MATH 595 - LECTURE 22

We want to use orbit spaces of Lie groupoids to describe singular spaces.

Given $\mathcal{G} \rightrightarrows M$ we have at least the following relevant transversal data:

- $M/\mathcal{G}$ = orbit space w/ quotient topology
- Normal space to $\mathcal{O} \in M/\mathcal{G}$ at $x \in \mathcal{O}$ ("tangent space to $\mathcal{O}$")
$$\nu_x(\mathcal{O}) := T_x M / T_x \mathcal{O}$$
- $\mathcal{G}_x \curvearrowright \nu_x(\mathcal{O})$ - normal representation
$$g \cdot [v] := [d_g t(\tilde{v})], \quad \text{where } \tilde{v} \in T_g \mathcal{G},\ d_g s(\tilde{v}) = v$$

Exercise:

1) Show that this is well-defined & can also be described as follows:
If $b : M \to \mathcal{G}$ is any (local) bisection w/ $b(x) = g$ then:
$$g \cdot [v] = [d_x(t \circ b)(v)]$$

2) Show that there a normal rep of $\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O}$ on $\nu(\mathcal{O})$ defined by similar formula.

3) If $G \times M \rightrightarrows M$ is an action groupoid show that these are the usual normal rep $G_x \curvearrowright \nu_x(\mathcal{O})$

4) If $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ show that this is the linear holonomy action $\mathrm{Hol}_x^{lin} \curvearrowright \nu_x(L)$.

We seek notions of equivalence of Lie groupoids $\mathcal{G} \rightrightarrows M$ & $\mathcal{H} \rightrightarrows N$ inducing "isomorphisms" of their orbit spaces.

Def. Given a Lie groupoid morphism $\begin{matrix} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ M & \xrightarrow[\phi]{} & N \end{matrix}$ we say that is <u>preserves transversal data</u> if:

(i) $M/\mathcal{G} \to N/\mathcal{H}$, $\mathcal{O} \mapsto \phi(\mathcal{O})$, is homeomorphism

(ii)
$$\begin{matrix} \mathcal{G}_x & \xrightarrow{\Phi_x} & \mathcal{H}_{\phi(x)} \\ \circlearrowleft & & \circlearrowleft \\ \nu_x(\mathcal{O}) & \xrightarrow[d_x\phi]{} & \nu_{\phi(x)}(\phi(\mathcal{O})) \end{matrix}$$
is isomorphism of Reps

There are several possibilities of equivalence of groupoids that have this property. They are inspired by usual notions of equivalence in category theory.

<u>Isomorphism of groupoids</u>

· $\mathcal{C}_1$ & $\mathcal{C}_2$ are isomorphic categories if $\exists$ functors $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ & $\Psi : \mathcal{C}_2 \to \mathcal{C}_1$ such that $\Psi \circ \Phi = \mathrm{id}_{\mathcal{C}_1}$ & $\Phi \circ \Psi = \mathrm{id}_{\mathcal{C}_2}$.

This makes sense for Lie groupoids & gives usual isomorphism of Lie groupoids that we studied before.

<u>Exercise:</u> Show that isomorphism of groupoids preserves transversal data.

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This is too strong a notion:

if $G \curvearrowright M$ is proper & free action, so $N = M/G$ is smooth, the groupoids $G \times M \rightrightarrows M$ & $N \rightrightarrows N$ are not isomorphic. But their orbit spaces should represent same space N.

Strong (or Categorical) Equivalence of Groupoids

Recall a natural transformation between two functors $\Phi, \Psi : C_1 \to C_2$ is a map $\tau : \mathrm{Obj}(C_1) \to \mathrm{Arr}(C_2)$ such that:

$$x \xrightarrow{g} y \quad \Rightarrow \quad \begin{tikzcd} \phi(x) \arrow[r, bend left, "\Phi(g)"] \arrow[d, "\tau(x)"'] & \phi(y) \arrow[d, "\tau(y)"] \\ \psi(x) \arrow[r, bend right, "\Psi(g)"'] & \Psi(y) \end{tikzcd}$$

It is a natural isomorphism if $\tau(x)$ are invertible arrows, $\forall x$.

In this case we say that Φ & Ψ are isomorphic functors

· C_1 & C_2 are equivalent categories if $\exists$ functors $\Phi : C_1 \to C_2$ & $\Psi : C_2 \to C_1$ and natural isomorphisms $\Psi \circ \Phi \simeq \mathrm{id}_{C_1}$ & $\Phi \circ \Psi \simeq \mathrm{id}_{C_2}$

In this situation we also call Ψ a quasi-inverse of Φ.

For a groupoid, every natural transformation is automatically a natural isomorphism, and these notions have natural smooth versions:

Def: Two Lie groupoid morphisms $\Phi, \Phi' : \mathcal{G} \to \mathcal{H}$ are isomorphic if there exists a smooth natural transformation (= isomorphism) $\tau : \Phi \simeq \Phi'$. Two Lie groupoids $\mathcal{G}$ & $\mathcal{H}$ are called strongly equivalent if $\exists$ Lie groupoid morphisms $\Phi : \mathcal{G} \to \mathcal{H}$ & $\Psi : \mathcal{H} \to \mathcal{G}$ such that $\Psi \circ \Phi \underset{\tau}{\simeq} \mathrm{id}_{\mathcal{G}}$ $\Phi \circ \Psi \underset{\tau'}{\simeq} \mathrm{id}_{\mathcal{H}}$.

<u>Example</u> Consider a submersion $\phi : M \to N$:

$$\begin{tikzcd} M \times_N M \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & N \arrow[l, dashed, bend left, "\Psi"] \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\phi"'] & N \arrow[l, dashed, bend right, "\psi"'] \end{tikzcd}$$

$\Phi(p_1, p_2) = \phi(p_1) = \phi(p_2)$

If $\psi : N \to M$ is a section of ϕ, then we obtain a groupoid morphism: $\Psi : N \to M \times_N M$, $q \mapsto (\psi(q), \psi(q))$.

We have that:

$$\Phi \circ \Psi = id \qquad \Psi \circ \Phi(p_1, p_2) = (\psi(q), \psi(q)) \qquad q = \phi(p_1) = \phi(p_2)$$

$$\Rightarrow \Psi \circ \Phi \underset{\tau}{\cong} id$$

where $\tau : N \to M \times_N M$, $q \mapsto (\psi(q), \psi(q))$ is a natural iso.

Hence Ψ and Φ are quasi-inverses, so they are strong equivalences.

<u>Exercise</u>. Show that Ψ is a strong equivalence iff ϕ admits a global section.

<u>Theorem</u> If $\mathcal{G} \xrightarrow{\Phi} \mathcal{H}$ is a strong equivalence covering, then Φ preserves transversal data:

<u>Proof</u>: If $\Psi : \mathcal{H} \to \mathcal{G}$ is a quasi-inverse:

$$M/\mathcal{G} \rightleftarrows N/\mathcal{G} \qquad \begin{array}{l} \mathcal{O} \mapsto \phi(\mathcal{O}) \\ \psi(\mathcal{O}') \leftarrow\!\shortmid \mathcal{O}' \end{array}$$ are continuous & inverse to each other

$$\begin{tikzcd} g \arrow[d, mapsto] & \mathcal{G}_x \arrow[d] \arrow[r, "\Phi_x"] & \mathcal{H}_{\phi(x)} \arrow[dl, "\Phi_{\phi(x)}"] \\ \tau(x)\, g\, \tau(x)^{-1} & \mathcal{G}_{\psi(\phi(x))} & \end{tikzcd}$$

$\Rightarrow$ Φ_x is a group isomorphism

$$\begin{tikzcd} {[v]} \arrow[d] \\ T(h)[v] \end{tikzcd} \qquad \begin{tikzcd} \nu_x(\mathcal{O}) \arrow[rr, "d_x\phi"] \arrow[d] & & \nu_{\phi(x)}(\phi(\mathcal{O})) \arrow[dll, "d_{\phi(x)}\psi"] \\ \nu_{\psi(\phi(x))}(\mathcal{O}) & & \end{tikzcd} \quad \Rightarrow \text{Isomorphism of normal reps.}$$

Normal Rep

$\mathfrak{g}_{\mathcal{O}} \circlearrowright \nu(\mathcal{O})$

$\blacksquare$

This is too strong a notion:

Given submersion $\phi : M \to N$ the submersion groupoid $M \times_N M \rightrightarrows M$ has smooth orbit space $\phi(M)$. So it should represent N if ϕ is a surjective submersion.

But $\Phi : M \times_N M \to N$ is strong equivalence iff ϕ admits a section.

Morita (or weak) Equivalence of Groupoids

For arbitrary categories, using axiom of choice, one proves:

Proposition

$\mathcal{C}_1$ & $\mathcal{C}_2$ are equivalent iff there exists a functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ s.t.

(i) Φ is essential surjective: For every object y in $\mathcal{C}_2$ there is an object x in $\mathcal{C}_1$ and an arrow $(\phi(x) \xrightarrow{h} y) \in \mathcal{C}_2$

(ii) Φ is fully faithful: For any two objects x_1, x_2 in $\mathcal{C}_1$ Φ restricts to a bijection

$$\mathcal{C}_1 \supset \{x_1 \xrightarrow{g} x_2\} \xrightarrow{\Phi} \{\phi(x_1) \xrightarrow{h} \phi(x_2)\} \subset \mathcal{C}_2$$

In the smooth category, in particular for Lie groupoids, this is no longer true.

We introduce smooth versions of these two notions:

Def: A Lie groupoid morphism
$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{H} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$
is a <u>Morita map</u>

(i) $t \circ pr_2 : \mathcal{H} \times_N M \longrightarrow N$ is a surjective submersion

(ii) The square:
$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "t \times s"'] & \mathcal{H} \arrow[d, "t \times s"] \\ M \times M \arrow[r, "\phi \times \phi"'] & N \times N \end{tikzcd}$$
is a (good) pullback of manifolds

Two Lie groupoids $\mathcal{G}_1$ & $\mathcal{G}_2$ are <u>Morita (or weak) equivalent</u> if there exist Morita maps:
$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ \mathcal{G}_1 & & \mathcal{G}_2 \end{tikzcd}$$

We will prove later that this is indeed an equivalence relation.

<u>Rmk: pullbacks of manifolds</u>

In general, given smooth maps $\pi_i : M_i \to N$ $(i=1,2)$ we have the pullback diagram:
$$\begin{tikzcd} \{(p_1,p_2) : \pi_1(p_1) = \pi_2(p_2)\} = M_1 \times_N M_2 \arrow[r] \arrow[d] & M_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\pi_1"'] & N \end{tikzcd}$$
$$\{(p_1,p_2) : \pi_1(p_1) = \pi_2(p_2)\} \subset M_1 \times M_2$$

But $M_1 \times_N M_2$ may fail to be a manifold, or maybe a manifold and not have the expected tangent bundle (= pullback of tang. bundles)
$$T(M_1 \times_N M_2) = TM_1 \times_{TN} TM_2 \subset TM_1 \times TM_2$$

Ex:
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] & t \arrow[d, mapsto] \\ \mathbb{R} \arrow[r] & \mathbb{R}^2 & (t,t^3) \end{tikzcd}$$
$$t \longmapsto (t,0)$$

$0 = \{0\}$ is a manifold

$T0 = \{0\}$

$TM_1 \times_{TN} TM_2 = \mathbb{R}$

Ex:

$\alpha, \beta \notin \mathbb{Q}$ Rationally indep.

$$\begin{array}{ccc} D & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & S^1 \times S^1 \end{array} \qquad \begin{array}{c} t \\ \downarrow \\ (e^{it}, e^{i\alpha t}) \end{array}$$

$$t \longmapsto (e^{it}, e^{i\beta t})$$

- D = intersection of two curves
- If we consider discrete topology, it is a pullback of manifolds but not of topological spaces
- If we consider D has a topological pullback, it is not a manifold

We say that $M_1 \times_N M_2$ is a (good) pullback of manifolds if

(i) $M_1 \times_N M_2 \hookrightarrow M_1 \times M_2$ is embedded submanifold

(ii) $T(M_1 \times_N M_2) = TM_1 \times_{TN} TM_2$

Exercise: Show that if π_1 and π_2 are transverse:

$$\mathrm{Im}(d_{p_1}\pi_1) + \mathrm{Im}(d_{p_2}\pi) = T_pM, \quad \forall (p_1, p_2) \text{ w/ } \pi_1(p_1) = \pi_2(p_2) = p$$

Then $M_1 \times_N M_2$ is a good fiber product. In particular, this is alway the case if either π_1 or π_2 is a submersion.

Proposition.

A strong equivalence is a Morita map

Proof Let

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\Phi} & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ M & \xrightarrow[\phi]{} & N \end{array}$$

be a strong equivalence w/ quasi-inverse

$\Psi : \mathcal{H} \to \mathcal{G}$, so there are natural iso $\Psi \circ \Phi \underset{T}{\simeq} \mathrm{Id}_{\mathcal{G}}$ $\Phi \circ \Psi \underset{G}{\simeq} \mathrm{Id}_{\mathcal{H}}$

(i) $t \circ pr_1 : \mathcal{H} \times_N M \to N$ is surjective submersion:

It is surjective since $y \in N$ is image of $(G(y), \psi(y))$. To see that it is submersion we construct local section through any (h_0, x_0) mapping to y_0: look at arrow:

$$G(y_0)^{-1} h_0 : \phi(x_0) \to \phi(\psi(y_0)) \in \mathcal{H}$$

Since Φ is a categorical equivalence, there is a unique arrow $g_0 : x_0 \to \psi(y_0) \in \mathcal{G}$ with $\Phi(g_0) = \sigma(y_0)^{-1} h_0$. Choose local bisection $b : M \to \mathcal{G}$ w/ $b(x_0) = g_0$. Then:

$$\lambda \equiv t \circ b : M \to M \quad \text{local diffeo}, \quad \lambda(x_0) = \psi(y_0)$$

So define:

$$\theta : N \to \mathcal{H} \times_N M, \quad \theta(y) := (\sigma(y)\Phi(b(\lambda^{-1}\circ\psi(y))), \lambda^{-1}(\psi(y)))$$

It is a local section w/ $\theta(y_0) = (h_0, x_0)$.

(ii) By (i), we have a (good) pullback

$$\begin{tikzcd} M \times_N \mathcal{H} \times_N M \arrow[r, "pr_2"] \arrow[d, "pr_1 \times pr_3"'] & \mathcal{H} \arrow[d, "t\times s"] \\ M \times M \arrow[r] & N \times N \end{tikzcd}$$

Since Φ is categorical equivalence, $\mathcal{G} \to M \times_N \mathcal{H} \times_N N \quad g \mapsto (t(g), \Phi(g), s(g))$ is a bijection. It is also smooth and immersive (exercise!). Hence it is a diffeo, so we obtain a (good) pullback:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "t\times s"'] & \mathcal{H} \arrow[d, "t\times s"] \\ M\times M \arrow[r, "\phi\times\phi"'] & N\times N \end{tikzcd}$$

$\blacksquare$

Hence we have:

Φ groupoid isomorphism $\Rightarrow$ Φ strong equivalence $\Rightarrow$ Φ Morita map

The reverse implications do not hold:

<u>Example:</u> Consider again a submersion $\phi : M \to N$ and the groupoid morphism:

$$\begin{tikzcd} \mathcal{G} = M \times_N M \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & N = \mathcal{H} \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

We already know that:

1) Φ isomorphism $\Leftrightarrow$ ϕ diffeomorphism

2) Φ strong equivalence $\Leftrightarrow$ ϕ has a section

Now:

- Φ essential surjective:

$\Leftrightarrow$ $t \circ pr_2 : M \times_N \mathcal{H} \to N$ surjective submersion $\Leftrightarrow$ ϕ surjective submersion

($\overset{\text{SI}}{\phi}$, $\overset{\text{SI}}{M}$)

- Φ fully faithfull $\Leftrightarrow$ (good) pullback diagram:

$$\begin{tikzcd} M \times_N M \arrow[r] \arrow[d, "(s,t)"'] & N \arrow[d, "(s,t)"] \\ M \times M \arrow[r, "\phi \times \phi"'] & N \times N \end{tikzcd}$$

allways true ($\phi \times \phi$ is submersion)

<u>Conclusion</u>: Φ is Morita map $\Leftrightarrow$ ϕ is surjective submersion

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Morita equivalence is the "good" equivalence:

<u>Theorem (del Hoyo)</u>

A groupoid morphism $\Phi : \mathcal{G} \to \mathcal{H}$ is a Morita map iff it preserves transversal data.