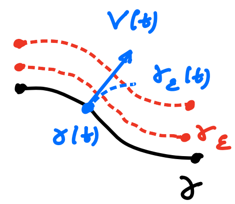
MATH 595 - Lecture 15

Last time:

- $\tilde{P}(M) = \{ \gamma : I \to M \text{, smooth} \}$ w/ C^1-topology is Banach Bundle
- Chart $\Gamma(\gamma^* TM) \supset U \longrightarrow \tilde{P}(M)$, $V(t) \longmapsto (t \mapsto \exp_{\gamma(t)}(V(t)))$
- $T_\gamma \tilde{P}(M) = \Gamma(\gamma^* TM) \cong$ vector fields along γ

  $\varepsilon \mapsto \gamma_\varepsilon$ smooth curve in $\tilde{P}(M)$:

$$V(t) := \frac{d}{d\varepsilon} \gamma(t) \Big|_{\varepsilon=0} \in T_{\gamma(t)} M$$



Want to apply this to space of A-paths:

$$P(A) = \{ a : I \to A \mid \rho(a(t)) = \dot{\gamma}_a(t) \} \subset \tilde{P}(A)$$

Namely, we want to show:

- $P(A) \subset \tilde{P}(A)$ is a Banach submanifold
- $\sim$ defines foliation $\mathcal{F}_A$ of $P(A)$ of codimension $\dim M + \operatorname{rank} A$
- A integrable $\Leftrightarrow$ Leaf space $\mathcal{F}_A$ is smooth and $\mathcal{G}(A) = P(A)/\sim$ w/ this smooth structure is Lie grpd

Fix TM-connection ∇ on A. Gives a splitting:

$$T_{\xi_x} A \simeq \underbrace{A_x}_{\text{vert}} \oplus \underbrace{T_x M}_{\text{horiz}}$$

Then we have identification:

$$\underset{\substack{\Vert \\ U \in \Gamma(a^* TA)}}{T_a \widetilde{P(A)}} \simeq \{ (u, \phi) : u : I \to A,\ \phi : I \to TM \text{ curves over } \gamma_a \}$$

$$U(t) \in T_{a(t)} A \xrightarrow{\sim} (u(t), \phi(t)) \in A_{\gamma_a(t)} \oplus T_{\gamma_a(t)} M$$

Recall that ∇ determines A-connections on A and TM:

$$\overline{\nabla}_\alpha \beta := \nabla_{\rho(\beta)}\alpha + [\alpha, \beta]$$

$$\overline{\nabla}_\alpha X := \rho(\nabla_X \alpha) + [\rho(\alpha), X]$$

Proposition

$P(A)$ is a Banach submanifold of $\widetilde{P}(A) = \{a : I \to A\}$

Given a TM-connection ∇ on A we have

$$T_a P(A) = \{ U = (u, \phi) \ : \ \rho(u(t)) = \overline{D}_a \phi(t) \}$$

where $\overline{D}$ is the derivative associated w/ A-connection $\overline{\nabla}$ on TM

Proof:

$$F : \widetilde{P}(A) \longrightarrow \widetilde{P}(TM)$$

$$a(t) \longmapsto \rho(a(t)) - \frac{d}{dt}\gamma_a(t)$$

This smooth map and:

$$P(A) = F^{-1}(O) \qquad \text{w/ } O = \{ O_\gamma : \gamma : I \to M \} \simeq \widetilde{P}(M)$$

Since $O \subset \widetilde{P}(TM)$ is Banach submanifold, we only need to check that $F \pitchfork O$. We compute

$$d_a F : T_a \widetilde{P}(A) \to T_{\gamma_a} \widetilde{P}(TM) \qquad (\text{at } P(A))$$

Using ∇ write $U(t) = (u(t), \phi(t)) \in T_{\gamma_a(t)}A \oplus T_{\gamma_a(t)}M$. Then:

$$d_a F(U)(t) \in T_{O_{\gamma_a(t)}}(TM) \simeq \underbrace{T_{\gamma_a(t)}M}_{\text{Vert}} \oplus \underbrace{T_{\gamma_a(t)}M}_{\text{horiz}}$$

Canonical !

Claim:

- $d_a F(U)^{\text{hor}} = \phi$
- $d_a F(U)^{\text{Vert}} = \rho(u) - D_a \phi$

This implies $F \pitchfork O$ and also gives formula for $T_a P(A)$.

By splitting a into small intervals, can do computation in local coordinates. Also decomposition is indep. of choice of ∇ so can assume ∇ is flat connection:

- (U, x^s) - local coordinates for M
- $\{\alpha_1, \dots, \alpha_k\}$ - local basis of section for $A|_U$, $\rho(\alpha_i) = B_i^s \frac{\partial}{\partial x^s}$
- $\nabla_{\frac{\partial}{\partial x^s}} \alpha_i = 0$
- $T_{0_x}(TM) = T_x M \oplus T_x M$ has horizontal basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\}$
  vertical basis $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^m}\}$
- $U(t) = \frac{d}{d\varepsilon} a_\varepsilon(t)\Big|_{\varepsilon=0}$, $a_0(t) = a(t)$

Let $\gamma_\varepsilon = \gamma_{a_\varepsilon}$ base path of a_ε. Then:

$$\begin{cases} a_\varepsilon(t) = a_\varepsilon^i(t)\, \alpha_i(\gamma_\varepsilon(t)) \\ \gamma_\varepsilon(t) = (\gamma_\varepsilon^1(t), \dots \gamma_\varepsilon^m(t)) \end{cases}$$

Then $U(t) = (u(t), \phi(t))$ where:

- $\phi(t) = \frac{d}{d\varepsilon}\gamma_\varepsilon(t)\Big|_{\varepsilon=0} = \frac{d\gamma_\varepsilon^s}{d\varepsilon}\Big|_{\varepsilon=0} \frac{\partial}{\partial x^s}\Big|_{\gamma_a(t)} \left(=: \phi^s(t) \frac{\partial}{\partial x^s}\Big|_{\gamma_a(t)}\right)$
- $u(t) = \frac{d}{d\varepsilon} a_\varepsilon^i(t)\Big|_{\varepsilon=0} \alpha_i\Big|_{\gamma(t)} \left(=: u^i(t)\, \alpha_i\Big|_{\gamma_a(t)}\right)$

$$\begin{aligned} d_a F(U)(t) &= \frac{d}{d\varepsilon} F(a_\varepsilon(t))\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\Big(\rho(a_\varepsilon(t)) - \dot\gamma_\varepsilon(t)\Big)\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\Big(B_i^s(\gamma_\varepsilon(t))\, a_\varepsilon^i(t) - \dot\gamma_\varepsilon^s(t)\Big)\Big|_{\varepsilon=0} \frac{\delta}{\delta x^s}\Big|_{\gamma_a(t)} + \\ &\qquad + \frac{d}{d\varepsilon}\gamma_\varepsilon^s(t)\Big|_{\varepsilon=0} \frac{\partial}{\partial x^s}\Big|_{\gamma_a(t)} = \end{aligned}$$

$$= \underbrace{\left(\frac{\partial B_i^s}{\partial x^r}(\gamma_a(t))\, \phi^r(t)\, a^i(t) + B_i^s(\gamma_a(t))\, u^i(t) - \dot{\phi}^s(t) \right)}_{d_a F(v)^{\text{vert}}} \frac{\delta}{\delta x^i}\Big|_{\gamma_a(t)}$$

$$+ \underbrace{\phi^s(t)}_{d_a F(v)^{\text{hor}}} \frac{\partial}{\partial x^s}\Big|_{\gamma_a(t)}$$

Now observe that:

$$\overline{\nabla}_{\alpha_i} \frac{\partial}{\partial x^r} = \rho\Big(\underbrace{\nabla_{\frac{\partial}{\partial x^r}} \alpha_i}_{0}\Big) + \left[\rho(\alpha_i), \frac{\partial}{\partial x^r}\right]$$

$$= \left[B_i^s \frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^r} \right] = - \frac{\partial B_i^s}{\partial x^r} \frac{\partial}{\partial x^s}$$

$$\Rightarrow \overline{D}_a \phi = a^i(t)\, \overline{D}_{\alpha_i}\left(\phi^r(t) \frac{\partial}{\partial x^r}\Big|_{\gamma(t)}\right) = \left(- a^i(t)\, \phi^r(t) \frac{\partial B_i^s}{\partial x^r}(\gamma(t)) + \dot{\phi}^s(t) \right) \frac{\partial}{\partial x^s}\Big|_{\gamma_a(t)}$$

so:

$$d_a F(v)^{\text{vert}} = \rho(u) - \overline{D}_a \phi$$

$\boxtimes$

Let

$$P_0 \Gamma(A) = \{ \beta_t \in \Gamma(A) : \beta_0 = \beta_1 = 0 \}$$

This is a Lie algebra with pointwise-bracket:

$$[\beta_\cdot, \beta'_\cdot]_t := [\beta_t, \beta'_t]_A$$

We define:

$$\sigma : P_0 \Gamma(A) \longrightarrow \mathfrak{X}(A), \quad \beta \longmapsto X_\beta$$

by:

$$X_\beta|_a = (u, \phi) \quad \text{w/} \quad \begin{cases} u = \overline{D}_a b \\ \phi = \rho(b) \end{cases} \qquad b(t) = \beta_t(\gamma_a(t))$$

<u>Lemma:</u> $[X_\beta, X_{\beta'}] = X_{[\beta, \beta']}, \quad \forall \beta, \beta' \in P_0 \Gamma(A)$

Proof is left as exercise.

Hence $\beta \mapsto X_\beta$ is a Lie algebra action on $P(A)$. Its image is the distribution $D \subset TP(A)$ given by:

$$D_a = \{ (\bar{D}_a b, \rho(b)) : b(t) \in A_{\gamma_a(t)},\ b(0) = b(1) = 0 \}$$

Proposition

The spaces D_a do not depend on choice of connection ∇:

$$X_{\beta,a}(t) = \frac{d}{d\varepsilon} \varphi^{\varepsilon,0}_{\beta_t}(a(t)) + \frac{d}{dt} \beta_t \Big|_{\gamma(t)}$$

where $\varphi^{\varepsilon,\varepsilon'}_{\beta_t} : A \to A$ is the flow of the section β_t. Moreover D_a has finite codimension, so defines a foliation $\mathcal{F}_A$ of $P(A)$ of codimension $m+k$, such that:

(i) $a_0 \sim a_1$ iff a_0 & a_1 belong to same leaf of $\mathcal{F}_A$

(ii) For any A-connection ∇, the exponential map

$$\mathrm{Exp}^\nabla : A \to P(A)$$

is transverse to $\mathcal{F}_A$.

Proof

To see that $D_a \subset T_a P(A)$ has codimension $\dim M + \operatorname{rank} A$, let $(u, \phi) \in T_a P(A)$, i.e., $\rho(u) = \bar{D}_a \phi$. Then

$$(u, \phi) \in D_a \quad \text{iff} \quad u = \bar{D}_a b,\ \phi = \rho(b),\ b(0) = b(1) = 0$$

So consider the linear map:

$$L : T_a P(A) \longrightarrow T_{\gamma(0)} M \oplus A_{\gamma_a(1)} \qquad \text{w/ } b(t) \text{ solution of}$$

$$(u, \phi) \longmapsto (\phi(0), b(1)) \qquad \begin{cases} \bar{D}_a b = u \\ b(0) = 0 \end{cases} \quad (*)$$

Since $(*)$ has unique solution, $D_a = L^{-1}(0)$ so

$$\operatorname{codim} D_a = \dim\left(T_{\gamma(0)} M \oplus A_{\gamma_a(1)}\right) = \dim M + \operatorname{rank} A$$

By Frobenius, $\exists\ \mathcal{F}_A$ with $T_a \mathcal{F}_A = D_a$, $\forall a \in P(A)$.

• <u>$X_{\beta, a} = \frac{d}{d\varepsilon} a_\varepsilon \big|_{\varepsilon=0}$ FOR SOME A-path homotopy a_ε w/ $a_0 = a$.</u>

RECALL That $\Phi = \Phi_1(t,\varepsilon)\,dt + \Phi_2(t,\varepsilon)\,d\varepsilon$ covering $\gamma(t,\varepsilon)$ DETERMINES A-homotopy iff $\rho \circ \Phi = d\gamma$ AND:

$$\left(\frac{d\alpha_{t,\varepsilon}}{d\varepsilon} - \frac{d\beta_{t,\varepsilon}}{dt} \right)\Bigg|_{\gamma(t,\varepsilon)} = [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\Big|_{\gamma(t,\varepsilon)}$$

w/ $\alpha_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_1(t,\varepsilon)$ $\quad \beta_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_2(t,\varepsilon)$

$\Phi_2(0,\varepsilon) = \Phi_2(0,\varepsilon) = 0$

So Given $\beta_t \in \Gamma(A)$ w/ $\beta_0 = \beta_1 = 0$ AND A-path $a : I \to A$ let $\alpha_t \in \Gamma(A)$ such that $\alpha_t(\gamma_a(t)) = a(t)$.

The solution OF:

$$\begin{cases} \dfrac{d\alpha_{t,\varepsilon}}{d\varepsilon} = \dfrac{d\beta_t}{dt} + [\alpha_{t,\varepsilon}, \beta_t] \\[2ex] \alpha_{t,0} = \alpha_t \end{cases}$$

IS Given by:

$$(*) \qquad \alpha_{t,\varepsilon} = \int_0^\varepsilon (\varphi_{\beta_t}^{\varepsilon,\varepsilon'}) \left(\frac{d\beta}{dt}\right) d\varepsilon' + (\varphi_{\beta_t}^{\varepsilon,0})(\alpha_t)$$

(DIRECT computation!)

HENCE IF we let

$$\gamma(t,\varepsilon) := \varphi^{\varepsilon,0}_{\rho(\beta_t)}(\gamma_a(t))$$

Then:

$$a_\varepsilon(t) = \Phi_1(t,\varepsilon) := \alpha_{t,\varepsilon}\Big|_{\gamma(t,\varepsilon)}$$

is A FAMILY OF A-homotopic A-paths with $a_0 = a$ AND DEFINING A TANGENT vector

$$\frac{d}{d\varepsilon} a_\varepsilon \Big|_{\varepsilon=0} \in T_a P(A)$$

Given some connection ∇, the vertical component is:

$$\mathrm{Vert} = \left(\frac{d}{d\varepsilon}\Phi_1(\varepsilon,t)\Big|_{\varepsilon=0}\right)^{\mathrm{Vert}} = \left(\nabla_{\rho(\beta_t)}\alpha_{t,\varepsilon} + \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\Big|_{\gamma(t,0)}$$

$$= \left(\overline{\nabla}_{\alpha_{t,\varepsilon}}\beta_t - [\alpha_{t,\varepsilon},\beta_t] + \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\Big|_{\gamma(t,0)}$$

$$= \left(T^{\overline{\nabla}}(\alpha_{t,\varepsilon},\beta_t) + \overline{\nabla}_{\beta_t}\alpha_{t,\varepsilon} + \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\Big|_{\gamma(t,0)}$$

Geometric interp. of torsion

$$= \left(T^{\overline{\nabla}}(\Phi_1,\Phi_2) + \overline{D}_{\Phi_2}\Phi_1\right)\Big|_{\varepsilon=0}$$

$$= \overline{D}_{\Phi_1}\Phi_2\Big|_{\varepsilon=0} = \overline{D}_a b \quad \text{w/} \quad b(t) = \Phi_2(t,0) = \eta_t(\gamma_a(t))$$

The horizontal component is:

$$\mathrm{hor}: \quad \frac{d}{d\varepsilon}\gamma(t,\varepsilon)\Big|_{\varepsilon=0} = \rho(b(t))$$

Hence:

$$\frac{d}{d\varepsilon}a_\varepsilon\Big|_{\varepsilon=0} = X_{\beta_t}\big|_a$$

and from (*) we obtain:

$$X_{\beta_t}\big|_a = \frac{d\beta_t}{dt}(\gamma_a(t)) + \frac{d}{d\varepsilon}\varphi^{\varepsilon,0}_{\beta_t}(a(t))\Big|_{\varepsilon=0}$$

- <u>Given A-path homotopy a_ε w/ $a_0 = a$ there exists $\beta_t \in P_0\Gamma(A)$:</u>

$$\beta_t\big|_a = \frac{d}{d\varepsilon}a_\varepsilon\Big|_{\varepsilon=0} \quad (*)$$

By definition, $a_\varepsilon(t) = \Phi_1(t,\varepsilon)$ of some $\Phi = \Phi_1(t,\varepsilon)dt + \Phi_2(t,\varepsilon)d\varepsilon$. Now extend Φ_1 & Φ_2 to time-dependent sections $\alpha_{t,\varepsilon}$ & $\beta_{t,\varepsilon}$. Take $\beta_t := \beta_{t,0}$. It follows from equation for homotopy that (*) holds. And $\beta_0 = \beta_1 = 0$.

Hence, a_0, a_1 belong to same leaf of $\mathcal{F}_A$ iff they are A-path homotopic.

To prove (iv) can assume that ∇ is trivial A-connection in some local basis of sections / coordinates and $a = 0_x$

Exercise: show this implies general form of (iv)

If a_ε is curve of geodesics starting at $a_0 = 0_x$:

$$\begin{cases} \frac{d}{dt} a^i_\varepsilon(t) = 0 \\ \dot{\gamma}^s_\varepsilon(t) = B^s_i(\gamma_\varepsilon(t))\, a^i_\varepsilon(t) \end{cases} \Rightarrow \begin{cases} \dot{u}^i = 0 \\ \dot{\phi}^s(t) = B^s_i(x)\, u^i(t) \end{cases}$$

So:

$$T_a(\mathrm{Exp}_\nabla(A)) = \{ (u_0, \phi_0 + t\rho(u_0)) : u_0 \in A_x, \phi_0 \in T_x M \}$$

which has dim = dim M + rank A. If $(u,\phi) \in T_a(\mathrm{Exp}(A)) \cap T_a \mathcal{F}_A$ then:

- $\phi(0) = 0 \Rightarrow \phi_0 = 0$
- $\begin{cases} u_0 = \overline{D}_a b \\ b(0) = b(1) = 0 \end{cases} \Rightarrow \begin{cases} \frac{db}{dt} = u_0 \\ b(0) = b(1) = 0 \end{cases} \Rightarrow \begin{cases} u_0 = 0 \\ b(t) \equiv 0 \end{cases}$

So $(u,\phi) = 0$

∎

Rmk: An A-path homotopy was defined as a map $\Phi = \Phi_1 dt + \Phi_2 d\varepsilon$. Henceforth, we shall refer to a curve $a_\varepsilon \in P(A)$ has an A-path homotopy iff $a_\varepsilon(t) = \Phi_1(t,\varepsilon)$ for some A-path homotopy $\Phi = \Phi_1 dt + \Phi_2 d\varepsilon$ the differential equation shows that Φ_2 is uniquely determined.