MATH 595 – Lecture 13

Geodesics

Def. Given an A-connection ∇ on A a geodesic for ∇ is an A-path $a: I \to A$ such that $D_a a = 0$.

In general, geodesics exist only for a small time:

$$\nabla_{\alpha_i} \alpha_j = \Gamma_{ij}^k \alpha_k \quad , \quad a(t) = a^i(t)\, \alpha_i(\gamma_a(t))$$

$$\rho(\alpha_i) = B_i^s \frac{\partial}{\partial x^s} \, , \quad \gamma_a(t) = (\gamma_a^1(t), \dots, \gamma_a^m(t))$$

$$\begin{cases} \dfrac{da^k}{dt}(t) = -\Gamma_{ij}^k(\gamma_a(t))\, a^i(t)\, a^j(t) \\[2ex] \dfrac{d\gamma_a^s}{dt}(t) = B_i^s(\gamma_a(t))\, a^i(t) \end{cases} \qquad \left(\begin{array}{l} k = 1, \dots, r \\ s = 1, \dots, m \end{array} \right)$$

So geodesics are integral curves of vector field

$$X^\nabla = B_i^s(x)\, \xi^i \frac{\partial}{\partial x^s} - \Gamma_{ij}^k(x)\, \xi^i \xi^j \frac{\partial}{\partial \xi^k} \in \mathfrak{X}(A)$$

This is called the geodesic spray of ∇

Proposition The geodesic spray is well-defined (independent of choice of coordinates) and satisfies:

(i) $d_a p_A(X_a^\nabla) = \rho(a)$, $\forall\, a \in A$

(ii) $(m_t)_* X^\nabla = \frac{1}{t} X^\nabla$, $\forall\, t > 0$

Conversely, any vector field satisfying (i) and (ii) is the geodesic spray of a unique torsion-free connection.

Sketch of proof:

Given vector field:

$$X = u^s(x,\xi)\frac{\partial}{\partial x^s} + v^k(x,\xi)\frac{\partial}{\partial \xi^k}$$

$$(i) + (ii) \Rightarrow \begin{cases} u^s(x,\xi) = B^s_i(x)\,\xi^i \\ v^k(x,\xi) = v^k_{ij}(x)\,\xi^i\xi^j \end{cases}$$

To find $\nabla_{\alpha_i}\alpha_j = \Gamma^k_{ij}\alpha_k$ with zero torsion and $X^\nabla = X$ solve the system:

$$\begin{cases} \Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij} & \left([\alpha_i,\alpha_j] = c^k_{ij}\alpha_k \right) \\ \Gamma^k_{ij} + \Gamma^k_{ij} = v^k_{ij} \end{cases}$$

Using partition of unity $\Rightarrow \nabla$ w/ $X^\nabla = X$. ∎

Corollary Given an A-connection ∇ on A there is a unique connection $\tilde{\nabla}$ with same geodesics and zero torsion

Exponential map

$$\left.\begin{array}{l} \nabla \equiv \text{A-connection on } A \\ \phi^t_{X^\nabla} \equiv \text{Flow of Geodesic spray} \end{array}\right\} \qquad \begin{array}{l} \exp^M_\nabla : \underset{\cup}{\overset{A}{}} V \longrightarrow M \\ \exp^M_\nabla(a) = pr(\phi^1_{X^\nabla}(a)) \end{array}$$

Exercise: Show that $\exp^M_\nabla|_{A_x \cap V}$ is a submersion onto a open neighborhood $x \in U \subset O_x$.

This is not quite the "right" exponential map:

Prop. Let $\mathcal{G} \rightrightarrows M$ Lie groupoid w/ $\mathrm{Lie}(\mathcal{G}) = A$. There is a map:

$$\exp_\nabla : V \longrightarrow \mathcal{G} \quad \text{w/} \quad O_M \subset V \subset A \ \text{open}$$

which is a local diffeo if V is sufficiently small neighborhood of O_M.

Proof

The left-invariant vector fields generates all vector fields tangent to t-fibers $\Rightarrow$ $\exists$ unique Koszul-connection $\tilde{\nabla}$ s.t.:

$$\tilde{\nabla}_{\overleftarrow{\alpha}} \overleftarrow{\beta} = \overleftarrow{\nabla_\alpha \beta}$$

One can think of $\tilde{\nabla}$ has a (smooth) family of ordinary connections on t-fibers.

$$\exp_\nabla : V \longrightarrow \mathcal{G}, \quad \exp_\nabla \Big|_{V \cap A_x} := \exp_{\tilde{\nabla}} \Big|_{t^{-1}(x)}$$

$\boxtimes$

Note that:

$$s \circ \exp_\nabla = pr_M \qquad t \circ \exp_\nabla = \exp^M_\nabla$$

Rmk. As for Lie Groups, one can express the groupoid structure in "exponential coordinates". The formulas now depend on choice of ∇, so there is no "universal" Baker-Campbell-Hausdorff formula.

Still it is possible to use this approach to show that to every Lie algebroid there is a "Local Lie groupoid" integrating it.

A-path homotopy Groupoid (a.k.a. Weinstein Groupoid)

Aim: Given algebroid $A \to M$ construct groupoid:

$$\mathcal{G}(A) := \frac{\text{A-paths}}{\text{A-path homotopy}}$$

Lemma 1

A-path homotopy is an equivalence relation

Proof

- Reflexive: ε-constant A-path homotopy $\Rightarrow$ $a_0 \sim a_0$
- Symmetry: $a_0 \sim a_1$ via Φ then $a_1 \sim a_0$ via

$$\bar{\Phi}(t,\varepsilon) = \Phi_1(t,1-\varepsilon)\,dt - \Phi_2(t,1-\varepsilon)\,d\varepsilon$$

- Transitivity: $a_0 \sim a_1$ via Φ and $a_1 \sim a_2$ via Φ' then

$a_0 \sim a_2$ via:

$$\Phi_{02}(t,\varepsilon) = \begin{cases} \Phi_1^{\phi}(t,2\varepsilon)\,dt + 2\,\Phi_2^{\phi}(t,2\varepsilon)\,d\varepsilon & 0 \le t \le 1,\quad 0 \le \varepsilon \le 1/2 \\ \Phi_1'^{\phi}(t,2\varepsilon-1)\,dt + 2\,\Phi_2'^{\phi}(t,2\varepsilon-1)\,d\varepsilon & 0 \le t \le 1,\quad 1/2 \le \varepsilon \le 1 \end{cases}$$

with $\phi : [0,1] \to [0,1]$ reparametrization in ε-direction with

$$\begin{cases} \phi(\varepsilon) = 0 & \text{if } 0 \le \varepsilon \le 1/3 \\ \phi(\varepsilon) = 1 & \text{if } 1/3 \le \varepsilon \le 1 \end{cases}$$

$\blacksquare$

Lemma 2 If $\phi : [0,1] \to [0,1]$ is a reparametrization then a & a^{ϕ} are A-path homotopic.

Proof. Define $\Phi : T(I \times I) \to A$ by:

$$\Phi(t,\varepsilon) = \big((1-\varepsilon)+\varepsilon\phi'(t)\big)\,a\big((1-\varepsilon)t+\varepsilon\phi(t)\big)\,dt + \big(-t+\phi(t)\big)\,a\big((1-\varepsilon)t+\varepsilon\phi(t)\big)\,d\varepsilon$$

Need to check this is A-path homotopy:

$$\left.\begin{array}{l} - \ \Phi_1(t,0) = a(t) \\ - \ \Phi_1(t,1) = \phi'(t)\,a(\phi(t)) = a^{\phi}(t) \\ - \ \Phi_2(0,\varepsilon) = 0 = \Phi_2(1,\varepsilon) \end{array}\right\} \ \gamma(t,\varepsilon) = \gamma_a\big((1-\varepsilon)t+\varepsilon\phi(t)\big)$$

If $\alpha_t \in \Gamma(A)$ is any section extending $a(t)$, then:

$$\alpha_{t,\varepsilon} = \big((1-\varepsilon)+\varepsilon\phi'(t)\big)\,\alpha_{(1-\varepsilon)t+\varepsilon\phi(t)} \qquad \beta_{t,\varepsilon} = \big(-t+\phi(t)\big)\,\alpha_{(1-\varepsilon)t+\varepsilon\phi(t)}$$

are extensions of Φ_1 and Φ_2 which satisfy:

$$\left.\left(\frac{d}{dt}\beta_{t,\varepsilon} - \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\right|_{\gamma(t,\varepsilon)} = -\left[\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}\right]\Big|_{\gamma(t,\varepsilon)}$$

$\blacksquare$

- $P(A) = \{ A\text{-paths} \}$ "$\sim$" $\equiv$ A-path homotopy

- $\mathcal{G}(A) := P(A)/_{\sim} \rightrightarrows M$ w/ structure maps

Source/target maps: $s([a]) = \gamma_a(0)$, $t([a]) = \gamma_a(1)$

Unit map: $u(x) = [0_x]$

Inverse map: $i([a]) = [\bar{a}]$

Multiplication: Fix a reparametrization w/ $\phi^{(n)}(0) = \phi^{(n)}(1) = 0$

$$[a_1] \cdot [a_2] := [a_1^{\phi} \circ a_2^{\phi}]$$

Topology: On $P(A)$ consider C^1-topology:

$$d(a_1, a_2) = \max \{ \sup_{t \in [0,1]} d^A(a_1(t), a_2(t)), \sup_{t \in [0,1]} d^{TA}(a_1'(t), a_2'(t)) \}$$

where d^A & d^{TA} are distances in A & TA. Then consider quotient topology on $\mathcal{G}(A)$.

Theorem

$\mathcal{G}(A)$ is a t-simply connected topological groupoid, independent of choice of reparametrization ϕ. Whenever A is integrable, $\mathcal{G}(A)$ has a compatible smooth structure such that it is a Lie groupoid integrating A.

Rmk:

- $\mathcal{G}(A)$ is called the Weinstein groupoid of A.
- $\mathcal{G}(-)$ is a functor: if $\Phi: A_1 \to A_2$ is a Lie algebroid morphism then we obtain a morphism of topolog. gpds

$$\mathcal{G}(\Phi): \mathcal{G}(A_1) \to \mathcal{G}(A_2), \quad [a] \mapsto [\Phi \circ a]$$

If A_1 & A_2 are integrable, this is a smooth morphism integrating Φ.

· <u>$\mathcal{G}(A)$ is topological grpd</u>: structure maps are continuous since they are continuous at the level of A-paths. One still needs to check that source/target are open maps. This follows from:

<u>Lemma</u>: $P(A) \to P(A)/_\sim \cong \mathcal{G}(A)$ is open map

Given $D \subset P(A)$ open, we need to check that its saturation

$$\tilde{D} = \{ a' \in P(A) : a' \sim a \in D \}$$

is open. This follows by showing that if $a_0 \sim a_1$ there exists a homeomorphism $T : P(A) \to P(A)$ with $T(a_0) = a_1$.

Let $\Phi = \Phi_1 dt + \Phi_2 d\varepsilon$ be A-homotopy from a_0 to a_1.

Let $\alpha_{t,\varepsilon}$ & $\beta_{t,\varepsilon} \in \Gamma(A)$ such that:

$$\alpha_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_1(t,\varepsilon)$$

$$\beta_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_2(t,\varepsilon)$$

So that

$$\left(\frac{d}{dt} \beta_{t,\varepsilon} - \frac{d}{d\varepsilon} \alpha_{t,\varepsilon} \right) \Big|_{\gamma(t,\varepsilon)} = - [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}] \Big|_{\gamma(t,\varepsilon)}$$

We can assume that $\beta_{t,\varepsilon}$ is compactly supported (since $\gamma(I \times I) \subset M$ is compact).

Given A-path $\tilde{a} : I \to A$ let $\tilde{\alpha}^0_t \in \Gamma_c(A)$ be time-dependent section w/ compact support: $\tilde{\alpha}^0_t(\gamma_{\tilde{a}}(t)) = \tilde{a}(t)$

Let $\tilde{\alpha}_{t,\varepsilon}$ be the solution of ODE:

$$\begin{cases} \dfrac{d \tilde{\alpha}_{t,\varepsilon}}{d\varepsilon} = \dfrac{d}{dt} \beta_{t,\varepsilon} + [\tilde{\alpha}_{t,\varepsilon}, \beta_{t,\varepsilon}], \\ \tilde{\alpha}_{t,0} = \tilde{\alpha}^0_t \end{cases}$$

Then if

$$\tilde{\gamma}(t,\varepsilon) := \Phi^{\varepsilon,0}_{\rho(\beta_{t,\varepsilon})}(\gamma_{\bar{a}}(t))$$

we see that

$$\tilde{\overline{\Phi}} = \tilde{\alpha}_{t,\varepsilon}(\tilde{\gamma}(t,\varepsilon))\,dt + \beta_{t,\varepsilon}(\tilde{\gamma}(t,\varepsilon))\,d\varepsilon$$

is an A-homotopy starting at a. We then set:

$$T(a) := \tilde{\alpha}_{t,1}(\tilde{\gamma}(t,1)).$$

- $\mathcal{G}(A)$ as 1-connected t-fibers (= s-fibers)

Fix $x \in M$ and let $\gamma_s = [a_s]$ be a loop in $s^{-1}(x)$ based at 1_x. $a_s : I \to A$ is a family of A-paths with $[a_0] = [a_1] = 1_x$ (no assumption on s-dependence). We can assume that $a_0(t) = a_1(t) = 0_x$. Then we define

$$H : I \times I \to s^{-1}(x) \subset \mathcal{G}(A)$$
$$(s,\varepsilon) \longmapsto [\varepsilon\, a_s(\varepsilon\,\cdot)]$$

Claim: H is a path-homotopy in $\mathcal{G}(A)$ between $s \mapsto \gamma_s$ and 1_x

- H is continuous:
- For fixed s, $\varepsilon \mapsto H(s,\varepsilon)$ is continuous because $[0,1] \to P(A)$, $\varepsilon \mapsto \varepsilon a_s(\varepsilon,\cdot)$ is continuous
- For fixed ε, $s \mapsto H(s,\varepsilon)$ is continuous because

$$\left.\begin{array}{l}\Phi^{\varepsilon} : P(A) \to P(A),\ a \mapsto \varepsilon a(\varepsilon\,\cdot) \text{ is continuous} \\ \text{and satisfies } a_0 \sim a_1 \Rightarrow \Phi^{\varepsilon}(a_0) \sim \Phi^{\varepsilon}(a_1)\end{array}\right\} \Rightarrow \overline{\Phi}^{\varepsilon} : \mathcal{G}(A) \to \mathcal{G}(A) \text{ is continuous}$$

$\Rightarrow s \mapsto H(s,\varepsilon) = \overline{\Phi}^{\varepsilon} \circ \gamma_s$ is continuous

- Boundary conditions:

$H(s,0) = [0_x] = 1_x$ $\qquad$ $H(s,1) = [a_s] = \gamma_s$

$H(0,\varepsilon) = [\varepsilon a_0(\varepsilon\,\cdot)] = [0_x] = 1_x$ $\qquad$ $H(1,\varepsilon) = [\varepsilon a_1(\varepsilon\,\cdot)] = [0_x] = 1_x$