MATH 595 - LECTURE 19

II - Singular Spaces

Aim: Differential Geometry on spaces which are singular, i.e., which are not smooth manifolds.

- Often such singular spaces arise as quotient spaces of smooth manifolds, but to be able to work on them we need to keep track of extra structure.

- Our point of view:

$$\text{singular spaces} \equiv \text{orbit spaces of Lie groupoids}$$

where:

(i) Extra structure is codified by groupoid (e.g., isotropy groups) but groupoid also contains irrelevant extra-information

(ii) Two groupoids can present the same singular space

(i) & (ii) $\Rightarrow$ Morita equivalence of Lie groupoids

As a warm-up we consider a special case of singular spaces, namely

Orbifolds

Idea: An orbifold is a topological space X where each $x_0 \in X$ has a neighborhood $V_{x_0} \simeq U/G$ with $U \subset \mathbb{R}^n$ open and $G \subset \mathrm{Diff}(U)$ a finite group

To formalie this:

Def: Let X be a toplocical space.

(i) An orbifold chart of dimension m fa X is a triple (U, G, ϕ) where $U \subset \mathbb{R}^m$ is a connected open set $G \subset \mathrm{Diff}(U)$ is a finite subgroup and and $\phi : U \to X$ is a G-invariant open map inducing a homeomorphism

$$U/G \to \phi(U) \subset X$$

(ii) An embedding of orbifold charts $(V, H, \psi) \hookrightarrow (U, G, \phi)$ is an embedding $\lambda : V \to U$ such that

$$\begin{tikzcd} V \arrow[d, "\lambda"'] \arrow[dr, "\psi"] & \\ U \arrow[r, "\phi"'] & X \end{tikzcd}$$

(iii) Two orbifold charts (U_1, G_1, ϕ_1) & (U_2, G_2, ϕ_2) are said to be compatible if for any $x \in \phi_1(U_1) \cap \phi_2(U_2)$ there exists embeddings of orbifold charts $\lambda_i : (V, H, \psi) \to (U_i, G_i, \phi_i)$ with $x \in \psi(V)$.

(iv) An orbifold atlas of dim m for X is a collection of pairwise compatible orbifold charts of dimension m

$$\mathcal{U} = \{(U_i, G_i, \phi_i) : i \in I\} \quad w/ \quad X = \bigcup_{i \in I} \phi_i(U_i)$$

Two orbifold atlas $\mathcal{U}_1$ & $\mathcal{U}_2$ for X are compatible if $\mathcal{U}_1 \cup \mathcal{U}_2$ is an orbifold atlas

(v) An orbifold of dimension m is a pair $(X, \mathcal{U})$ where X is a second countable, Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas.

Rmks

1) Any orbifold atlas $\mathcal{U}$ defines an orbifold ($\mathcal{U}$ is contained in a unique maximal orbifold atlas)

2) Every orbifold is locally compact and paracompact

3) A smooth function $f: X \to \mathbb{R}$ is a continuous map such that for any orbifold chart (U, G, ϕ), $f \circ \phi : U \to \mathbb{R}$ is smooth

4) Similarly, a smooth map $f: X \to Y$ between to orbifolds is a continuous map such that $\forall x \in X$ there are orbifold charts (U, G, ϕ) w/ $x \in \phi(U)$ and (V, H, ψ) w/ $f(x) \in \psi(V)$, and a smooth map $\bar{f}: U \to V$, such that

$$f \circ \phi = \psi \circ \bar{f}$$

<u>Notation</u>: For any manifold M and $G \subset \mathrm{Diff}(M)$ we write:

$$\lambda_g : M \to M, \quad x \longmapsto gx \quad (\text{action by } g \in G)$$

$$\Sigma_g := \{ x \in M : gx = x \}$$

$$\Sigma_G := \bigcup_{g \neq e} \Sigma_g = \{ x \in M : G_x \neq 1 \}$$

$$G_S := \{ g \in G : gS = S \}$$

A subset $S \subset M$ is called <u>G-stable</u> if either:

$$gS = S \quad \text{or} \quad gS \cap S = \phi$$

<u>Exercise</u>:

G-stable sets are the connected components of G-invariant sets. If G is finite, the open G-stable sets give basis for topology of M.

We will look at finite subgroups $G \subset \mathrm{Diff}(M)$ and we will show that:

- If (U, G, ϕ) is orbifold chart and $V \subset U$ is a G-stable open subset, then $(V, G_V, \phi|_V)$ is an orbifold chart compatible w/ (U, G, ϕ).
- Given two orbifold charts (U, G, ϕ), (V, H, ψ) and $x \in \phi(p) \cap \psi(q)$, one has that:

  (i) $p \in \Sigma_G$ iff $q \in \Sigma_H$.

  (ii) There are faithfull representations

$$G_p \to GL(n, \mathbb{R}), \qquad g \mapsto d_p \lambda_g$$

$$H_q \to GL(n, \mathbb{R}), \qquad h \mapsto d_p \lambda_h$$

  and images are conjugate subgroups

Def. For an orbifold X:

(i) $x \in X$ is called a singular point if for some chart (U, G, ϕ) $x = \phi(p)$ with $p \in \Sigma_G$. The singular locus of X is denoted Σ_X

(ii) The isotropy type of x is the conjugacy class in $Gl(n, \mathbb{R})$ of the image $G_p \to GL(n, \mathbb{R})$ for some chart (U, G, ϕ), and is denoted $\mathrm{Iso}_x(X)$.

Hence:

$$\Sigma_X = \{ x \in X : \mathrm{Iso}_x(X) \neq 1 \}.$$

We will see that $\Sigma_X \subset X$ is a closed subset with empty interior.

Examples

1. Smooth manifolds = orbifolds w/ empty singular locus
 = orbifolds w/ $Iso_x(X) = 1$, $\forall x$

2. If $G \subset Diff(M)$ is a finite group then $X = M/G$ with quotient topology has a natural orbifold structure, with:

$$\Sigma_X = \{ xG : \text{with } G_x \neq 1 \}, \quad Iso_x(X) \simeq G_x$$

To construct orbifold charts on $\pi : M \to M/G$, given $p_0 \in M$ there exists a chart (V, φ) for M centered at p_0 such that:

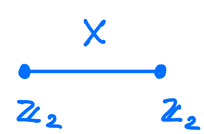
- V is G_{p_0}-invariant;
- $gV \cap V = \phi$ is $g \notin G_{p_0}$.

We obtain $G_{p_0} \simeq H \subset Diff(\varphi(V))$, so we can define an orbifold chart $(\varphi(V), G_{p_0}, \pi \circ \varphi^{-1})$.

An orbifold isomorphic to M/G for some finite group $G \subset Diff(M)$ is called a global quotient or a good orbifold.

Ex: $M = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, $G = \mathbb{Z}_2 \curvearrowright S^1$ complex conjugation

Then:
$$X = S^1/\mathbb{Z}_2 \simeq [-1, 1]$$
$$\Sigma_X = \{ -1, 1 \}$$
$$Iso_1(X) \simeq Iso_{-1}(X) \simeq \mathbb{Z}_2$$



Ex: $M = S^n$, $G = \mathbb{Z}_k$ acting by rotations $\frac{2\pi}{k}$

Then:
$$X = S^n/\mathbb{Z}_k \simeq S^n$$
$$\Sigma_X = \{ p_N, p_S \}$$
$$Iso_{p_N}(X) \simeq Iso_{p_S}(X) \simeq \mathbb{Z}_k$$

3. An orbifold which is not a global quotient:

Take $X = S^2$ has a topological space. Consider two orbifold charts:

(a) $(D, 1, \phi)$: $\phi : D \xrightarrow{\sim} S^2 - \{p_N\}$

(b) $(D, \mathbb{Z}_m, \psi)$: $\mathbb{Z}_m = \{\text{rotations by } \frac{2\pi}{m}\} \circlearrowright \mathbb{R}^2$

$$\psi : D \longrightarrow D/\mathbb{Z}_m \simeq S^2 - \{p_S\}$$

These charts are compatible: If we consider the orbifold chart $(D-\{0\}, 1, \tau)$:

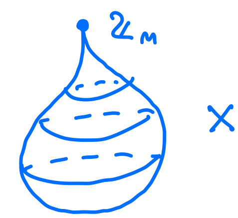
$$\tau : D - \{0\} \longrightarrow S^2 - \{p_N, p_S\}$$

We have maps:

$$(D-\{0\}, 1, \tau) \xrightarrow{id} (D, 1, \phi)$$

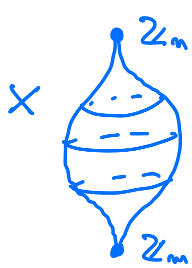
$$(D-\{0\}, 1, \tau) \longrightarrow (D, \mathbb{Z}_m, \psi), \quad z \mapsto z^m$$

The first map is an embedding. The second map is $m:1$ cover. So restricting to sectors $U_\alpha \subset D-\{0\}$ we obtain embedding of orbifold charts. Giving compatibility of the charts.



"Tear drop"

This orbifold has singular set $\Sigma_X = \{p_N\}$ and isotropy $Iso_{p_N}(X) = \mathbb{Z}_m$.



Similarly, one can construct a 2-dim orbifold $X \simeq S^2$ with two singular pts and isotropies $\mathbb{Z}_n$ & $\mathbb{Z}_m$. It is a global quotient iff $n = m$. One can show that orbifold structures on S^2 with 3 or more singular points are always good.