MATH 595 - Lecture 12

A-Homotopy

In this discussion we fix a Lie algebroid $A \to M$

Def. If Σ is some manifold (possibly w/ boundary) a Lie algebroid morphism

$$\begin{tikzcd} T\Sigma \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\ \Sigma \arrow[r, "\phi"'] & M \end{tikzcd}$$

is called an A-map.

Rmks:

1) The condition on Φ is that it intertwines the Lie algebroid differential and the de Rham differential

$$\Phi^* d_A = d\,\Phi^*$$

In particular, we have a map in cohomology:

$$\Phi^*: H^\bullet(A) \to H^\bullet(\Sigma)$$

2) In degree 0, we have "preserving anchors"

$$\begin{tikzcd} & A \arrow[d, "\rho"] \\ T\Sigma \arrow[ur, "\Phi"] \arrow[r, "d\phi"'] & TM \end{tikzcd} \qquad \rho \circ \Phi = d\phi \qquad (*)$$

It follows that if Σ is connected, then $\phi(\Sigma) \subset \mathcal{O}$ (an orbit of A).

3) When Σ is 1-dim $(*)$ is the only condition on Φ. When $\Sigma = [0,1]$ we obtain A-paths. When $\Sigma = S^1$ we obtain A-loops.

4) When $\dim \Sigma \geq 2$ we have additional conditions, that we have already seen in discussion last time. More to come!

—— / ——

A-maps can be pre-composed w/ ordinary maps:

$$\begin{cases} \psi : \Sigma_0 \to \Sigma \\ \Phi : T\Sigma \to A \end{cases} \Rightarrow \quad \Phi \circ d\psi : T\Sigma_0 \to A$$

For example, reparametrization of A-paths is an instance of this!

<u>Def</u> Given an A-map $\Phi : T\Sigma \to A$ its <u>restriction</u> to a submanifold $i : \Sigma_0 \hookrightarrow \Sigma$ is the A-map

$$\Phi|_{\Sigma_0} := \Phi \circ di : T\Sigma_0 \to A$$

For example, if Σ is manifold w/ $\partial\Sigma$ we have:

$$\partial\Phi := \Phi|_{\partial\Sigma} : T(\partial\Sigma) \to A$$

<u>Def:</u> Let Σ be manifold without boundary. Two A-maps $\Phi_0, \Phi_1 : T\Sigma \to A$ are called <u>A-homotopic</u> if there exists A-map $\Phi : T(\Sigma \times [0,1]) \to A$ such that

$$\Phi|_{\Sigma \times \{0\}} = \Phi_0 \qquad \Phi|_{\Sigma \times \{1\}} = \Phi_1$$

<u>Proposition.</u> Two A-homotopic maps $\Phi_0, \Phi_1 : T\Sigma \to A$ induce the same map in cohomology:

$$\Phi_0^* = \Phi_1^* : H^\bullet(A) \to H^\bullet(\Sigma)$$

<u>Proof.</u>

Choose A-homotopy $\Phi : T(\Sigma \times I) \to A$ between Φ_0 & Φ_1. Let $i_0 : \Sigma \hookrightarrow \Sigma \times I$, $i_1 : \Sigma \hookrightarrow \Sigma \times I$ be inclusions $i_0(x) = (x,0)$, $i_1(x) = (x,1)$. We have:

$$\begin{tikzcd} H^\bullet(A) \arrow[r, "\Phi^*"] \arrow[rr, bend left, dashed, "\Phi_1^*"] \arrow[rr, bend right, dashed, "\Phi_0^*"'] & H^\bullet(\Sigma \times I) \arrow[r, shift left, "i_1^*"] \arrow[r, shift right, "i_0^*"'] & H^\bullet(\Sigma) \end{tikzcd} \qquad i_0^* = i_1^* \Rightarrow \Phi_1^* = \Phi_0^*.$$

$\blacksquare$

Integration can be defined for A-maps:

Def. Let Σ be a compact, oriented, k-dim manifold. The <u>integral of $\omega \in \Omega^k(A)$</u> along an A-map $\Phi : T\Sigma \to A$ is

$$\int_{\Phi} \omega := \int_{\Sigma} \Phi^* \omega$$

Note that:

$$\int_{\Phi} : \Omega^k(A) \xrightarrow{\Phi^*} \Omega^k(\Sigma) \xrightarrow{\int_\Sigma} \mathbb{R}$$

so if $\partial \Sigma = \phi$ we have a map in cohomology

$$\int_{\Phi} : H^k(A) \xrightarrow{\Phi^*} H^k(\Sigma) \xrightarrow{\int_\Sigma} \mathbb{R}$$

From homotopy invariance:

Corollary. If $\Phi_0, \Phi_1 : T\Sigma \to A$ are A-homotopic maps on a compact, oriented, k-manifold Σ and $\omega \in \Omega^k(A)$ a closed k-form:

$$\int_{\Phi_0} \omega = \int_{\Phi_1} \omega$$

We also have:

Stokes Thm

Let Σ be a compact, oriented, k-manifold with boundary. For any A-map $\Phi : T\Sigma \to A$ and $\omega \in \Omega^{k-1}(A)$:

$$\int_{\Phi} d_A \omega = \int_{\partial \Phi} \omega$$

Proof:

$$\int_{\Phi} d_A \omega = \int_{\Sigma} \Phi^* d_A \omega = \int_{\Sigma} d \Phi^* \omega = \int_{\partial \Sigma} \Phi^* \omega = \int_{\partial \Phi} \omega \quad \blacksquare$$

For an A-path $a: I \to A$ and a 1-form $\omega \in \Omega^1(A)$ the integral is:

$$\int_a \omega := \int_I (a\,dt)^* \omega = \int_0^1 \langle \omega_{\gamma_a(t)}, a(t) \rangle \, dt$$

and we obtain the following properties:

Properties:

(i) Linearity: $\int_a \lambda\omega_1 + \mu\omega_2 = \lambda \int_a \omega_1 + \mu \int_a \omega$

(ii) Invariance under reparametrization: $\int_{a^\phi} \omega = \int_a \omega$

(iii) Additivity relative to concatenation: $\int_{a_1 \circ a_2} \omega = \int_{a_1} \omega + \int_{a_2} \omega$

(iv) Change of sign under reversion: $\int_{\bar{a}} \omega = -\int_a \omega$
$\bar{a}(t) := -a(1-t)$

(v) For exact 1-forms: $\int_a d_A f = f(\gamma_a(1)) - f(\gamma_a(0))$

In particular: $\int_a : H^1(A) \longrightarrow \mathbb{R}$ (linear function)

Note however that:

$$\underset{(\partial\Sigma = \phi)}{\text{A-homotopy}} \;\not\Rightarrow\; \underset{(\Sigma = [0,1],\ \partial\Sigma \neq \phi)}{\text{A-path homotopy}} \; !$$

In this language an A-path homotopy between $a_0, a_1 : I \to A$ is an A-map $\Phi : T(I \times I) \to A$ such that

$$\partial\Phi(t,0) = a_0(t), \quad \partial\Phi(t,1) = a_1(t), \quad \forall t \in [0,1]$$

$$\partial\Phi(0,\varepsilon) = 0 = \partial\Phi(1,\varepsilon), \quad \forall \varepsilon \in [0,1]$$

Exercise. Show that A-path homotopy is an equivalence relation.

Hint: For transitivity use $Id \times \phi : [0,1] \times [0,1] \to [0,1] \times [0,1]$ where $\phi : [0,1] \to [0,1]$ is reparametrization w/ $\phi(t) = \begin{cases} 0 & \text{in } [0, 1/3] \\ 1 & \text{in } [1/3, 1] \end{cases}$

Theorem Let $a_0, a_1 : I \to A$ be A-path homotopic and $\omega \in \Omega^1(A)$ a closed 1-form. Then:

$$\int_{a_0} \omega = \int_{a_1} \omega$$

We will need the following important proposition characterizing A-homotopies:

Proposition

Let $\Phi : T(I \times I) \to A$, $\Phi = \Phi_1 dt + \Phi_2 d\varepsilon$, be a bundle map which is anchor preserving (i.e., $\rho \circ \Phi = d\gamma$). The following are equivalent:

(i) Φ is a Lie algebroid morphism

(ii) For every $\omega \in \Omega^1(A)$,

$$\frac{d}{dt}\langle \omega, \Phi_2 \rangle - \frac{d}{d\varepsilon}\langle \omega, \Phi_1 \rangle = d_A \omega(\Phi_1, \Phi_2)$$

(iii) For every families of sections $\alpha_{t,\varepsilon}, \beta_{t,\varepsilon} \in \Gamma(A)$ extending Φ_1 & Φ_2:

$$\alpha_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_1(t,\varepsilon), \quad \beta_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_2(t,\varepsilon)$$

one has:

$$\left(\frac{d}{dt}\beta_{t,\varepsilon} - \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\Bigg|_{\gamma(t,\varepsilon)} = -[\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\Bigg|_{\gamma(t,\varepsilon)}$$

(iv) For any A-connection ∇ on A

$$T(\Phi_1, \Phi_2) = D_{\Phi_1}\Phi_2 - D_{\Phi_2}\Phi_1$$

Proof. (i) $\Leftrightarrow$ (ii)

For $f \in C^\infty(M) = \Omega^0(A)$,

$$\Phi^* d_A f = d\,\Phi^* f \iff \Phi^* \rho^* df = d(f \circ \gamma)$$

$$\iff df \circ (\rho \circ \Phi) = df \circ d\gamma$$

So in deg 0: $\Phi^* d_A = d\Phi^*$ $\Leftrightarrow$ preserving anchors

For $\omega \in \Omega^1(A)$:

$$\left(\Phi^* d_A\omega - d\Phi^*\omega\right)\left(\frac{d}{dt}, \frac{d}{d\varepsilon}\right) =$$

$$= d_A\omega\left(\Phi_1, \Phi_2\right) - \frac{d}{dt}\left(\Phi^*\omega\left(\frac{d}{d\varepsilon}\right)\right) + \frac{d}{d\varepsilon}\left(\Phi^*\omega\left(\frac{d}{dt}\right)\right) - \Phi^*\omega\left(\left[\frac{d}{dt}, \frac{d}{d\varepsilon}\right]\right)$$

$$= d_A\omega\left(\Phi_1, \Phi_2\right) - \frac{d}{dt}\langle\omega, \Phi_2\rangle + \frac{d}{d\varepsilon}\langle\omega, \Phi_1\rangle$$

Since $\dim(I\times I) = 2$, these are the only conditions (actually, if $\Phi^* d_A\omega = d\Phi^*\omega$ holds in degree 0 & 1 $\Leftrightarrow$ holds in any degree).

(ii) $\Leftrightarrow$ (iii)

$$d_A\omega\left(\Phi_1, \Phi_2\right) = d_A\omega\left(\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}\right)\Big|_{\gamma(t,\varepsilon)}$$

$$= \Big(\rho(\alpha_{t,\varepsilon})(\langle\omega, \beta_{t,\varepsilon}\rangle) - \rho(\beta_{t,\varepsilon})(\langle\omega, \alpha_{t,\varepsilon}\rangle) - \langle\omega, [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\rangle\Big)\Big|_{\gamma(t,\varepsilon)}$$

$$= \langle d\langle\omega, \beta_{t,\varepsilon}\rangle, \frac{d\gamma}{dt}\rangle - \langle d\langle\omega, \alpha_{t,\varepsilon}\rangle, \frac{d\gamma}{d\varepsilon}\rangle - \langle\omega, [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\rangle\Big|_{\gamma(t,\varepsilon)}$$

$$= \frac{d}{dt}\langle\omega, \beta_{t,\varepsilon}\rangle(\gamma(t,\varepsilon)) - \langle\omega, \frac{d\beta_{t,\varepsilon}}{dt}\rangle\Big|_{\gamma(t,\varepsilon)}$$

$$- \frac{d}{d\varepsilon}\langle\omega, \alpha_{t,\varepsilon}\rangle(\gamma(t,\varepsilon)) + \langle\omega, \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\rangle\Big|_{\gamma(t,\varepsilon)}$$

$$- \langle\omega, [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\rangle\Big|_{\gamma(t,\varepsilon)}$$

$$= \frac{d}{dt}\langle\omega, \Phi_2\rangle - \frac{d}{d\varepsilon}\langle\omega, \Phi_1\rangle - \langle\omega, \frac{d\beta_{t,\varepsilon}}{dt} - \frac{d}{d\varepsilon}\alpha_{t,\varepsilon} + [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\rangle\Big|_{\gamma(t,\varepsilon)}$$

(iii) $\Leftrightarrow$ (iv) For any connection ∇:

$$D_{\Phi_1}\Phi_2 = \left(\nabla_{\alpha_{t,\varepsilon}}\beta_{t,\varepsilon} + \frac{d}{dt}\beta_{t,\varepsilon}\right)\Big|_{\gamma(t,\varepsilon)}$$

$$D_{\Phi_2}\Phi_1 = \left(\nabla_{\beta_{t,\varepsilon}}\alpha_{t,\varepsilon} + \frac{d}{d\varepsilon}\alpha_{t,\varepsilon}\right)\Big|_{\gamma(t,\varepsilon)}$$

So:

$$T(\Phi_1, \Phi_2) - D_{\Phi_1}\Phi_2 + D_{\Phi_2}\Phi_1 = \left(-\frac{d}{dt}\beta_{t,\varepsilon} + \frac{d}{d\varepsilon}\alpha_{t,\varepsilon} - [\alpha_{t,\varepsilon}, \beta_{t,\varepsilon}]\right)\Big|_{\gamma(t,\varepsilon)}$$

∎

Proof of Theorem:

If $\Phi : T(I\times I) \to A$ is A-homotopy connecting a_0 and a_1

Then for $a_\varepsilon(t) = \Phi_1(t,\varepsilon)$, by (ii):

$$\frac{d}{d\varepsilon}\int_{a_\varepsilon}\omega = \frac{d}{d\varepsilon}\int_0^1 \langle \omega, \Phi_1\rangle \, dt$$

$$= \int_0^1 \frac{d}{dt}\langle \omega, \Phi_2\rangle - d_A\omega(\Phi_1, \Phi_2)\, dt$$

$$= \langle \omega, \Phi_2(1,\varepsilon)\rangle - \langle \omega, \Phi_2(0,\varepsilon)\rangle - \int_0^1 d_A\omega(\Phi_1,\Phi_2)\, dt$$

So if $d_A\omega = 0$ we obtain that $\int_{a_0}\omega = \int_{a_1}\omega$.

∎

Rmk: Another proof: pullback to $[0,1]\times[0,1]$ and apply Stokes Thm for manifolds w/ corners.

Application. $(\mathbb{L}_M, \nabla)$ = orientable rank 1 Rep of $A \to M$

$\mu \in \Gamma(\mathbb{L}_M)$ = non-vanishing section

$$\Rightarrow \omega_\mu \in \Omega^1_{cl}(A) \quad \text{w/} \quad \nabla_\alpha \mu = \omega_\mu(\alpha)\,\mu$$

Proposition For any A-path:

$$\tau_a(\mu_{\gamma_a(0)}) = \exp\left(-\int_a \omega_\mu\right)\mu_{\gamma_a(1)}$$

In particular, for an A-loop based at x_0:

$$\tau_a : \mathbb{L}_{x_0} \to \mathbb{L}_{x_0}, \quad \tau_a = \exp\left(-\int_a c(\mathbb{L}_M, \nabla)\right)$$

Proof.

A path $u : I \to \mathbb{L}$ covering $\gamma_a : I \to M$ takes the form
$$u(t) = g(t)\,\mu_{\gamma_a(t)}\ ,\qquad g : I \to \mathbb{R}$$
We have that:
$$D_a u(t) = \Big(\underbrace{\frac{dg}{dt}(t) + \omega_\mu(a(t))\,g(t)}_{=0}\Big)\,\mu_{\gamma_a(t)}$$
$$\Rightarrow\quad g(t) = \exp\Big(-\int_0^t \omega_\mu(a(t))\,dt\Big)\,g(0)$$
$$\Rightarrow\quad \tau_a(\cdot) = \exp\Big(-\int_0^1 \omega_\mu(a(t))\,dt\Big)\cdot = \exp\Big(-\int_a \omega_\mu\Big)\cdot$$

$\blacksquare$

Exercise. If $(\mathbb{L}, \nabla)$ is possibly non-oriented Rep of rank 1, show that for any A-loop based at x:
$$\tau_a = (-1)^{\langle \omega_1(\mathbb{L}), \gamma_a\rangle} \exp\Big(-\int_a c(\mathbb{L}, \nabla)\Big) \qquad (*)$$
where $\omega_1(\mathbb{L}) \in H^1(M, \mathbb{Z}_2)$ is the 1st Stiefel-Whitney class of $\mathbb{L}$

Rmk. For any A-path $a : I \to A$ we have the linear holonomy of a. For a loop $a : I \to A$ $(*)$ yields:
$$\det(\mathrm{hol}(a)) = (-1)^{\langle \omega_1, \gamma_a\rangle} \exp\Big(-\int_a \mathrm{mod}\,A\Big)$$
where

- $\mathrm{mod}\,A \in H^1(A)$ is the modular class of A.
- $\omega_1 \in H^1(M, \mathbb{Z}_2)$ is the 1st Stiefel-Whitney class of $\mathbb{L} = \Lambda^{top} A \otimes \Lambda^{top} T^*M$