MATH 595 - Lecture 2

I.1 Lie Groupoids: Definition & Examples

Def: A Lie groupoid is a groupoid $\mathcal{G} \rightrightarrows M$ where $\mathcal{G}$ & M are manifolds, s & t are submersions, and m, u, i are smooth.

Notation: We have spaces of composable arrows:

$$\mathcal{G}^{(0)} = M, \quad \mathcal{G}^{(1)} = \mathcal{G}, \quad \mathcal{G}^{(2)} = \mathcal{G} \, {}_s\!\times_t \mathcal{G} = \{(g,h) : s(g) = t(h)\}$$

$$\ldots \quad \mathcal{G}^{(k)} = \mathcal{G} \, {}_s\!\times_t \cdots {}_s\!\times_t \mathcal{G} = \{(g_1, \ldots, g_k) : s(g_i) = t(g_{i+1})\}$$

s & t are submersions $\Rightarrow$ $\mathcal{G}^{(k)}$ is a manifold

In particular, it makes sense to say $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is smooth.

Def: A morphism from a Lie groupoid $\mathcal{G} \rightrightarrows M$ to a Lie groupoid $\mathcal{H} \rightrightarrows N$ is a pair of smooth maps $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ and $f : M \to N$ which are compatible w/ the structure maps.

Compatability $\equiv$ $(\mathcal{F}, f)$ functor

- If $y \xleftarrow{g} x$ in $\mathcal{G}$ then $f(y) \xleftarrow{\mathcal{F}(g)} f(x)$ in $\mathcal{H}$
- If $(g,h) \in \mathcal{G}^{(2)}$ then $\mathcal{F}(gh) = \mathcal{F}(g)\mathcal{F}(h)$
- If $x \in M$, then $\mathcal{F}(1_x) = 1_{f(x)}$
- If $y \xleftarrow{g} x$ in $\mathcal{G}$ then $\mathcal{F}(g^{-1}) = \mathcal{F}(g)^{-1}$

The last property follows from the others

Convention:

Manifolds are assumed Hausdorff and 2^{nd} countable.

We do not assume this for the space of arrows $\mathcal{G}$.

But we still assume that M and the fibers of s and t are Hausdorff and 2^{nd} countable (see examples).

Rmk: Because $s^{-1}(x)$ and $t^{-1}(y)$ a closed, embedded, Hausdorff and 2^{nd} countable, for most purposes one can work with $\mathcal{G}$ as if it was Hausdorff and 2^{nd} countable.

Exercise: Show that for a Lie groupoid $\mathcal{G} \rightrightarrows M$:

- $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is a submersion
- $i : \mathcal{G} \to \mathcal{G}$ is a diffeo
- $u : M \hookrightarrow \mathcal{G}$ is an embedding, which is closed if $\mathcal{G}$ is Hausdorff

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Proposition

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

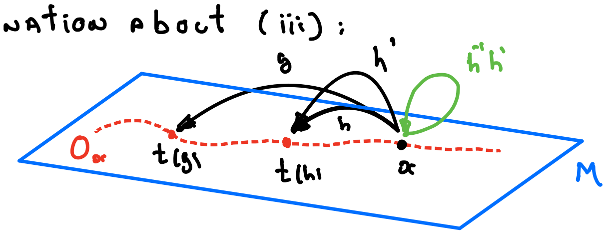
(i) $s^{-1}(x) \cap t^{-1}(y)$ are closed embedded submanifolds of $\mathcal{G}$

(ii) The isotropy groups G_x are Lie groups

(iii) $t : s^{-1}(x) \to O_x$ is a principal $\mathcal{G}_x$-bundle

(iv) The orbits O_x are immersed submanifolds in M

Explanation about (iii):



Examples:

1) Lie groups $\equiv$ Lie groupoids over $M = \{*\}$

$$\begin{array}{c} \mathcal{G} \\ t \downarrow\downarrow s \\ \{*\} \end{array}$$

one orbit / one isotropy group

2) Bundles of Lie groups $\equiv$ Lie groupoids with $s = t$

$$\begin{array}{c} \mathcal{G} \\ t \downarrow\downarrow s \\ M \end{array}$$

Orbits = pts of M Isotropy groups = fibers of $t = s$

Very special case: identity groupoid

$$\begin{array}{c} M \\ \mathrm{id} \downarrow \\ M \end{array}$$

Rmk. A bundle of groups need not be locally trivial neither as a bundle nor as a group bundle:

$$\begin{array}{c} \mathcal{G} = \mathbb{R} \times \mathbb{R}^2 \\ \downarrow pr_1 \\ M = \mathbb{R} \end{array} \qquad (t, x_1, y_1) * (t, x_2, y_2) := (x_1 + x_2, y_1 + e^{t x_1} y_2)$$

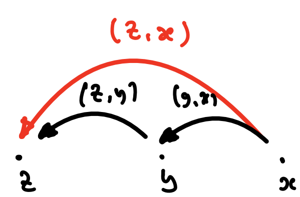
$$\begin{cases} t = 0, & \mathcal{G}_0 \text{ is abelian} \\ t \neq 0, & \mathcal{G}_t \text{ is non-abelian} \end{cases}$$

$$\begin{array}{cc} \mathbb{R}^2 & (t, x) \\ \downarrow pr_1 & \downarrow \\ \mathbb{R} & t \end{array} \qquad \Lambda = \{ (t, \tfrac{m}{t}) : m \in \mathbb{Z}, t \neq 0 \} \cup \{(0,0)\} \subset \mathbb{R}^2$$

$$\rightsquigarrow \begin{array}{c} \mathcal{G} = \mathbb{R}^2 / \Lambda \\ t \downarrow\downarrow s \\ M = \mathbb{R} \end{array} \qquad \begin{cases} t = 0: & \mathcal{G}_0 = \mathbb{R} \\ t \neq 0: & \mathcal{G}_t \simeq S^1 \end{cases}$$

3) Pair Groupoid. For any mfld M:

$$\begin{array}{c} M \times M \\ t = pr_1 \downarrow\downarrow pr_2 = s \\ M \end{array}$$



one orbit / isotropy groups are all trivial

4) <u>Submersion Groupoids.</u> For a submersion $\mu : M \to N$

$$M \times_\mu M \rightrightarrows M \qquad y \xleftarrow{(y,x)} x \quad \text{if } \mu(y) = \mu(x)$$

Orbits = Fibers of μ / isotropy groups are trivial

• <u>Rmk</u>: If $\mu = \mathrm{id} : M \to M$ we recover pair groupoid

5) <u>Equivalence Relations.</u> Any equivalence $R \subset M \times M$ defines a subgroupoid of the pair groupoid:

$$\begin{array}{c} R \\ pr_1 \downarrow\downarrow pr_2 \\ M \end{array}$$

This is a Lie groupoid if $R \subset M \times M$ is an immersed submanifold and pr_1, pr_2 restrict to submersions. We say that R is <u>smooth</u>

• For any Lie groupoid $\mathcal{G} \rightrightarrows M$ one has a Lie groupoid morphism, called the <u>anchor</u> of $\mathcal{G}$:

$$\Phi : \mathcal{G} \xrightarrow{(t,s)} M \times M$$

The image of Φ is the equivalence relation groupoid associated w/ orbit equivalence relation (not Lie, in general)

<u>Exercise:</u> Show that a Lie groupoid $\mathcal{G} \rightrightarrows M$ is isomorphic to an equivalence relation groupoid iff its isotropy groups are all trivial.

Def: A <u>Lie subgroupoid</u> of $\mathcal{G} \rightrightarrows M$ is a Lie groupoid $\mathcal{H} \rightrightarrows N$ together with a Lie groupoid morphism:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ N \arrow[r, "\phi"'] & M \end{tikzcd}$$

which is an injective immersion. If $N = M$ we call the Lie subgroupoid <u>wide</u>.

- An equivalence relation is the same thing as a wide Lie subgroupoid of $M \times M$.
- An isotropy group $\mathcal{G}_x \hookrightarrow \mathcal{G}$ is a Lie subgroupoid which is not wide.

6) <u>Action Groupoids</u>. Any Lie group action

$$G \times M \to M, \quad (g, x) \mapsto gx$$

$$\begin{tikzcd} G \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \end{tikzcd} \qquad \begin{tikzcd} gx & x \arrow[l, bend right, "{(g,x)}"'] \end{tikzcd} \qquad \begin{array}{c} (h,y)\cdot(g,x) = (hg, x) \\ \text{if } y = gx \end{array}$$

Orbits = orbits of action

Isotropy = isotropy groups of action

7) <u>Flow of a vector field</u>. For $X \in \mathfrak{X}(M)$ take flow:

- ϕ_X^t with domain $D(X) \subset \mathbb{R} \times M$ (open set)

$$(t, x) \longmapsto \phi_X^t(x)$$

$$\begin{tikzcd} D(X) \arrow[d, shift right] \arrow[d, shift left] \\ M \end{tikzcd} \qquad \begin{tikzcd} \phi_X^t(x) & x \arrow[l, bend right, "{(t,x)}"'] \end{tikzcd} \qquad \begin{array}{c} (s,y)\cdot(t,x) = (s+t, x) \\ \text{if } y = \phi_X^t(x) \end{array}$$

· Orbits = orbits of vector field

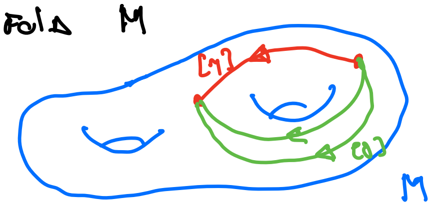
· Isotropy group of x $\cong$ $\begin{cases} \mathbb{R} & \text{if } x \text{ is zero} \\ \mathbb{Z} & \text{if } x \text{ lies in periodic orbit} \\ \{1\} & \text{otherwise} \end{cases}$

Rmk:

X is complete $\Leftrightarrow$ $D(X) = \mathbb{R} \times M$ $\Rightarrow$ Flow defines $\mathbb{R}$-action on M so flow groupoid becomes action gpd.

8) Homotopy Groupoid. For any manifold M

$$\Pi_1(M) \rightrightarrows M \qquad \gamma(1) \xleftarrow{[\gamma]} \gamma(0)$$



· $[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \circ \gamma_2]$

w/ $\gamma_1 \circ \gamma_2(t) := \begin{cases} \gamma_2(2t), & 0 \leq t \leq 1/2 \\ \gamma_1(2t-1), & 1/2 \leq t \leq 1 \end{cases}$

This is a Lie Groupoid. Assume M connected:

$$\left.\begin{array}{l}\text{Universal} \\ \text{covering} \\ \text{space}\end{array}\right\} \quad \tilde{M} = \{ [\gamma] \mid \gamma : [0,1] \to M, \ \gamma(0) = x_0 \} \xrightarrow{\pi} M$$

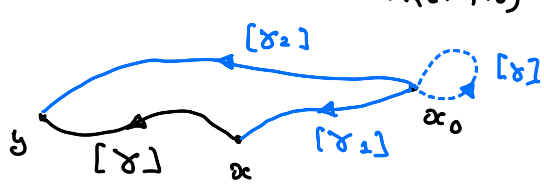
$$\circlearrowleft \ \pi_1(M, x_0) \qquad\qquad [\gamma] \longmapsto \gamma(1)$$

· $\tilde{M}$ is a smooth manifold

$\pi_1(M, x_0) \circlearrowleft \tilde{M}$ is proper & free action $\Big\}$ $\pi : \tilde{M} \to M$, $\circlearrowleft \pi_1(M, x_0)$

Principal bundle

· $\Pi_1(M) \cong \left(\tilde{M} \times \tilde{M} \right) / \pi_1(M, x_0)$



Conclusion:

$$\begin{array}{c} \tilde{M} \times \tilde{M} \\ \downdownarrows \\ \tilde{M} \end{array} \quad \circlearrowleft \pi_1(M,x) \quad \Rightarrow \quad \begin{array}{c} \Pi_1(M) \cong (\tilde{M}\times\tilde{M})/_{\pi_1(M,x_0)} \\ \downdownarrows \\ M \simeq \tilde{M}/_{\pi_1(M,x_0)} \end{array}$$

Pair groupoid

Proper & free action by groupoid automorphisms

Orbits = connected components of M / isotropy at $x = \pi_1(M,x)$

<u>Exercise:</u>

$\begin{cases} \mathcal{G} \rightrightarrows M \text{ Lie groupoid,} \\ \mathcal{G} \circlearrowleft K \text{ Lie group action by grpd automorphisms} \end{cases}$

$$\Rightarrow \quad \begin{array}{c} \mathcal{G}/K \\ \downdownarrows \\ M/K \end{array} \quad \text{is a Lie groupoid}$$

9) <u>Gauge Groupoid.</u> $\begin{array}{c} P \circlearrowleft G \\ \pi \downarrow \\ M \end{array}$ principal G-bundle

$$\begin{array}{c} (P\times P)/_G \\ \downdownarrows \\ M \end{array}$$

(quotient of pair groupoid $P\times P \rightrightarrows P$ by diagonal action of G: $(p,q)g = (pg,qg)$)

$$\pi(q) \xleftarrow{[q,p]} \pi(p)$$

$[r,q']\cdot[q,p] := [rg,p]$

if $q' = qg$

One orbit / isotropy groups $\simeq G$

<u>Remark:</u> When M is connected, $\Pi_1(M)$ is an example of a Gauge groupoid (associated with $\begin{array}{c} \tilde{M} \circlearrowleft \pi_1(M,x_0) \\ \downarrow \\ M \end{array}$)

Def. A groupoid is called **transitive** if it has only one orbit.

- Gauge groupoid of $\overset{\circlearrowleft G}{P} \to M$ is transitive
- $\mathcal{G} \rightrightarrows M$ transitive $\Rightarrow$ $t : s^{-1}(x_0) \longrightarrow M$ is principal $\mathcal{G}_{x_0}$-bundle

If $\mathcal{G} \rightrightarrows M$ is transitive then:

$$\begin{array}{ccc} \left(s^{-1}(x_0) \times s^{-1}(x_0)\right)/\mathcal{G}_{x_0} & \longrightarrow & \mathcal{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & = & M \end{array} \qquad [h_1, h_2] \longmapsto h_1 h_2^{-1}$$

is a Lie groupoid isomorphism.