MATH 595 - Lecture 17

Last time:

- Monodromy map: $\partial_x : \pi_2(O_x) \longrightarrow \mathcal{G}(\mathfrak{g}_x)$
- Extended Monodromy Groups:

$$\widetilde{N}_x(A) = \operatorname{Im} \partial_x = \{ [g] \in \mathcal{G}(\mathfrak{g}_x) : g \sim 0_x \text{ has A-paths} \} \subset Z(\mathcal{G}(\mathfrak{g}_x))$$

- Monodromy Groups:

$$N_x(A) = \{ v \in Z(\mathfrak{g}_x) : v \sim 0_x \text{ has A-paths} \}$$

These are related via exponential map $\exp : \mathfrak{g}_x \to \mathcal{G}(\mathfrak{g}_x)$

$$\exp : N_x(A) \xrightarrow{\sim} \widetilde{N}_x(A) \cap Z(\mathcal{G}(\mathfrak{g}_x))^0$$

- $N(A) = \bigcup_{x \in M} N_x(A)$

Main Theorem

A Lie algebroid A is integrable iff there exists an open $U \subset A$ containing zero section 0_M s.t.

$$N(A) \cap U = \{ 0_M \} \qquad (*)$$

Before sketching a proof we discuss:

- How to compute monodromy groups?

Note that $N_x(A) = N_x(A_{O_x})$, so to simplify notation we assume that A is transitive:

$$0 \to \mathfrak{g}_M \to A \underset{\sigma}{\overset{\rho}{\rightleftarrows}} TM \to 0$$

Fix splitting $\sigma : TM \to A$:

- Curvature of splitting:

$$\Omega^\sigma \in \Omega^2(M; \mathfrak{g}_M), \quad \Omega^\sigma(X,Y) := \sigma([X,Y]) - [\sigma(X), \sigma(Y)]$$

- TM-connection on $\mathfrak{g}_M$:

$$\nabla^\sigma_X \alpha := [\sigma(X), \alpha] \qquad (\alpha \in \Gamma(\mathfrak{g}_M))$$

This connection, in general, is not flat:

$$R^{\nabla^\sigma}(X,Y)\alpha = [\Omega^\sigma(X,Y), \alpha]$$

Assumption: Curvature is center-valued:

$$\Omega^\sigma \in \Omega^2(M, \mathfrak{z}(\mathfrak{g}_M))$$

Under this assumption, ∇^σ is flat so we can integrate forms $\omega \in \Omega^2(M, \mathfrak{g}_M)$:

- $\gamma \in \pi_2(M, x)$: $\displaystyle\int_\gamma \omega := \int_{S^2} \gamma^* \omega \in \mathfrak{g}_x$, with $\begin{array}{c} \gamma^* \mathfrak{g}_M \simeq S^2 \times \mathfrak{g}_x \\ \downarrow \\ S^2 \end{array}$

Rmk: An equivalent way to define the integral is to consider the holonomy cover:

$$p : \tilde{M} \longrightarrow M, \qquad \text{Deck transformations} =$$
$$= \ker(\mathrm{hol}^\nabla_x) \subset \pi_1(M, x)$$

Then $p_* : \pi_2(\tilde{M}, y) \xrightarrow{\sim} \pi_2(M, p(y))$ and $p^* \mathfrak{g}_M \simeq \tilde{M} \times \mathfrak{g}$, so we can set:

$$\int_\gamma \omega := \int_{p_*^{-1}(\gamma)} p^* \omega$$

Proposition Under assumption:

$$N_x(A) = \left\{ \int_\gamma \Omega^6 : [\gamma] \in \pi_2(M,x) \right\} \subset Z(\mathfrak{g}_x)$$

Moreover, $\exp : N_x(A) \to \tilde{N}_x(A)$ is an isomorphism.

Proof: By passing to holonomy cover $p:\tilde{M} \to M$, we can assume that $\mathfrak{g}_M \simeq M \times \mathfrak{g}$. The splitting then gives

$A \simeq TM \times \mathfrak{g}$, $\rho = pr_{TM} : A \to TM$

$\Gamma(A) \simeq \mathfrak{X}(M) \times C^\infty(M,\mathfrak{g})$, $[(X,f),(Y,g)] = ([X,Y], [f,g]_{\mathfrak{g}} + \nabla^6_X g - \nabla^6_Y f - \Omega^6(X,Y))$

A-paths: $a = (\gamma, \phi)$ with $\gamma : I \to M$, $\phi : I \to \mathfrak{g}$.

Choose some TM-connection ∇^M on TM.

- $\nabla = (\nabla^M, \nabla^6)$ is a TM-connection on A & $\overline{\nabla}$ has torsion:

$$T^{\overline{\nabla}}((X,f),(Y,g)) = (T^{\nabla^M}(X,Y), \Omega^6(X,Y) - [f,g]_{\mathfrak{g}})$$

- A-path homotopy covering γ:

$$\Phi = \Phi_1 \, dt + \Phi_2 \, d\varepsilon \qquad \Phi_1 = \left(\frac{d\gamma}{dt}, \phi_1\right), \ \Phi_2 = \left(\frac{d\gamma}{d\varepsilon}, \phi_2\right)$$

w/ $\phi_1, \phi_2 : I \times I \to \mathfrak{g}$ satisfying:

$$\begin{cases} \dfrac{d}{dt}\phi_2 - \dfrac{d}{d\varepsilon}\phi_1 = \Omega^6\left(\dfrac{d\gamma}{dt}, \dfrac{d\gamma}{d\varepsilon}\right) - [\phi_1, \phi_2] \\ \phi_2(0,\varepsilon) = \phi_2(1,\varepsilon) = 0 \end{cases}$$

Definition of ∂_x: Given $[\gamma] \in \pi_2(M,x)$ choose lift $\Phi : T(I \times I) \to A$ of $d\gamma$ satisfying $\Phi_1(t,0) = 0_x$. If we choose $\phi_2 \equiv 0$:

$$\phi_1(t,\varepsilon) = -\int_0^\varepsilon \Omega^6\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\varepsilon}\right) d\varepsilon \Rightarrow$$

$$\Rightarrow \partial_x[\gamma] = \left[t \mapsto -\int_0^1 \Omega^6\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\varepsilon}\right) d\varepsilon\right] = \exp\left(-\int_0^1\int_0^1 \Omega^6\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\varepsilon}\right) d\varepsilon\, dt\right) = \exp\left(-\int_\gamma \Omega^6\right)$$

$a : I \to Z(\mathfrak{g}) \Rightarrow [a] = \exp\left(\int_0^1 a(t)\, dt\right)$

∎

Examples and Applications

1) $\omega \in \Omega^2_{cl}(M)$ determines prequantization algebroid:

- $A = TM \oplus \mathbb{L}_M$, $\rho = pr_{TM}$
- $[(X,f),(Y,g)] = ([X,Y], X(g) - Y(f) + \omega(X,Y))$

This is naturally split with $\Omega^c = \omega$. Hence:

$$N_x(A) = \left\{ \int_\gamma \omega : \gamma \in \pi_2(M,x) \right\} \subset \mathbb{R}$$

If $M = S^2 \times S^2$, $\omega = pr_1^* \omega_{S^2} + \lambda\, pr_2^* \omega_{S^2}$ $\quad \int_{S^2} \omega_{S^2} = 1$

$\pi_2(S^2 \times S^2) \simeq \mathbb{Z} \times \mathbb{Z}$ w/ generators $\gamma_1 = S^2 \times \{y_0\}$, $\gamma_2 = \{x_0\} \times S^2$

$$\partial_x[\gamma_1] = 1, \quad \partial_x[\gamma_2] = \lambda$$

$$\Rightarrow N_x(A) = \{ n_1 + n_2 \lambda : n_1, n_2 \in \mathbb{Z} \} \subset \mathbb{R}$$

Conclusion: A is integrable iff $\lambda \in \mathbb{Q}$.

2) $A = \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ w/ global basis of sections $\{e_1, e_2, e_3\}$

Set:

$$[e_1, e_2] = a e_1 + b\, x_1 m$$
$$[e_2, e_3] = a e_2 + b\, x_2 m \qquad m := x_1 e_1 + x_2 e_2 + x_3 e_3$$
$$[e_3, e_1] = a e_3 + b\, x_3 m$$
$$\rho(e_i) = a X_i$$

where:

$$X_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}, \quad X_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

If $a = a(R)$ and $b = b(R)$ this is a Lie algebroid with orbits the origin & 2-spheres:

$$S_R = \{ (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = R^2 \}$$

Also, $\operatorname{Ker} \rho_x = \mathbb{R}\, \overline{m}$ if $m \neq 0$. $\operatorname{Ker} \rho_0 = \mathbb{R}^3$.

At origin:
$$N_0(A) = \{0\}$$

For sphere w/ $R>0$: restrict A_{S_R} and consider splitting of action:
$$\sigma(X_i) = \frac{1}{a}\left(e_i - \frac{x_i}{R}\bar{m}\right)$$

A tedious computation gives:
$$\Omega^6 = \frac{bR^2-a}{a^2R^4}\,\omega\,\bar{m}, \qquad \omega = x_1 dx_2\wedge dx_3 + x_2 dx_3\wedge dx_1 + x_3 dx_1\wedge dx_2$$

Since $\int_{S_R}\omega = 4\pi R^3$ we obtain:
$$N_x = 4\pi\frac{bR^2-a}{a^2R}\,\mathbb{Z}\,\bar{m}_x$$

Hence:

$b=1,\ a=1 \Rightarrow N_x = 4\pi\frac{R^2-1}{R}\mathbb{Z}\,\bar{m}_x$ Discrete for all x

Not uniformly discrete in neighborhood of $R=1$

$a=R^2,\ b=R^3+1 \Rightarrow N_x = 4\pi\,\mathbb{Z}\,\bar{m}_x$ Discrete for all x

Not uniformly discrete in any neighborhood of 0

$$\Rightarrow \begin{cases} A|_{\mathbb{R}^3-\{0\}} \text{ is integrable} \\ A|_{B(\varepsilon)} \text{ is not integrable for any } \varepsilon>0\end{cases}$$

3) In previous examples, we have $Z(\mathfrak{g})$-valued splitting so $\exp: N_x \xrightarrow{\sim} \tilde{N}_x$. Consider action groupoid:
$$SU(3)\ltimes su(3) \rightrightarrows su(3) \qquad g\cdot\alpha = \mathrm{Ad}_g\cdot\alpha = g\alpha g^{-1}$$

Any element $X\in su(3)$ is diagonalizable:
$$gXg^{-1} = \begin{pmatrix} i\lambda_1 & 0 & 0 \\ 0 & i\lambda_2 & 0 \\ 0 & 0 & i\lambda_3\end{pmatrix}, \quad g\in SU(3) \qquad (\lambda_1+\lambda_2+\lambda_3=0)$$

$\Rightarrow$ Orbits = isospectral sets

- All eigenvalues distinct:
  - Isotropy group $\cong \begin{pmatrix} e^{i\theta_1} & & 0 \\ & e^{i\theta_2} & \\ 0 & & e^{i\theta_3} \end{pmatrix}$ $(\theta_1+\theta_2+\theta_3=0) \cong \mathbb{S}^1\times\mathbb{S}^1$
  - Orbits $\cong SU(3)/_{\mathbb{S}^1\times\mathbb{S}^1}$ (6-dimensional)
- Two equal eigenvalues:
  - Isotropy group $\cong \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \det A^{-1} \end{array} \right) : A \in U(2) \right\}$
  - Orbits $\cong SU(3)/_{U(2)} \cong \mathbb{C}P(2)$ (4-dimensional)
- Origin: Isotropy group $= SU(3)$

Take a 4-dim orbit. Then:

$$\mathfrak{g}_\alpha = \left\{ \left(\begin{array}{c|c} X & 0 \\ \hline 0 & -\mathrm{tr}\,X \end{array} \right) : X \in \mathfrak{u}(2) \right\}$$

We obtain:

$$U(2) = G_{\mathfrak{g}_\alpha}\ G\ \begin{array}{c} S^{-1}(\alpha) \\ \cong \\ SU(3) \end{array} \longrightarrow \begin{array}{c} O_\alpha \\ \cong \\ \mathbb{C}P(2) \end{array}$$

$$\Rightarrow \quad \begin{array}{c} \pi_2(O_\alpha) \\ \cong \\ \mathbb{Z} \end{array} \xrightarrow{\partial_\alpha} \begin{array}{c} \mathcal{G}(\mathfrak{g}_\alpha) \\ \cong \\ \mathbb{R}\times SU(2) \end{array} \xrightarrow{p_\alpha} \begin{array}{c} \mathcal{G}_\alpha \\ \cong \\ \mathbb{S}^1\times SU(2) : U(2) \end{array} \xrightarrow{q_\alpha} \begin{array}{c} \pi_1(\mathcal{G}_\alpha) \\ \cong \\ 1 \end{array} \longrightarrow 1$$

- $Z(SU(2)) = \{I, -I\}$
- $\partial_\alpha(m) = (\pi m, (-1)^m I)$
- $p_\alpha(\theta, A) = e^{i\theta} A$
- $Z(\mathbb{R}\times U(2)) = \mathbb{R}\times\{\pm I\}$

$$\Rightarrow \left\{ \begin{array}{l} \tilde{N}_\alpha \subset \pi\mathbb{Z}\times\{\pm I\} \\ \cancel{\cong} \\ N_\alpha \cong 2\pi\mathbb{Z}\times\{I\} \end{array} \right.$$

$\Rightarrow \nexists$ $\mathbb{Z}$-valued splitting

Another argument for $\nexists$ 2-valued splitting

2-valued splitting $\Rightarrow$ Flat connection on $\ker \rho|_{G_x} \simeq \nu(\mathbb{CP}(2))$
$$\begin{array}{c}\nu(\mathbb{CP}(2)) \\ \cap \\ su(3) \simeq \mathbb{R}^8\end{array}$$

$\mathbb{CP}(2)$ 1-connected $\Rightarrow$ $\nu(\mathbb{CP}(2))$ is trivial

But total Stiefel-Whitney class of $\mathbb{CP}(2)$ is non-trivial

$\Rightarrow$ " " " " $\nu(\mathbb{CP}(2))$ " " . Contradiction!

4) Regular Lie algebroids are locally integrable

$A \to M$ regular Lie algebroid : $\mathrm{Im}\,\rho = T\mathcal{F}$

choose $U \subset M$ foliated chart : $U \simeq \mathbb{R}^p \times \mathbb{R}^q$, $q = \mathrm{codim}\,\mathcal{F}$

$\Rightarrow$ $A|_U \to U$ has 2-connected leaves $\Rightarrow$ integrable

$\Rightarrow$ $A|_U \simeq T\mathbb{R}^p \times \mathfrak{g} \to U$