phi}) \rightrightarrows M$  is Morita.
- $G \times \pi \rightrightarrows M$  action:  $(G \times \pi)|_T \rightrightarrows T \hookrightarrow (G \times \pi) \rightrightarrows M$  is Morita.

3) Composition of Morita maps is a Morita map

Def:  $G_1 \cong M_1$  &  $G_2 \cong M_2$  are Morita equivalent if  $\exists \mathcal{H} \cong M$   
& Morita maps



To prove this is an equivalence relation, we need **WEAK**  
**pullback of groupoids:**

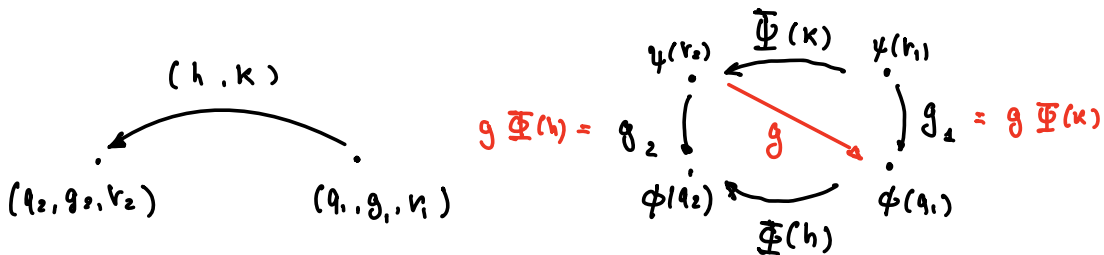


Where:

- Objects of  $\mathcal{H} \times_{\mathcal{G}} \mathcal{H} := \mathcal{Q} \times_{\mathcal{N}} \mathcal{G} \times_{\mathcal{M}} \mathcal{R}$ , i.e.:  
 $(q, g, r)$  with  $\begin{cases} \psi(r) = s(g) \\ \phi(q) = t(g) \end{cases}$

- Arrows of  $\mathcal{H} \times_{\mathcal{G}} \mathcal{H}$ : Arrow from  $(q_1, g_1, r_1)$  to  $(q_2, g_2, r_2)$  is  
a pair  $(h, k)$  such that

$$\Phi(h)g_1 = g_2\Psi(k) \quad (**)$$



Hence, we can also identify arrows w/  $(h, g, k) \in \mathcal{H} \times_{\mathcal{M}} \mathcal{G} \times_{\mathcal{N}} \mathcal{H}$  with:

$$\begin{cases} s(g) = \psi(t(k)) \\ t(g) = \phi(s(h)) \end{cases}$$

The diagram  $(*)$  is not commutative:  $(**)$  says that  $\text{id}: \mathcal{G} \rightarrow \mathcal{G}$   
is a natural isomorphism between two objects of  $(*)$ . In other words,  
The diagram  $(*)$  is weakly commutative.

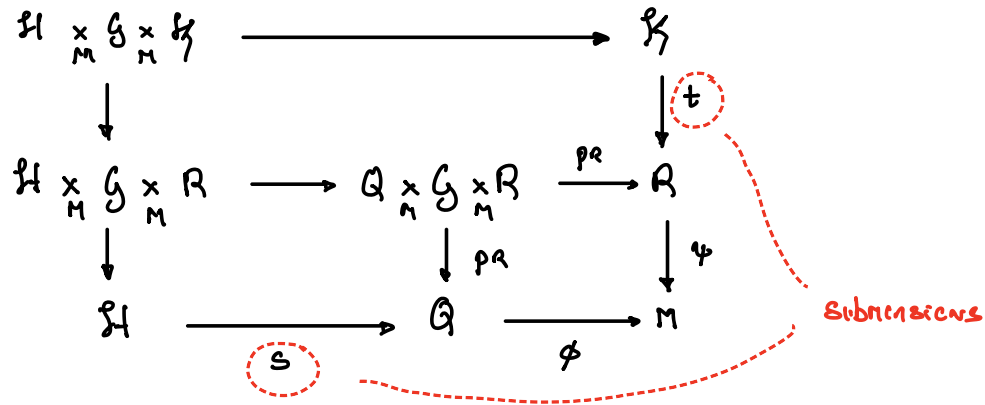
In general, weak pullback is not smooth.

Proposition

If the fibred product  $Q \times_M^G \times_M R$  exists, then the weak pullback is a Lie groupoid. This is the case, e.g., if  $\widehat{\Phi}$  (resp.,  $\Phi$ ) is a Morita map, and in this case the base change of  $\widehat{\Phi}$  (resp.,  $\Phi$ ) is a Morita map.

Proof:

The first part follows from diagram:



For the second part, observe that if  $\Phi$  is Morita, then

$$t \circ p_{R_2} : Q \times_M^G \times_M R \rightarrow M \quad \Rightarrow \quad \begin{array}{ccc} Q \times_M^G \times_M R & \rightarrow & R \\ \downarrow & & \downarrow \psi \\ Q \times_M^G & \xrightarrow{t \circ p_{R_2}} & M \end{array} \quad \begin{array}{l} \text{cos} \\ \text{pullback} \end{array}$$

is submersive

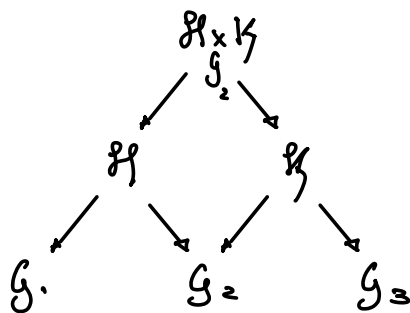
so weak pullback is smooth by first part.

Exercise: Verify that  $\widehat{\Phi}$  is Morita. □

Corollary: Morita Equivalence is an equivalence relation

Proof: Reflexive and symmetric is obvious.

For transitive, by Proposition, we can use weak pullback:



and the fact that composition of Morita maps is Morita.

□

### Examples:

1) A Lie groupoid  $\mathfrak{G} \rightrightarrows M$  is Morita equivalent to a manifold  $N \rightrightarrows N$  iff  $\exists$  surjective submersion  $\phi: M \rightarrow N$  & a groupoid isomorphism  $\mathfrak{G} \cong M \times_N M$ .

2) An action groupoid  $G \ltimes M \rightrightarrows M$  is Morita equivalent to a manifold  $N \rightrightarrows N$  iff the action is proper & free.

3) A Lie groupoid  $\mathfrak{G} \rightrightarrows M$  is Morita equivalent to a Lie group  $G$  iff it is transitive and any of its isotropy groups is isomorphic to  $G$ .

4) A Lie groupoid  $\mathfrak{G} \rightrightarrows M$  is Morita equivalent to a discrete group iff it is transitive & any of its isotropy groups is discrete iff  $t \times s: \mathfrak{G} \rightarrow M \times M$  is a covering.

—————/—————

We will discuss later what properties of a Lie groupoid are invariant under Morita Equivalences.

Morita Equivalence can also be seen as isomorphisms in a certain category:

Def. A generalized morphism  $\Phi/\underline{\Phi} : \mathcal{G} \dashrightarrow \mathcal{H}$  is given by a pair of Lie algebra morphisms:

$$\begin{array}{ccc} \Phi & \xrightarrow{s} & \Psi \\ \mathcal{G} & & \mathcal{H} \end{array} \quad \text{w/ } \underline{\Phi} \text{ Morita map.}$$

Two generalized morphisms  $\frac{\Phi_1}{\underline{\Phi}_1}, \frac{\Phi_2}{\underline{\Phi}_2} : \mathcal{G} \dashrightarrow \mathcal{H}$  are identical

$$\Phi_1/\underline{\Phi}_1 = \Phi_2/\underline{\Phi}_2$$

if there is a third generalized morphism  $\Phi_3/\underline{\Phi}_3 : \mathcal{G} \dashrightarrow \mathcal{H}$

fitting into a weakly commutative diagram:

$$\begin{array}{ccccc} \mathcal{G} & \xleftarrow{\sim} & \mathcal{K}_1 & \xrightarrow{\Psi_1} & \mathcal{H} \\ \parallel & & \uparrow & & \parallel \\ \mathcal{G} & \xleftarrow{\sim} & \mathcal{K}_3 & \xrightarrow{\Psi_3} & \mathcal{H} \\ \parallel & & \downarrow & & \parallel \\ \mathcal{G} & \xleftarrow{\sim} & \mathcal{K}_2 & \xrightarrow{\Psi_2} & \mathcal{H} \end{array}$$

(This is an equivalence relation on the set of gen. morphisms)

Morita equivalences  $\equiv$  generalized isomorphisms  
(invertible generalized morphisms)

Exercise: Show that

(i) every generalized map  $\frac{\Phi}{\underline{\Phi}} : \mathcal{G}_1 \dashrightarrow \mathcal{G}_2$  has a representative  $\frac{\Phi'}{\underline{\Phi}'}$  :  $\mathcal{G}_1 \dashrightarrow \mathcal{G}_2$ , where  $\underline{\Phi}'$  covers a surjective submersion

(ii) every Morita equivalence can be represented by a pair of Morita maps which are surjective submersions.

Hint: Given Morita map  $\Phi : \mathcal{H} \rightarrow \mathcal{G}$  consider weak pullback

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{G}} \mathcal{H} & \rightarrow & \mathcal{H} \\ s \downarrow & & s \downarrow \Phi \\ \mathcal{G} & \xrightarrow{id} & \mathcal{G} \end{array}$$