MATH 595 - Lecture 14

Last time:

$$\mathcal{G}(A) := \frac{A\text{-paths}}{A\text{-path homotopy}}$$

Theorem

$\mathcal{G}(A)$ is a t-simply connected topological groupoid, independent of choice of reparametrization ϕ. Whenever A is integrable, $\mathcal{G}(A)$ has a compatible smooth structure such that it is a Lie groupoid integrating A.

To finish proof assume A integrable. Let $\mathcal{G}$ be t-simply connected with:

$$\mathrm{Lie}(\mathcal{G}) \simeq A$$

Proposition There is a groupoid homeomorphism

$$\mathcal{G} \simeq \mathcal{G}(A)$$

To construct homeomorphism, we introduce:

Def: The Maurer-Cartan form is the vector bundle map:

$$\begin{array}{ccc} \mathrm{Ker}\, dt & \xrightarrow{\Theta} & A \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{t} & M \end{array} \qquad \Theta_g(v) = d\, L_{g^{-1}}(v)$$

Rmks

- For a Lie group, Θ is a left-invariant form on G with values in $\mathfrak{g}$ satisfying the Maurer-Cartan equation

$$\Theta : TG \to \mathfrak{g}, \qquad d\Theta + \tfrac{1}{2}[\Theta, \Theta]_{\mathfrak{g}} = 0 \quad (*)$$

Exercise: Show that (*) is equivalent to:

(*) $\Theta : TG \to \mathfrak{g}$ is a Lie algebroid morphism: $\Theta^* d_{\mathfrak{g}} = d\,\Theta^*$

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• For a Lie groupoid, Θ is a left-invariant form with values in A (Recall left-invariant forms are only defined for vectors tangent to t-fibers).

Exercise: Show that the groupoid MC-form is a Lie algebroid morphism:

$$\Theta : \ker dt \to A, \qquad \Theta^* d_A = d_{\mathcal{F}_t}\Theta^*$$

(so $d_{\mathcal{F}_t}$ is the t-foliated de Rham differential)

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Lemma There are 1:1 correspondences

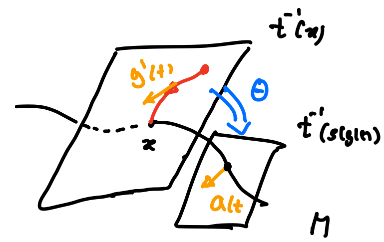
$$\left\{ \begin{array}{l} \text{paths } g : I \to \mathcal{G} \\ g(1) = 1_x,\ g(t) \in t^{-1}(x) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} A\text{-paths} \\ a : TI \to A \end{array} \right\} \qquad a = \Theta \circ dg$$

$$\left\{ \begin{array}{l} \text{path-homotopies} \\ H : I \times I \to \mathcal{G} \\ H(1,\varepsilon) = 1_x,\ H(t,\varepsilon) \in t^{-1}(x) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} A\text{-path} \\ \text{homotopies} \\ \Phi : T(I\times I) \to A \end{array} \right\} \qquad \Phi = \Theta \circ dH$$

Proof The maps are well-defined since they are compositions of Lie algebroid morphisms:

$$a : TI \xrightarrow{dg} \ker dt \xrightarrow{\Theta} A$$

$$\Phi : T(I\times I) \xrightarrow{dH} \ker dt \xrightarrow{\Theta} A$$



Moreover, we have the right boundary conditions:

$$\begin{cases} H(1,\varepsilon) = 1_x \\ H(0,\varepsilon) = g \end{cases} \Rightarrow \quad \Phi_2(0,\varepsilon) = \Phi_2(1,\varepsilon) = \Theta \circ \frac{\partial H}{\partial \varepsilon}\, d\varepsilon = 0$$

<u>1st map has inverse:</u> given $a : I \to A$ the corresponding $g : I \to \mathcal{G}$ is the solution of:

$$\begin{cases} \frac{dg}{dt}(t) = d L_{g(t)}(a(t)) \\ g(1) = 1_x \qquad (\text{w/ } a(1) \in A_x) \end{cases}$$

If we extend $a(t)$ to time-dependent section α_t:

$$\alpha_t(\gamma_a(t)) = a(t)$$

We can choose α_t such that there exists $K \subset M$ compact:

$$\operatorname{supp}(\alpha_t) \subset K, \quad \forall t \in [0,1]$$

Then $g(t)$ is integral curve of $X_t = \overleftarrow{\alpha_t}$ which is a complete vector field (exercise!)

<u>2nd map has inverse:</u> Let Φ be A-path homotopy. Apply inverse of 1st map to $a_\varepsilon = \Phi_1(-,\varepsilon)$ to obtain $H(t,\varepsilon) \in t^{-1}(x)$ with $H(1,\varepsilon) = 1_x$. Need to show that $H(0,\varepsilon)$ is independent of ε. Set $\tilde{\Phi} := \Theta \circ dH$. We claim that $\tilde{\Phi}(t,\varepsilon) = \Phi(t,\varepsilon)$ so that

$$0 = \tilde{\Phi}(0,\varepsilon) = \Theta \circ \frac{\partial H}{\partial \varepsilon}(0,\varepsilon) \Rightarrow \frac{\partial H}{\partial \varepsilon}(0,\varepsilon) = 0$$

The claim follows because both $\tilde{\Phi}$ and Φ satisfy the same ODE w/ the same initial condition (proposition A-path homotopies).

$\blacksquare$

By the lemma we obtain a bijection

$$\Phi : \mathcal{G} \longrightarrow \mathcal{G}(A), \quad g \longmapsto [a] \quad \text{w/} \quad a = \Theta \circ d\tilde{g}$$

$$\text{w/ } \tilde{g}(t) \in t^{-1}(t(g)), \ \tilde{g}(1) = g, \ \tilde{g}(0) = 1_{t(g)}$$

Then:

- Φ is homeomorphism : $\mathcal{G} \simeq P(\mathcal{G}, M)/_{\text{path-homotopy}}$ (C^k-topology)
- Φ preserves source/target & unit section
- Φ preserves multiplication :

Let $\tilde{g}_1, \tilde{g}_2 : I \to \mathcal{G}$ be paths in t-fibers w/

$$\begin{cases} \tilde{g}_1(0) = g_1 \\ \tilde{g}_1(1) = 1_{t(g_1)} \end{cases} \quad \begin{cases} \tilde{g}_2(0) = g_2 \\ \tilde{g}_2(1) = 1_{t(g_2)} \end{cases} \qquad \tilde{g}_i^{(n)}(0) = \tilde{g}_i^{(n)}(1) = 0 \quad i = 1,2$$

Then define $\tilde{g} : I \to \mathcal{G}$:

$$\tilde{g}(t) = \begin{cases} g_1 \tilde{g}_2(2t), & 0 \le t \le 1/2 \\ \tilde{g}_1(2t-1), & 1/2 \le t \le 1 \end{cases}$$

We find:

$$\tilde{g}(0) = g_1 g_2, \quad \tilde{g}(1) = 1_{t(g_1 g_2)}, \quad \Theta \circ dg = a_1 \circ a_2$$

$$\Rightarrow \Phi(g_1 g_2) = \Phi(g_1) \cdot \Phi(g_2) \qquad \blacksquare$$

- $\mathcal{G}(-)$ is a functor : if $\Phi : A_1 \to A_2$ is a Lie algebroid morphism then we obtain a morphism of topolog. gpds

$$\mathcal{G}(\Phi) : \mathcal{G}(A_1) \to \mathcal{G}(A_2), \quad [a] \longmapsto [\Phi \circ a]$$

If A_1 & A_2 are integrable, this is a smooth morphism integrating Φ.

So we conclude:

1) If $\Phi : A_1 \to A_2$ is a morphism between integrable Lie algebroids and $\mathcal{G}_1$ & $\mathcal{G}_2$ are Lie groupoids w/ $\mathcal{G}_1$ target 1-connected, $\exists^1$ Lie groupoid morphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ w/ $\Phi_* = \Phi$

Recall that:

- (E,∇) is Rep of A $\Leftrightarrow$ $\nabla: A \to \mathfrak{gl}(E)$ Lie alg morphism
- $\mathcal{G} \circlearrowright E$ is Rep $\Leftrightarrow$ $\mathcal{G} \to GL(E)$ Lie grp morphism

So we also obtain:

2) Every Rep (E,∇) of A integrates to a Rep of the target 1-connected $\mathcal{G}$ integrating A.

Also, if ∇ is an A-connection on A; we have exponential

3) The exponential map of ∇ is:

$$\exp^{\nabla}: \underset{A}{\underset{\cap}{U}} \to \mathcal{G}(A), \quad a_0 \longmapsto [a] \quad \text{w/ } a = \text{unique geodesic w/ } a(0)=a_0$$

<u>Rmk</u>: The functor $\mathcal{G}(-)$ shows that 1), 2) & 3) hold even for non-integrable Lie algebroids!

<u>Integrability</u>

- When is a Lie algebroid A integrable?

$\Leftrightarrow$ • When is the Weinstein grpd $\mathcal{G}(A)$ smooth?

<u>Detour</u>. Smooth structure on space of paths $\tilde{P}(M)$

- <u>$M = \mathbb{R}^m$</u>:

$\tilde{P}(\mathbb{R}^m) = \{\gamma: I \to \mathbb{R}^m \text{ smooth}\}$ is a Banach space:

$$\|\gamma\|_1 = \max\{\sup_{t\in I}\|\gamma(t)\|, \sup_{t\in I}\|\dot{\gamma}(t)\|\}$$

(This is a norm; uniform limit of C^0-functions is C^0)

<u>Rmk</u>: This true for any k-norm $\|\cdot\|_k$. If we want to control all derivatives need Fréchet space.

- M = any manifold w/ $m = \dim M$

$\widetilde{P}(M) = \{ \gamma : I \to M \text{ smooth} \}$ is a Banach manifold for C^1-topology defined by:

$$d_1(\gamma, \eta) := \max \left\{ \sup_{t \in I} d^M(\gamma(t), \eta(t)), \sup_{t \in I} d^{TM}(\dot\gamma(t), \dot\eta(t)) \right\}$$

It is locally modeled on $\widetilde{P}(\mathbb{R}^m)$.

Fix Riemannian metric g on M. Given $\gamma_0 \in \widetilde{P}(M)$ a chart around γ_0:

- $\gamma_0^* TM$ = trivial vector bundle $\simeq I \times \mathbb{R}^m \to I$
- Choose $\gamma_0 \in U \subset P(M)$ open small enough that for every $\gamma \in U$, $\gamma(t)$ belongs to domain of injectivity of $\exp_{\gamma_0(t)}$
- $\phi : U \longrightarrow \Gamma(\gamma_0^* TM) \simeq \widetilde{P}(\mathbb{R}^m)$

$$\gamma \longmapsto V \quad \text{w/} \quad \exp_{\gamma_0(t)}(V(t)) = \gamma(t)$$

Exercise: Check that (U, ϕ) form an atlas.

Notice that

$$T_{\gamma_0} \widetilde{P}(M) = \Gamma(\gamma_0^* TM)$$

i.e., a tangent vector at γ_0 is a vector field along γ_0.

From now on $\widetilde{P}(A) = \{ a : I \to A \mid \text{smooth} \}$ viewed as a Banach manifold as above. We will see that:

- $P(A) \subset \widetilde{P}(A)$ is a Banach submanifold
- $\sim$ defines foliation $\mathcal{F}$ of $P(A)$ of codimension $\dim M + \operatorname{rank} A$
- A integrable $\iff$ leaf space $\mathcal{F}$ is smooth and $\mathcal{G}(A) = P(A)/\sim$ w/ this smooth structure is Lie gpd