MATH 595 – Lecture 18

<u>Examples (cont.)</u>

5) Infinitesimal $\mathfrak{g}$-actions are allways integrable

- $\sigma : \mathfrak{g} \to \mathfrak{X}(M) \equiv \mathfrak{g}$-action
- $A = \mathfrak{g} \ltimes M \to M$,

$$\rho(\alpha, x) = \sigma(\alpha)_x$$

$$[f, g]_A(x) = [f(x), g(x)]_{\mathfrak{g}} - (\mathcal{L}_{\sigma(f(x))} g)(x) + (\mathcal{L}_{\sigma(g(x))} f)(x)$$

- Fix $x \in M$: $\mathfrak{g}_x = \operatorname{Ker} \rho_x$ = isotropy of $\mathfrak{g}$-action

$G_x \subset G(\mathfrak{g})$ = connected Lie group w/ Lie algebra $\mathfrak{g}_x$

$$\begin{tikzcd}
\pi_2(\mathcal{O}_x) \arrow[r, "\partial_x"] & G(\mathfrak{g}_x) \arrow[r, "P_x"] \arrow[d] & G(A)_x \arrow[dl] \\
\mathfrak{g} \supset \mathfrak{g}_x \arrow[ur, dashed, "\exp"] \arrow[r, dashed, "\exp_{\mathfrak{g}}"'] & G_x \subset G(\mathfrak{g}) &
\end{tikzcd}$$

$\Rightarrow \tilde{N}_x(A) \subset \pi_1(G_x)$

$\Rightarrow N_x(A) \cap U = \{0\}$

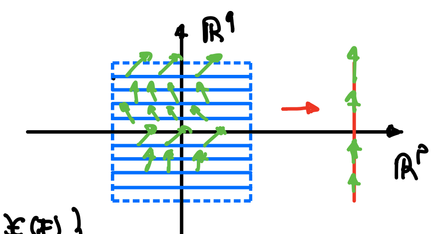
$U \equiv$ domain of injectivity of $\exp : \mathfrak{g} \to G(\mathfrak{g})$

6) Transversally parallelizable foliations

- $(M, \mathcal{F}) \equiv$ foliated manifold (assume M compact to simplify)
- $C^\infty_{bas}(M, \mathcal{F}) \equiv$ smooth functions constant on leaves
- $\ell(M, \mathcal{F}) \equiv$ transverse vector fields

$$= \left\{ X \in \Gamma(\nu(\mathcal{F})) : \text{locally projectable along submersions defining foliation} \right\}$$

$$= L(M, \mathcal{F}) / \mathfrak{X}(\mathcal{F})$$



where:

$$L(M, \mathcal{F}) = \{ Y \in \mathfrak{X}(M) : [\mathfrak{X}(\mathcal{F}), Y] \subset \mathfrak{X}(\mathcal{F}) \}$$

- $\ell(M,\mathcal{F})$ is a Lie algebra for usual Lie bracket of v.f.
- $\ell(M,\mathcal{F})$ is a $C^\infty_{basic}(M,\mathcal{F})$-module:

$$X \in \ell(M,\mathcal{F}),\ f \text{ basic} \Rightarrow fX \in \ell(M,\mathcal{F})$$

Def: $(M,\mathcal{F})$ is <u>$\perp$-parallelizable</u> if $\nu(\mathcal{F})$ admits global frame consisting of transverse vector fields

<u>Rmk:</u> Since we are assuming M compact, a $\perp$-parallelizable foliation $(M,\mathcal{F})$ is homogeneous, i.e., for any $x,y \in M$ there exists $\phi: M \to M$ a diffeo w/ $\phi(\mathcal{F}) = \mathcal{F}$ and $\phi(x) = y$. This implies very strong properties:

(i) All leaves of $\mathcal{F}$ are diffeomorphic

(ii) Closure of leaves form a foliation $\mathcal{F}_{bas}$ w/ smooth leaf space

(iii) $C^\infty_{basic}(M,\mathcal{F}) = C^\infty_{bas}(M,\mathcal{F}_{bas}) \simeq C^\infty(M/\mathcal{F}_{basic})$

(iv) $\ell(M,\mathcal{F}) \longrightarrow \ell(M,\mathcal{F}_{bas}) \simeq \mathfrak{X}(M/\mathcal{F}_{bas})$ is surjective

If $(M,\mathcal{F})$ is $\perp$-parallelizable then:

(v) $\ell(M,\mathcal{F})$ is a free $C^\infty_{bas}(M,\mathcal{F})$-module: a basis is any $\perp$-parallelism $\{X_1,\dots,X_q\}$

$$\Rightarrow \begin{cases} \ell(M,\mathcal{F}) \text{ is the space of sections of a Lie algebroid} \\ \qquad\qquad A(M,\mathcal{F}) \to M_{bas} := M/\mathcal{F}_{bas} \\ \text{with anchor } \ell(M,\mathcal{F}) \longrightarrow \ell(M,\mathcal{F}_{bas}) \simeq \mathfrak{X}(M_{bas}) \end{cases}$$

<u>Theorem</u> (Almeida-Molino)

$(M,\mathcal{F})$ is developable (i.e., pullback of $\mathcal{F}$ to $\widetilde{M}$ is simple) iff $A(M,\mathcal{F})$ is integrable.

Exercise: Let G be compact 1-connected (e.g., $SU(3)$)
Take $H \subset G$ a non-closed subgroup (e.g., $H=$ irrational line in $\mathbb{T}^2 \subset SU(3)$)
Show that $\mathcal{F} := \{ gH : g \in G \}$ is $\perp$-parallelizable & not developable.

Main Theorem

A Lie algebroid A is integrable iff there exists an open $U \subset A$ containing zero section 0_M s.t.

$$N(A) \cap U = \{ 0_M \} \qquad (*)$$

Historical Remark:

- Pradines 1960's: formulated Lie functor for Lie groupoids/algebroids following work of Ehresmann, Kumpera & Spencer. Stated Lie III was valid
- McKenzie 1980-85: Look at transitive case and found cohomological obstruction; tried to showed that obstruction always vanishes
- Almeida & Molino 1985: while working on transversely parallelizable foliations found a non-integrable transitive Lie algebroid
- 1985–2003: Many examples of integrable and non-integrable Lie algebroids using various ad-hoc methods
- Crainic & RLF 2003: General theory of integrability giving the obstructions and explaining/improving all previous results

Sketch of Proof

($\Rightarrow$) Assume $\mathcal{G}(A)$ is Lie groupoid. Choose TM-connection ∇ on A, and consider

$$\exp_p^{\bar{\nabla}} : U \longrightarrow \mathcal{G}(A), \quad M \subset U \subset A \text{ open}$$

Note that $\exp^{\tilde\nabla}|_{\mathfrak{g}_x} = \exp : \mathfrak{g}_x \to \mathcal{G}(A)_x$ (usual group exp)

so that:

$$\begin{tikzcd} & \mathcal{G}(\mathfrak{g}_x) \arrow[d, "p_x"] \\ \mathfrak{g}_x \arrow[ur, "\exp"] \arrow[r, "\exp^{\tilde\nabla}|_{\mathfrak{g}_x}"'] & \mathcal{G}(A)_x \end{tikzcd} \qquad \tilde{N}_x(A) = \ker p_x$$

$$v \in N_x(A) \Rightarrow \exp(v) \in \ker p_x$$

We can choose U small enough so that $\mathrm{Exp}^{\tilde\nabla}$ is injective.

For such a U we obtain:

$$U \cap N_x(A) = \{0_x\}, \quad \forall x \in M$$

($\Leftarrow$) We need to show that $P(A)/\mathcal{F}_A$ is smooth, i.e.,

<u>Proposition</u> If $N(A)$ is uniformly discrete then for each $a \in P(A)$ there exists $S_a \subset P(A)$ a transversal to $\mathcal{F}_A$ that intersects each leaf at most once.

Fix $a \in P(A)$ and let $x = \gamma_a(1)$.

<u>Step 1:</u> We may assume $a = 0_x$

Choose sections $\alpha_t \in \Gamma_c(A)$ with $\alpha_t(\gamma_a(t)) = a_t$.

Define $b_\alpha : M \to P(A)$ (think bisection!)

$$b_\alpha(y)(t) := \alpha_t(\varphi^{t,0}_{\rho(\alpha_t)}(y)) \quad (t \in [0,1])$$

Note that $b_\alpha(y)$ is an A-path w/ initial point y. Then we have a map $T : P(A) \to P(A)$ given by left multiplication by b_α:

$$\tilde{a} \longmapsto b_\alpha(\gamma_{\tilde a}(1)) \circ \tilde{a}$$

This is a smooth, injective, immersion.

If S_x is transversal through 0_x as in Prop, then $T(S_x)$ is transversal through $a \circ 0_x$ as in Prop. Using holonomy along any path in $\mathcal{F}_A$ connecting $a \circ 0_x$ and a we obtain the desired transversal S_a. ∎

- Fix $x \in M$ and prove Prop for $a = 0_x$
- (x^s) = local coord around x;
- $\{\alpha_1, \dots, \alpha_r\}$ = local basis of sections for A;
- ∇ = canonical flat TM-connection $\nabla_{\frac{\partial}{\partial x^s}} \alpha_i = 0$
- $0_x \in U \subset A$ open so that $\exp^{\tilde{\nabla}} : U \to P(A)$ is transverse to $\mathcal{F}_A$

Proposition will follow by showing that if U is small enough then $\exp^{\tilde{\nabla}}(U)$ intersects each leaf at most once.

Step 2. Can choose U so that

$$v \in U \cap \mathfrak{g}_y, \quad \exp^{\tilde{\nabla}}(v) \sim 0_y \implies v \in Z(\mathfrak{g}_y)$$

Exercise: If $|\cdot|$ is a norm in a Lie algebra $\mathfrak{g}$ satisfying

$$|[v,w]| \leq |v||w|$$

Then

$$|v| < \pi, \ \operatorname{Exp}(\operatorname{ad} v) = \operatorname{Id} \implies \operatorname{ad} v = 0$$

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Can find $|\cdot|$ on A such that $|[v,w]| \leq |v||w|$, $\forall v,w \in \mathfrak{g}_y$ y in a neighborhood of x. Now

$$\operatorname{Exp}^{\tilde{\nabla}}(v) \sim 0_y \implies \tau_v = \operatorname{Id} \implies \tau_v|_{\mathfrak{g}_y} = \operatorname{Exp}(\operatorname{ad} v) = \operatorname{Id}$$

$$\implies \operatorname{ad} v = 0 \implies v \in Z(\mathfrak{g}_y)$$

(if U is small so $|v| < \pi$)

Step 3 May choose U so that

$$v \in U \cap \mathfrak{g}_y,\ \mathrm{Exp}^{\bar\nabla}(v) \sim 0_y \implies v = 0_y$$

By step 2, this is just a restatement of the assumption that $N(A)$ are uniformly discrete

Step 4 May choose U so that

$v \in U$, $\exp^{\nabla}(v)$ has base path closed $\implies v \in \mathfrak{g}_y$

Recall $\exp^{\nabla}(v) \equiv$ geodesic with initial condition v. Eqs for geodesics:

$$\begin{cases} \dot{x}^s = B^s_i(x(t))\, a^i(t) \\ \dot{a}^i(t) = 0 \end{cases} \implies \dot{x}^s = B^s_i(x(t))\, v^i \quad (*)$$

Period Bounding Lemma: Given open set D any non-trivial periodic solution of $(*)$ w/ $x(0) \in D$ has period

$$T \geq \frac{2\pi}{M_D} \quad \text{w/} \quad M_D = \sup_{\substack{x \in D \\ 1 \leq s,k \leq m}} \left| \frac{\partial B^s_i}{\partial x^k}(x)\, v^i \right|$$

So it is enough to choose $U \subset A_D$, with D neighborhood of x, so that $M_D < 2\pi$. Then base path of $\exp^{\nabla}(v)$ is $y \iff v \in \mathfrak{g}_y$.

Step 5 Consider pairs (U, O) where U satisfy conditions in previous steps, O is a foliation chart, $\exp^{\bar\nabla}: U \to O$ intersects each plaque in O only once. Can find such pairs (U_1, O_1) (U_2, O_2) and $V \subset U_2$ neighborhood of x such that

$$O_1 \cdot O_1 \subset U, \quad O_2 \cdot O_2 \subset O_2, \quad \overline{O_i} = O_i$$

$$\forall v, w \in V: \ 0_x \cdot \mathrm{Exp}^{\bar\nabla}(v) \sim_O \exp^{\bar\nabla}(v), \quad \exp^{\bar\nabla}(v)\, \overline{\exp^{\bar\nabla}(w)}\, \exp^{\nabla}(w) \sim_O \exp^{\bar\nabla}(v)$$

Here "$\cdot$" means concatenation, $\overline{}$ means reversing A-path. These are continuous operations in $P(A)$, so first set of relations hold.

Here $\sim_{\mathcal{O}}$ means A-path homotopy in $\mathcal{O}$. To prove the second set of relations one observes that there are "natural" homotopies $h : I \times U \to P(A)$ connecting $\exp^{\bar{\nabla}}(v)$ & $0_x \cdot \exp^{\bar{\nabla}}(v)$:

$$h(0,v) = \exp^{\bar{\nabla}}(v), \quad h(1,v) = 0 \cdot \exp^{\bar{\nabla}}(v), \quad h(\varepsilon, 0_x) = 0_x$$

Since I is compact and $\mathcal{O}$ is open, we can find $V \subset U$ neighborhood of x such that $h(I \times V) \subset \mathcal{O}$, so result follows (second identity is similar)

<u>Step 6</u> $\operatorname{Exp}^{\bar{\nabla}} : V \to P(A)$ intersects each leaf of $\mathcal{F}_A$ at most in one point.

Assume $v, w \in V$, $\exp^{\bar{\nabla}}(v) \sim \exp^{\bar{\nabla}}(w)$

$$\underset{(\text{step } 5)}{\Longrightarrow} \quad a_1 = \exp^{\bar{\nabla}}(v) \cdot \overline{\exp^{\bar{\nabla}}(w)} \in \mathcal{O}_1 \text{ is homotopic to } 0_y$$

$$\underset{(\text{choice of } (U_1, \mathcal{O}_1))}{\Longrightarrow} \quad a_1 \sim_{\mathcal{O}_1} \exp^{\nabla}(u) \quad \text{for unique } u \in U_1$$

Since $\exp^{\nabla}(u) \sim 0_y$ its base path is closed

$$\underset{(\text{Step } 4)}{\Longrightarrow} \quad u \in \mathfrak{g}_y \quad \underset{(\text{Step } 3)}{\Longrightarrow} \quad u = 0_y, \quad \text{so} \quad a_1 \sim_{\mathcal{O}_1} 0_y$$

$$\underset{(\text{step } 5)}{\Longrightarrow} \quad \exp^{\nabla}(v) \sim_{\mathcal{O}} \underbrace{\exp^{\bar{\nabla}}(v)\, \overline{\exp^{\nabla}(w)}}_{a_1}\, \exp^{\bar{\nabla}}(w) \sim_{\mathcal{O}} 0_y \cdot \exp^{\nabla}(w) \sim_{\mathcal{O}} \exp^{\nabla}(w)$$

By construction of $\mathcal{O}$, we conclude that $v = w$. $\blacksquare$

To conclude the proof one needs to show that the groupoid operations on $\mathcal{G}(A)$ are smooth and that $\operatorname{Lie}(\mathcal{G}(A)) \simeq A$. We leave

The first part as an exercise. For the second part:

- Definitions $\Rightarrow$ $A \simeq \operatorname{Ker} d_M t$ & $\rho = ds|_A$
- For Lie bracket, observe that:

  (i) Lie bracket on A is determined by flow of sections:

  $$[\alpha, \beta]_A = \frac{d}{dt} (\phi^t_\alpha)^*(\beta) \Big|_{t=0}$$

  (ii) $\operatorname{Exp}: \Gamma(A) \to \operatorname{Bis}(\mathcal{G}(A))$ is injective in neighborhood of zero section and $\phi^t_\alpha \leftrightarrow \phi^t_{\overleftarrow{\alpha}}$

  Hence $[\alpha, \beta]_A \leftrightarrow [\overleftarrow{\alpha}, \overleftarrow{\beta}]$.

∎

Finally, from the proof one also concludes that:

Thm Every Lie algebroid integrates to a local Lie groupoid

Proof:

As in proof, one can choose connection ∇, opens $0_M \subset U \subset A$ & $\mathcal{O} \subset P(A)$ with $\mathcal{O} = \overline{\mathcal{O}}$ and

$$\exp^\nabla : U \longrightarrow \mathcal{O},$$ intersects each plaque of $\mathcal{F}_A$ in $\mathcal{O}$ only once

Then $\mathcal{G}^{loc} := \mathcal{O}/_{\mathcal{N}_\mathcal{O}} \simeq U$ is a local Lie groupoid integrating A.

∎