MATH 595 - LECTURE 16

<u>Last time:</u> $\mathcal{F}_A \subset P(A)$ foliation determined by A-path homotopy

When is A integrable?

$\Leftrightarrow$ When is the leaf space $\mathcal{G}(A) = P(A)/\mathcal{F}_A$ smooth?

<u>An obstruction to integrability:</u>

Assume A integrable $\Leftrightarrow$ $\mathcal{G}(A) = P(A)/\mathcal{F}_A$ smooth

$\Rightarrow$ $\mathcal{G}(A)_x$ is a Lie group w/ Lie algebra $\mathfrak{g}_x = \operatorname{Ker} \rho_x$

$\Rightarrow$ $q_x : G(\mathfrak{g}_x) \longrightarrow \mathcal{G}(A)_x^0$ is a covering map

<u>Lemma:</u> $\widetilde{N}_x = \operatorname{Ker} q_x \subset G(\mathfrak{g}_x)$ is a discrete subgroup of $Z(G(\mathfrak{g}_x))$

<u>Proof:</u> q_x is covering & homomorph. $\Rightarrow$ $\widetilde{N}_x$ is discrete and normal.

Every normal, discrete subgroup D of a connected Lie group G is contained in $Z(G)$:

$g \in G$, $d \in D$. Choose path $g(t) \in G$ w/ $g(0) = e$, $g(1) = g$. Then:

$g(t)\, d\, g(t)^{-1} \in D$ must be constant $\Rightarrow$ $g d g^{-1} = g(0)\, d\, g(0)^{-1} = d$.

$\Rightarrow$ $g d g^{-1} = g(0)\, d\, g^{-1}(0) = d$ $\Leftrightarrow$ $d \in Z(G)$. ∎

Hence:

$$\mathcal{G}(A)_x^0 = G(\mathfrak{g}_x)/\widetilde{N}_x \quad \& \quad \pi_1(\mathcal{G}(A)_x^0) = \widetilde{N}_x$$

$t: s^{-1}(x) \to O_x$ = principal $\mathcal{G}(A)_x$-bundle w/ $s^{-1}(x)$ 1-connected, so:

$$\cdots \longrightarrow \pi_2(O_x) \xrightarrow{\partial_x} \pi_1(\mathcal{G}(A)_x) \longrightarrow 1 \longrightarrow \pi_1(O_x) \longrightarrow \pi_0(\mathcal{G}(A)_x) \longrightarrow 1$$

It follows that we have short exact sequence:

$$\pi_2(G_x) \xrightarrow{\partial_x} \mathcal{G}(\mathfrak{g}_x) \xrightarrow{q_x} \mathcal{G}(A)_x \longrightarrow \pi_1(O_x) \longrightarrow 1 \qquad (*)$$

where $\mathrm{Im}\,\partial_x = \widetilde{N}_x$.

(*) still exists in the non-integrable case!!

This leads to:

Main Obstruction to integrability:

If A is integrable, $\widetilde{N}_x = \mathrm{Im}\,\partial_x \subset \mathcal{G}(\mathfrak{g}_x)$ is discrete

We will see that $\widetilde{N}_x$ can be computed in many cases. First we see how it can be defined for any Lie algebroid:

Proposition For any Lie algebroid A, there is a short exact sequence of groups:

$$\pi_2(G_x) \xrightarrow{\partial_x} \mathcal{G}(\mathfrak{g}_x) \xrightarrow{q_x} \mathcal{G}(A)_x \xrightarrow{p_x} \pi_1(O_x) \longrightarrow 1$$

where:

(i) p_x maps $[a] \mapsto [\gamma_a]$

(ii) $q_x = \mathcal{G}(i)$, $i: \mathfrak{g}_x \hookrightarrow A$

(iii) ∂_x maps $[\sigma]$ to $[a]$ where $a: I \to \mathfrak{g}_x$ is A-path homotopic to 0_x via A-path homotopy covering σ.

RMK: What we are doing is working out explicitly the first terms of long exact sequence of $\tilde{s}^{-1}(x) \to O_x$, $\tilde{s}^{-1}(x) \subset \mathcal{G}(A)$. Since $\tilde{s}^{-1}(x)$ can be very pathological, we are not allowed to use the result that principal bundle is a Serre fibration.

We defer the proof for later.

Defn. The map $\partial_x : \pi_2(\mathcal{O}_x) \to \mathcal{G}(\mathfrak{g}_x)$ is called the **monodromy map** of A and $\tilde{N}_x(A) := \operatorname{Im} \partial_x$ is called the **monodromy group** at $x \in M$.

Note that $\tilde{N}_x(A)$ is a normal subgroup of $\mathcal{G}(\mathfrak{g}_x)$. But it may fail to be closed (obstruction!). Still:

Lemma $\tilde{N}_x(A) \subset Z(\mathcal{G}(\mathfrak{g}_x))$

Proof.

Each $g \in \tilde{N}_x(A)$ is represented by an A-path $a : I \to \mathfrak{g}_x$ which is A-path homotopic to 0_x. Working on orbit $\mathcal{O} \ni x$, we have Rep $(A_\mathcal{O})$ on isotropy $\mathfrak{g}_\mathcal{O}$ ("Bott connection")

$$\nabla_\alpha \beta = [\alpha, \beta] \qquad (\alpha \in \Gamma(A|_\mathcal{O}),\ \beta \in \Gamma(\mathfrak{g}_\mathcal{O}))$$

This Rep $(A_\mathcal{O})$ restricts to $\mathrm{ad} : \mathfrak{g}_x \to \mathfrak{gl}(\mathfrak{g}_x)$. Then //-transp gives Ad on $\mathcal{G}(\mathfrak{g}_x)$. Hence, since $a \sim 0_x$:

$$\left.\begin{array}{l} \cdot\ \tau_a = \mathrm{Ad}_g : \mathfrak{g}_x \to \mathfrak{g}_x \\ \cdot\ \tau_a = \tau_{0_x} = \mathrm{Id} \end{array}\right\} \Rightarrow \mathrm{Ad}_g = \mathrm{Id} \Leftrightarrow g \in Z$$

$\boxtimes$

Note that $Z(\mathcal{G}(\mathfrak{g}_x))$ integrates $Z(\mathfrak{g}_x)$ but may fail to be connected. Passing to connected component of identity:

$$\exp : (Z(\mathfrak{g}_x), +) \longrightarrow (Z(\mathcal{G}(\mathfrak{g}_x))^0, \cdot)$$

This leads to a version of monodromy living in $\mathfrak{g}_x \subset A_x$, which is better for computations.

Proposition: Set:

$$N_x(A) = \exp^{-1}(\widetilde{N}_x(A) \cap Z^0)$$

TFAE:

(i) $\widetilde{N}_x(A) \subset \mathcal{G}(\mathfrak{g}_x)$ is closed

(ii) $\widetilde{N}_x(A) \subset \mathcal{G}(\mathfrak{g}_x)$ is discrete

(iii) $N_x(A) \subset \mathfrak{g}_x$ is closed

(iv) $N_x(A) \subset \mathfrak{g}_x$ is discrete

Proof: For a 1-connected Lie group G w/ Lie algebra $\mathfrak{g}$

$$\exp : \mathfrak{g} \to G$$

restricts to a group isomorphism

$$\exp : Z(\mathfrak{g}) \to Z(G)^0$$

Since $\pi_2(O_x)$ is countable, $\widetilde{N}_x(A)$ and $N_x(A)$ are countable and the equivalences follow.

$\blacksquare$

Note that:

- $\widetilde{N}_x(A) = \{ [g] \in \mathcal{G}(A)_x : g \sim 0_x \}$
- $N_x(A) = \{ v \in \mathfrak{g}_x : v \sim 0_x \}$

Exercise: Show that if x, y belong to same orbit O of A then $N_x(A) \simeq N_y(A)$ and $\widetilde{N}_x(A) \simeq \widetilde{N}_y(A)$ (canonically!) Moreover, there is a bundle isomorphism $\tau : \mathfrak{g}_O \to \mathfrak{g}_O$ such that $\tau : N_x(A) \xrightarrow{\sim} N_y(A)$.

Hint: Use the Bott A_O-connection on $\mathfrak{g}_O$ as in proof above

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Set:
$$N(A) = \bigcup_x N_x(A) \subset \operatorname{Ker}\rho \subset A$$

<u>Theorem (Crainic-F)</u>

A Lie algebroid A is integrable iff there exists an open $U \subset A$ containing zero section 0_M s.t.
$$N(A) \cap U = \{0_M\} \quad (*)$$

<u>Rmks</u>

- Fixing $x \in M$, $U_x := U \cap \mathfrak{g}_x$ is open in $\mathfrak{g}_x$, so $(*)$ gives $N_x(A) \cap U_x = \{0\}$ $\Rightarrow$ $N_x(A) \subset \mathfrak{g}_x$ are discrete
- Condition $(*)$ says that $N_x(A)$ are "uniformly" discrete
- If x, y belong to same orbit: $N_x(A) \simeq N_y(A)$ and N_x is discrete iff N_y is discrete (exercise above)
- If A is transitive, $N_x(A) \subset \mathfrak{g}_x$ discrete $\Rightarrow$ $(*)$ (Again by exercise above!)
- For general A, we can think that condition $(*)$ has two components:

  (a) Along leaves: $N_x(A) \subset \mathfrak{g}_x$ is discrete

  (b) transverse to leaves: $d(N_x(A) - \{0_x\}, 0_x)$ stays bounded away from 0 when x varies in transverse direction.

<u>Corollary</u>. Any Lie algebroid w/ trivial monodromy groups $N_x(A)$ is integrable. For example, this happens if:

(i) $\mathfrak{g}_x$ has trivial center, $\forall x \in M$;

(ii) orbits O_x have finite π_2, $\forall x \in M$;

(iii) For every orbit $O \subset M$, there is a splitting $\sigma: TO \to A_O$ of the anchor $\rho: A_O \to TO$ preserving Lie brackets.

Still we would like to compute the monodromy groups. We will see that next lecture.

Proof of Proposition:

We want to show exactness of sequence of group homomorphisms:

$$\pi_2(O_x) \xrightarrow{\partial_x} \mathcal{G}(\mathfrak{g}_x) \xrightarrow{q_x} \mathcal{G}(A)_x \xrightarrow{p_x} \pi_1(O_x) \longrightarrow 1$$

First one needs to check that maps ∂_x & q_x are well-defined:

- $q_x : \mathcal{G}(\mathfrak{g}_x) \to \mathcal{G}(A)_x$: this is just $\mathcal{G}(i)$ where $i : \mathfrak{g}_x \hookrightarrow A$

- $\partial_x : \pi_2(O_x) \longrightarrow \mathcal{G}(\mathfrak{g}_x)$: Let $\sigma : I \times I \to O_x$ w/ $\sigma(\partial(I\times I)) = x$

Then $\partial_x[\sigma] = [a]$ where $a : I \to \mathfrak{g}_x$ is A-path homotopic to 0_x via $\Phi : T(I\times I) \to A$ covering σ.

Exercise: Show that Φ exists (Hint: see proofs last time and use splitting of anchor)

Now similarly to ordinary homotopy theory, one shows:

(i) if $a : I \to A$ has base path γ_a, and $\gamma : I\times I \to O_x$ is homotopy starting at γ_a, $\exists$ A-path homotopy Φ covering γ & starting at a

(ii) two $\mathfrak{g}_x$-path $a_0, a_1 : I \to \mathfrak{g}_x$ are $\mathfrak{g}_x$-homotopic iff $\exists$ A-path homotopy whose base path $\sigma : I\times I \to O_x$ is the trivial class $[\sigma] \in \pi_2(O_x)$.

- $\operatorname{Im}\partial_x = \operatorname{Ker} q_x$: $\subset$ is obvious from definition. For $\supset$ let $a : I \to \mathfrak{g}_x$ represent $[a] \in \mathcal{G}(\mathfrak{g}_x)$ in $\operatorname{Ker} q_x$. This means $\exists\ \Phi$ A-path homotopy giving $a \sim 0_x$. But base γ of this homotopy defines $[\gamma] \in \pi_2(O_x)$ w/ $\partial_x[\gamma] = [a]$.

• $\underline{\mathrm{Im}\, q_x = \mathrm{Ker}\, p_x}$: $\subset$ is obvious from definitions. For $\supset$ let $[a] \in \mathcal{G}(A)_x$ be in $\mathrm{Ker}\, p_x$, i.e., γ_a is contractible in orbit $\mathcal{O}_x$. Choose path-homotopy $\sigma : I \times I \to \mathcal{O}_x$ giving $\gamma_a \sim \underline{x}$. Can find A-path homotopy $\Phi : T(I \times I) \to A$ covering σ with $\Phi_1(t,0) = a(t)$.

Then $a_1(t) \equiv \Phi_1(t,1)$ is $\mathfrak{g}_x$-path which is A-path homotopic to a. So:

$$q_x([\sigma]) = [a_1] = [a]$$

• $\underline{p_x \text{ is surjective:}}$ Any loop $\gamma : I \to \mathcal{O}_x$, $\gamma(0) = \gamma(1) = x$ is the base path of an A-path $a : I \to A$ (e.g., use splitting $\sigma : T\mathcal{O} \to A|_{\mathcal{O}}$ of the anchor)

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