# MATH 595 - LECTURE 23

Last time: Different notions of equivalence

A Lie groupoid morphism
$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{H} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$
is called a:

- Isomorphism: $\exists\ \Psi: \mathcal{H} \to \mathcal{G}$ w/ $\Psi \circ \Phi = id_{\mathcal{G}}$, $\Phi \circ \Psi = id_{\mathcal{H}}$
- Strong Equivalence: $\exists\ \Psi: \mathcal{H} \to \mathcal{G}$ w/ $\Psi \circ \Phi \underset{\tau}{\simeq} id_{\mathcal{G}}$, $\Phi \circ \Psi \underset{\sigma}{\simeq} id_{\mathcal{H}}$
- Morita map:

  (i) Ess. surjective: $t \circ pr_1 : \mathcal{H} \times_N M \to N$ is surjective submersion

  (ii) Fully Faithful:
$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "t\times s"'] & \mathcal{H} \arrow[d, "t\times s"] \\ M\times M \arrow[r, "\phi\times\phi"'] & N\times N \end{tikzcd}$$
  is (good) pullback

We saw that:

Isomorphism $\Rightarrow$ strong equivalence $\Rightarrow$ Morita map

Morita is the right notion of equivalence:

**Theorem (del Hoyo)**

A groupoid morphism $\Phi: \mathcal{G} \to \mathcal{H}$ is a Morita map iff it preserves transversal data

**Proof:**

(i) $\Phi$ fully faithful $\Rightarrow$
$$\begin{cases} \mathcal{G}_x \simeq \mathcal{H}_{\phi(x)} \\ M/\mathcal{G} \to N/\mathcal{H} \text{ injective} \\ d_x\phi : \nu_x(\mathcal{O}) \to \nu_{\phi(x)}(\phi(\mathcal{O})) \text{ injective} \end{cases}$$

Since $\Phi$ is categorical equivalence (set theoretically) first two follow

For last one, use the pullback diagram:

$$\begin{tikzcd} T_{u(x)}\mathcal{G} \arrow[r, "d\Phi"] \arrow[d, "dt \times ds"'] & T_{u(\phi(x))}\mathcal{H} \arrow[d, "dt \times ds"] \\ T_xM \times T_xM \arrow[r, "d\phi \times d\phi"'] & T_{\phi(x)}N \times T_{\phi(x)}N \end{tikzcd}$$

$$\left.\begin{array}{l} v \in T_xM \\ d\phi(v) \in T_{\phi(x)}\phi(O) \end{array}\right\} \Rightarrow (0, d\phi(v)) \in \mathrm{Im}(dt \times ds) \Rightarrow (0,v) \in \mathrm{Im}(dt \times ds)$$

$$\Rightarrow v \in T_xO$$

(ii) $\Phi$ ess surjective $\Leftrightarrow \begin{cases} M/\mathcal{G} \to N/\mathcal{H} & \text{surjective \& open} \\ d_x\phi : \nu_x(O) \to \nu_{\phi(x)}(\phi(O)) & \text{surjective} \end{cases}$

For first item note that we have diagram:

$$\begin{tikzcd} \mathcal{H} \times_N M \arrow[r, "t \circ pr_1"] \arrow[d, "\text{open, surjective}"'] & N \arrow[d, "\text{open, surjective}"] \\ M/\mathcal{G} \arrow[r] & N/\mathcal{H} \end{tikzcd}$$

Hence:

$t \circ pr_1$ surjective/open $\Leftrightarrow$ $M/\mathcal{G} \to N/\mathcal{H}$ surjective/open

On the other hand, the diagram:

$$\begin{tikzcd} T_{(h,x)}(\mathcal{H} \times_N M) \arrow[r, "dpr_1"] \arrow[d, "dpr_2"'] & T_h\mathcal{H} \arrow[r, "dt"] \arrow[d, "ds"] & T_{t(h)}N \arrow[d, "d\lambda_{h^{-1}}"] \\ T_xM \arrow[r, "d\phi"'] & T_{\phi(x)}N \arrow[r] & \nu_{\phi(x)}(\phi(O)) \end{tikzcd}$$

shows that

$d_x(t \circ pr_1)$ surjective $\Leftrightarrow$ $d\phi : T_xM \to \nu_{\phi(x)}(\phi(O))$ surjective

$\Leftrightarrow$ $d\phi : \nu_x(O) \to \nu_{\phi(x)}(\phi(O))$ surjective

(iii) $\Phi$ preserves transversal data $\Rightarrow$ $\Phi$ fully faithful

$$\left.\begin{array}{l} \mathcal{G}_x \simeq \mathcal{H}_{\phi(x)} \\ M/\mathcal{G} \xrightarrow{\sim} N/\mathcal{H} \end{array}\right\} \Rightarrow \Phi : s^{-1}(x) \cap t^{-1}(x) \to s^{-1}(\phi(x)) \cap t^{-1}(\phi(x)) \text{ is diffeo}$$

Groupoid anchors form set theoretical pullback

$d_x\phi : \nu_x(\mathcal{O}) \to \nu_{\phi(x)}(\phi(\mathcal{O}))$ surjective $\Rightarrow \begin{cases} \phi\times\phi : M\times M \to N\times N \\ t\times s : \mathcal{H} \to N\times N \end{cases}$ transverse maps

bijection

$$\Rightarrow \quad \begin{tikzcd} \mathcal{G} \arrow[r, "\sim"] \arrow[dr, "t\times s"'] & M\times M \times_{N\times N} \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[d, "t\times s"] \\ & M\times M \arrow[r, "\phi\times\phi"'] & N\times N \end{tikzcd}$$

(Good) pullback

It remains to check that diff. of $\mathcal{G} \to M\times M\times_{N\times N}\mathcal{H}$ is bijective. This follows from diagram chasing:

$$\begin{tikzcd} 0 \arrow[r] & T_g(t^{-1}(y)\cap s^{-1}(x)) \arrow[r] \arrow[d, "d_g\Phi"', "\cong"] & T_g\mathcal{G} \arrow[r, "dt\times ds"] \arrow[d, "d_g\Phi"'] & T_xM\times T_yM \arrow[r] \arrow[d, "\phi\times\phi"'] & \nu_x(\mathcal{O}) \arrow[r] \arrow[d, "\cong"', "d\phi"] & 0 \\ 0 \arrow[r] & T_{\phi(g)}(t^{-1}(\phi(y))\cap s^{-1}(\phi(x))) \arrow[r] & T_{\phi(g)}\mathcal{H} \arrow[r, "dt\times ds"] & T_{\phi(x)}N\times T_{\phi(y)}N \arrow[r] & \nu_{\phi(x)}(\phi(\mathcal{O})) \arrow[r] & 0 \end{tikzcd}$$

$\blacksquare$

## Examples

1) If $\mathcal{G} \rightrightarrows M$ is a transitive Lie groupoid, then for any $p\in M$, $\mathcal{G}_p \hookrightarrow \mathcal{G}$ is a Morita map: $M/\mathcal{G} = \{*\}$ & $\nu_p(M) = 0$

2) If $\mathcal{G} \rightrightarrows M$ is any groupoid and $T\subset M$ is a submanifold that intersects transversally every orbit of $\mathcal{G}$, then $\mathcal{G}|_T \rightrightarrows T$ is a Lie groupoid and $\mathcal{G}|_T \hookrightarrow \mathcal{G}$ is a Morita map.:

- $(M,\mathcal{F})$ foliation: $\mathrm{Hol}(M,\mathcal{F})|_T \rightrightarrows T \hookrightarrow \mathrm{Hol}(M,\mathcal{F}) \rightrightarrows M$ is Morita
- $G\ltimes M \rightrightarrows M$ action: $(G\ltimes M)|_T \rightrightarrows T \hookrightarrow (G\ltimes M)\rightrightarrows M$ is Morita.

3) Composition of Morita maps is a Morita map

**Def:** $\mathcal{G}_1 \rightrightarrows M_1$ & $\mathcal{G}_2 \rightrightarrows M_2$ are Morita equivalent if $\exists\ \mathcal{H} \rightrightarrows M$ & Morita maps

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd}$$

To prove this is an equivalence relation, we need weak pullback of groupoids:

$$\begin{tikzcd} \mathcal{H} \times_{\mathcal{G}} \mathcal{K} \arrow[r, "\hat{\Phi}"] \arrow[d, "\hat{\Psi}"'] & \mathcal{K} \arrow[d, "\Psi"] \\ \mathcal{H} \arrow[r, "\Phi"'] & \mathcal{G} \end{tikzcd} \qquad (*)$$

Where:

- Objects of $\mathcal{H} \times_{\mathcal{G}} \mathcal{K} := Q \times_M \mathcal{G} \times_M R$, i.e.:

  $(q, g, r)$ with $\begin{cases} \psi(r) = s(g) \\ \phi(q) = t(g) \end{cases}$

- Arrows of $\mathcal{H} \times_{\mathcal{G}} \mathcal{K}$: arrow from $(q_1, g_1, r_1)$ to $(q_2, g_2, r_2)$ is a pair $(h, K)$ such that

$$\Phi(h)\, g_1 = g_2\, \Psi(K) \qquad (**)$$

$$\begin{tikzcd} (q_2, g_2, r_2) & & (q_1, g_1, r_1) \arrow[ll, bend right, "(h, K)"'] \end{tikzcd}$$

$$\begin{tikzcd} \psi(r_2) \arrow[d, "g_2"'] \arrow[dr, "g"] & \psi(r_1) \arrow[l, "\Psi(K)"'] \arrow[d, "g_1"] \\ \phi(q_2) & \phi(q_1) \arrow[l, "\Phi(h)"] \end{tikzcd}$$

$g\, \Phi(h) = g_2$ $\qquad$ $g_1 = g\, \Psi(K)$

Hence, we can also identify arrows w/ $(h, g, K) \in \mathcal{H} \times_M \mathcal{G} \times_M \mathcal{K}$ with:

$$\begin{cases} s(g) = \psi(t(K)) \\ t(g) = \phi(s(h)) \end{cases}$$

The diagram $(*)$ is not commutative: $(**)$ says that $id: \mathcal{G} \to \mathcal{G}$ is a natural isomorphism between two sides of $(*)$. In other words, the diagram $(*)$ is weakly commutative.

In general, weak pullback is not smooth.

**Proposition**

If the fibred product $Q \times_M \mathcal{G} \times_M R$ exists, then the weak pullback is a Lie groupoid. This is the case, e.g., if $\Phi$ (resp., $\Psi$) is a Morita map, and in this case the base change of $\Phi$ (resp. $\Psi$) is a Morita map

**Proof:**

The first part follows from diagram:

$$\begin{tikzcd}
\mathcal{H} \times_M \mathcal{G} \times_M \mathcal{K} \arrow[rr] \arrow[d] & & \mathcal{K} \arrow[d, "t"] \\
\mathcal{H} \times_M \mathcal{G} \times_M R \arrow[r] \arrow[d] & Q \times_M \mathcal{G} \times_M R \arrow[r, "pr"] \arrow[d, "pr"] & R \arrow[d, "\psi"] \\
\mathcal{H} \arrow[r, "s"'] & Q \arrow[r, "\phi"'] & M
\end{tikzcd}$$

submersions

For the second part, observe that if $\Phi$ is Morita, then

$t \circ pr_2 : Q \times_M \mathcal{G} \longrightarrow M$ is submersion $\Rightarrow$

$$\begin{tikzcd}
Q \times_M \mathcal{G} \times_M R \arrow[r] \arrow[d] & R \arrow[d, "\psi"] \\
Q \times_M \mathcal{G} \arrow[r, "t \circ pr_2"'] & M
\end{tikzcd}$$

Coco pullback

So weak pullback is smooth by first part.

Exercise: Verify that $\hat{\Phi}$ is Morita. $\blacksquare$

**Corollary:** Morita equivalence is an equivalence relation

**Proof:** Reflexive and symmetry is obvious.

For transitive, by Proposition, we can use weak pullback:

$$\begin{tikzcd}
 & & \mathcal{H} \times_{\mathcal{G}_2} \mathcal{K} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{H} \arrow[dl] \arrow[dr] & & \mathcal{K} \arrow[dl] \arrow[dr] & \\
\mathcal{G}_1 & & \mathcal{G}_2 & & \mathcal{G}_3
\end{tikzcd}$$

and the fact that composition of Morita maps is Morita. $\boxtimes$

Examples.

1) A Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to a manifold $N \rightrightarrows N$ iff $\exists$ surjective submersion $\phi : M \to N$ & a groupoid isomorphism $\mathcal{G} \cong M \times_N M$.

2) An action groupoid $G \ltimes M \rightrightarrows M$ is Morita equivalent to a manifold $N \rightrightarrows N$ iff the action is proper & free.

3) A Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to a Lie group $G$ iff it is transitive and any of its isotropy groups is isomorphic to $G$.

4) A Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to a discrete group iff it is transitive & any of its isotropy groups is discrete iff $t \times s : \mathcal{G} \to M \times M$ is a covering.

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We will discuss later what properties of a Lie groupoid are invariant under Morita equivalence.

Morita equivalence can also be seen as isomorphisms in a certain category:

**Def.** A <u>generalized morphism</u> $\Psi/\Phi : \mathcal{G} \dashrightarrow \mathcal{H}$ is given by a pair of Lie groupoid morphisms:

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "\Phi"', "s"] \arrow[dr, "\Psi"] & \\ \mathcal{G} & & \mathcal{H} \end{tikzcd} \quad \text{w/ } \Phi \text{ Morita map.}$$

Two generalized morphisms $\frac{\Psi_1}{\Phi_1}, \frac{\Psi_2}{\Phi_2} : \mathcal{G} \dashrightarrow \mathcal{H}$ are identical

$$\Psi_1/\Phi_1 = \Psi_2/\Phi_2$$

if there is a third generalized morphism $\Psi_3/\Phi_3 : \mathcal{G} \dashrightarrow \mathcal{H}$ fitting into a weakly commutative diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[d, equal] & \mathcal{K}_1 \arrow[l, "\Phi_1"', "\sim"] \arrow[r, "\Psi_1"] & \mathcal{H} \arrow[d, equal] \\ \mathcal{G} \arrow[d, equal] & \mathcal{K}_3 \arrow[l, "\Phi_3"', "\sim"] \arrow[r, "\Psi_3"] \arrow[u] \arrow[d] & \mathcal{H} \arrow[d, equal] \\ \mathcal{G} & \mathcal{K}_2 \arrow[l, "\Phi_2"', "\sim"] \arrow[r, "\Psi_2"] & \mathcal{H} \end{tikzcd}$$

(This is an equivalence relation on the set of gen. morphisms)

Morita equivalences $\equiv$ generalized isomorphisms
(invertible generalized morphisms)

<u>Exercise:</u> Show that

(i) every generalized map $\frac{\Psi}{\Phi} : \mathcal{G}_1 \dashrightarrow \mathcal{G}_2$ has a representative $\frac{\Psi'}{\Phi'} : \mathcal{G}_1 \dashrightarrow \mathcal{G}_2$, where $\Phi'$ covers a surjective submersion

(ii) every Morita equivalence can be represented by a pair of Morita maps which are surjective submersions.

<u>Hint:</u> Given Morita map $\Phi : \mathcal{H} \to \mathcal{G}$ consider weak pullback

$$\begin{tikzcd} \mathcal{G} \times_{\mathcal{G}} \mathcal{H} \arrow[r] \arrow[d, "s"'] & \mathcal{H} \arrow[d, "s"', "\Phi"] \\ \mathcal{G} \arrow[r, "id"'] & \mathcal{G} \end{tikzcd}$$