MATH 595 - Lecture 11

Last time: Given Lie algebroid $A \to M$

- **A-connection on v.b. $E \to M$:**

  $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ $\mathbb{R}$-bilinear s.t. $\begin{cases} \nabla_{f\alpha} s = f \nabla_\alpha s \\ \nabla_\alpha (fs) = f \nabla_\alpha s + \rho(\alpha)(f) s \end{cases}$

  Curvature 2-form $R^\nabla \in \Omega^2(A; \mathrm{End}\, E)$

  When $E = A$, Torsion 2-form $T^\nabla \in \Omega^2(A; A)$

- **A-path:**

  $a : I \to A$ s.t. $\rho(a(t)) = \frac{d}{dt} \gamma_a(t)$ (γ_a = base path of a)

  $\iff$ $a\,dt : TI \to A$ Lie algebroid morphism

- **A-derivative:**

  $\left.\begin{array}{l} \text{A-path } a : I \to E \\ \text{path } u : I \to E \text{ above } \gamma_a : I \to M \end{array}\right\}$ $D_a u : I \to E$ path above γ_a

Rmk: A-paths can have "inner content" not revealed in γ_a.

- $A = TM$: A-path $\iff$ path in M
- $A = \mathfrak{g} \to \{*\}$: A-path is any path $a : I \to \mathfrak{g}$
- $A = T\mathcal{F}$: A-path is any path in a leaf of $\mathcal{F}$
- $A = TP/G$ Atiyah algebroid : A-path = path in P/G

Later we will look in more detail into space $P(A)$ of A-paths

Note: if $\gamma_a(t) = x_0 \iff a(t) \in \mathrm{Ker}\, \rho_{x_0} \Rightarrow D_a u(t) = \frac{du}{dt}(t)$

Geometric Interpretation of Curvature

Consider a Lie algebroid morphism

$$\Phi : T(I\times I) \longrightarrow A, \quad \Phi(t,\varepsilon) = \Phi_1(t,\varepsilon)dt + \Phi_2(t,\varepsilon)d\varepsilon$$

with base map $\gamma : I\times I \to M$ (parametrized surface of A-paths)

Fix path $u : I\times I \to E$ above γ.

$$\rho\circ\Phi = d\gamma \Rightarrow \begin{cases} \rho\circ\Phi_1(t,\varepsilon) = \frac{\partial\gamma}{\partial t}(t,\varepsilon) \\ \rho\circ\Phi_2(t,\varepsilon) = \frac{\partial\gamma}{\partial\varepsilon}(t,\varepsilon)\end{cases}$$

- Fixing $\varepsilon = \varepsilon_0$: $t \mapsto \Phi_1(t,\varepsilon_0)$ is A-path with base path $t\mapsto\gamma(t,\varepsilon_0)$

$\rightsquigarrow$ A-derivative $D_{\Phi_1}u : [0,1]\times[0,1] \to E$

w/ $(D_{\Phi_1}u)(t,\varepsilon_0) := \left(D_{\Phi_1(\cdot,\varepsilon_0)} u(\cdot,\varepsilon_0)\right)(t)$

- Fixing $t = t_0$: $\varepsilon \mapsto \Phi_2(t_0,\varepsilon)$ is A-path with base path $\varepsilon\mapsto\gamma(t_0,\varepsilon)$

$\rightsquigarrow$ A-derivative $D_{\Phi_2}u : [0,1]\times[0,1] \to E$

Proposition With these notations:

$$R^\nabla(\Phi_1,\Phi_2)u = D_{\Phi_1}D_{\Phi_2}u - D_{\Phi_2}D_{\Phi_1}u$$

Proof

We can pullback the connection via the algebroid map $\Phi : T(I\times I) \to A$, and we find:

$$D_{\Phi_1}u = \nabla_{\frac{d}{dt}}u \qquad D_{\Phi_2}u = \nabla_{\frac{d}{d\varepsilon}}u$$

The curvatures are related by pullback:

$$R^{\nabla^*}(\Phi^*\alpha,\Phi^*\beta)(\gamma^*s) = \gamma^*(R^\nabla(\alpha,\beta)(s))$$

$$\Rightarrow R^\nabla(\Phi_1,\Phi_2)u = R^{\nabla^*}\left(\frac{d}{dt},\frac{d}{d\varepsilon}\right)u$$

Since $\left[\frac{d}{dt},\frac{d}{d\varepsilon}\right] = 0$, the result follows. $\blacksquare$

Geometric Interpretation of Torsion

Consider a vector bundle map

$$\begin{tikzcd} T(I\times I) \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\ I\times I \arrow[r, "\gamma"'] & M \end{tikzcd} \qquad \Phi = \Phi_1\, dt + \Phi_2\, d\varepsilon$$

Assume that Φ preserves anchors:

$$\rho \circ \Phi = d\gamma$$

We still have the A-paths obtained by freezing ε/t as before.
(preservation of brackets was not used)

Proposition

Given an A-connection ∇ on A, a v.b. map $\Phi : T(I\times I) \to A$ compatible w/ anchors is a Lie algebroid morphism iff

$$T^{\nabla}(\Phi_1, \Phi_2) = D_{\Phi_1}\Phi_2 - D_{\Phi_2}\Phi_1$$

Proof.

(=>) Pullback ∇ along Φ to obtain classical connection on $T(I\times I)$ and apply classical result.

(<=) We will comeback to this result and its proof later.

∎

Parallel Transport

Def. Given $E \to M$ with A-connection ∇ we say that $u : I \to E$ is parallel along an A-path $a : I \to A$ if u covers γ_a and:

$$D_a u = 0$$

As in classical case, one can //-transport vectors:

Prop. Given (E,∇) and A-path $a: I \to A$, for each $u_0 \in E_{\gamma_a(0)}$ there exists a unique $/\!/$ curve $u: I \to E$ along a w/ $u(0) = u_0$. Moreover, the end-point $u(1)$ depends linearly on u_0.

Proof.

Pullback ∇ along A-path $a: I \to A$ to obtain classical connection on $\gamma_a^* E \to [0,1]$. This reduces result to existence of $/\!/$-transport for classical connection on $[0,1]$.

Alternative: In local coordinates, equation for $/\!/$-transport is:

$$\begin{cases} \dfrac{d}{dt} u^r(t) = - \Gamma^r_{es}(\gamma_a(t))\, a^e(t)\, u^s(t) \\ u(0) = u_0 \end{cases}$$

linear t-dependent ODE $\Rightarrow$ solutions exist for all t ∎

$/\!/$-transport map along A-path:

$$\tau_a : E_{\gamma_a(0)} \to E_{\gamma_a(1)}, \quad u_0 \longmapsto u(1)$$

(linear isomorphism between fibers)

Examples

0. $A = TM \Rightarrow$ usual $/\!/$-transport along paths

1. $A = T\mathcal{F}$, $E = (\nu(\mathcal{F}), \nabla^{Bott})$

$\gamma: I \to L$ path in leaf $L \in \mathcal{F}$ $\Leftrightarrow$ $a(t) = \dot{\gamma}(t): I \to TL$ is $T\mathcal{F}$-path

$\Rightarrow \tau_a = hol_{Lin}(\gamma): \nu_{\gamma(0)}(L) \to \nu_{\gamma(1)}(L)$ <u>Linear holonomy of γ</u>

2. More generally, for any Lie algebroid $A \to M$:

$a: I \to A$ A-path $\Rightarrow$ $\gamma_a(t) \in \mathcal{O}$ w/ $\mathcal{O}$ orbit of A

$\Rightarrow a: I \to A_{\mathcal{O}}$ is $A_{\mathcal{O}}$-path

Recall $(\nu(\mathcal{O}), \nabla^{Bot}) \in Rep(A_{\mathcal{O}})$, so:

$$\tau_a : \nu_{\gamma_a(0)}(\mathcal{O}) \to \nu_{\gamma_a(1)}(\mathcal{O})$$ <u>linear A-holonomy</u> of a

3. $A = \mathfrak{g} \to \{*\}$, $V \in \mathrm{Rep}(\mathfrak{g})$

$a : I \to \mathfrak{g}$ any path $\Rightarrow \tilde{\tau}_a : V \to V$

What is this map?

$\{\alpha_1, \dots, \alpha_d\}$ basis for $\mathfrak{g}$; $\{e_1, \dots, e_k\}$ basis for V

$\rho(\alpha_i)(e_r) = A^s_{ir} e_s \qquad a(t) = a^i(t)\alpha_i \qquad u(t) = u^s(t) e_s$

$$(D_a u)(t) = \left(\frac{du^s}{dt}(t) + A^s_{ir} a^i(t) u^r(t)\right) e_s$$

So $\tau_a(v) = u(1)$ where:

$$\begin{cases} \frac{du}{dt}(t) = -\rho(a(t))\, u(t) \\ u(0) = v \end{cases}$$

Integrating to group rep $G \to Gl(V)$ we obtain:

$$\tau_a(v) = \exp\left(-\int_0^1 a(t)\, dt\right) \cdot v$$

Ordinary //-transport satisfies 3 basic properties:

(i) Invariance under reparametrization of paths

(ii) Concatenation of paths = composition: $\tilde{\tau}_{\gamma_1 \cdot \gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2}$

(iii) Invariance under path-homotopy when connection is flat

We extend all these properties for A-connections.

<u>Prop.</u> Let $a : I \to A$ be A-path and $\tau : [0,1] \to [0,1]$, $\tau(0) = 0, \tau(1) = 1$.
Set

$$a^{\tau}(t) := \tau'(t)\, a(\tau(t))$$

Then $a^{\tau} : [0,1] \to A$ is A-path and for any A-connection

//-transport along a = //-transport along a^{τ}

Proof. Note that $\gamma_{a^\tau}(t) = \gamma_a(\tau(t))$ and:

$$\rho(a^\tau(t)) = \tau'(t)\,\rho(a(\tau(t))) = \tau'(t)\,\dot\gamma_a(\tau(t)) = \frac{d}{dt}\gamma_a(\tau(t))$$

so $a^\tau(t)$ is A-path.

To perform //-transport along a we can pull-back along $a : TI \to A$ and reduce to ordinary //-transport for pullback bundle $\gamma_a^* E \to I$ along id-path $\eta(t) = t$

$$\tau_a = \tau_\eta : (\gamma_a^* E)_0 \to (\gamma_a^* E)_1$$
$$(\gamma_a^* E)_0 = E_{\gamma_a(0)}, \quad (\gamma_a^* E)_1 \cong E_{\gamma_a(1)}$$

For reparametrized A-path, we have similar reduction. But then two reductions are just //-transport for TI-connection relative to η and reparametrization $\eta \circ \tau$. By ordinary result, the //-transport maps coincide. $\boxtimes$

- Using reparametrizations that satisfy $\tau^{(m)}(0) = \tau^{(m)}(1) = 0$, $\forall m \in \mathbb{N}$ we obtain A-path a^τ with $a(0) = 0_x$ $a(1) = 0_y$ and all its derivatives at $t = 0, 1$ vanish.
- $a_1 : I \to A$ & $a_2 : I \to A$ w/ all derivatives $= 0$ at $t = 0, 1$

  If $\gamma_{a_1}(0) = \gamma_{a_2}(0)$:

$$a_1 \circ a_2(t) = \begin{cases} 2a_2(2t), & 0 \le t \le 1/2 \\ 2a_1(2t-1), & 1/2 \le t \le 1 \end{cases}$$

Proposition. Under these conditions, for any A-connection:

$$\tau_{a_1 \circ a_2} = \tau_{a_1} \circ \tau_{a_2}$$

Proof. Exercise.

Def. Two A-paths $a_0, a_1 : I \to A$ are called <u>A-path homotopic</u> if there exists a Lie algebroid morphism

$$\Phi : T(I \times I) \longrightarrow A, \qquad \Phi = \Phi_1 \, dt + \Phi_2 \, d\varepsilon$$

$$\Phi_1(t, 0) = a_0(t), \qquad \Phi_1(t, 1) = a_1(t)$$

$$\Phi_2(0, \varepsilon) = \Phi_2(0, \varepsilon) = 0$$

Rmk.

- Since Φ is algebroid morphism, it satisfies:

$$\rho \circ \Phi = d\gamma$$

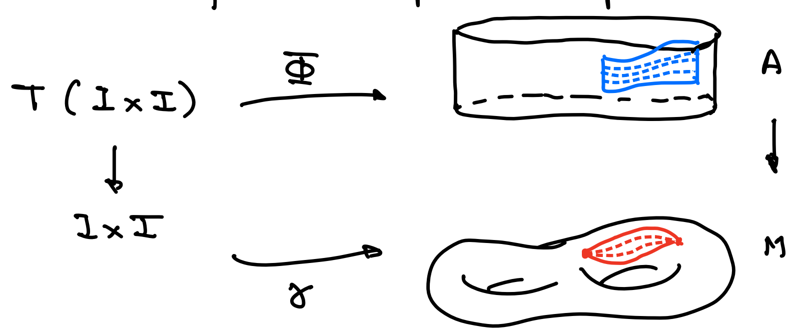
where $\gamma : [0,1] \times [0,1] \longrightarrow M$ is base path of Φ. As we saw before:

$$\begin{cases} \text{For each } \varepsilon: \; t \longmapsto \Phi_1(t, \varepsilon) \text{ is } A\text{-path} \\ \text{For each } t: \; \varepsilon \longmapsto \Phi_2(t, \varepsilon) \text{ is } A\text{-path} \end{cases}$$

- It follows also that

$$\begin{cases} \Phi_2(0, \varepsilon) = 0 \implies \dfrac{\partial \gamma}{\partial \varepsilon}(0, \varepsilon) = 0 \implies \gamma(0, \varepsilon) \text{ constant} \\ \Phi_2(1, \varepsilon) = 0 \implies \dfrac{\partial \gamma}{\partial \varepsilon}(1, \varepsilon) = 0 \implies \gamma(1, \varepsilon) \text{ constant} \end{cases}$$

So a_0 & a_1 have path-homotopic base paths



Proposition For a flat A-connection, given any two A-path homotopic paths a_0 & a_1,

$$\tilde{\tau}_{a_0} = \tilde{\tau}_{a_1}$$

Proof

- $\Phi : T(I \times I) \to A$ be an A-path homotopy between a_0 & a_1
- $\tau^{t,\varepsilon}_{\Phi_1} : \underset{\substack{\| \\ E_{\gamma(0,\varepsilon)}}}{E_{\gamma(0,0)}} \longrightarrow E_{\gamma(t,\varepsilon)}$ //-transport along $t \mapsto \Phi_1(t,\varepsilon)$

For any $u \in E_{\gamma(0,0)}$, $c(t,\varepsilon) := \tau^{t,\varepsilon}_{\Phi_1}(u)$ gives a curve above $\gamma : I \times I \to M$, satisfying: $D_{\Phi_1} c = 0$

Claim.

$$D_{\Phi_2} c = 0$$

Assuming claim, we have:

$$\frac{d}{d\varepsilon} c(1,\varepsilon) = (D_{\Phi_2} c)(1,\varepsilon) = 0$$

So $c(1,\varepsilon) = \tau^{1,\varepsilon}_{\Phi_1}(u)$ is constant, showing that

$$\tau_{a_0}(u) = \tau^{1,0}_{\Phi_1}(u) = \tau^{1,1}_{\Phi_1}(u) = \tau_{a_1}(u)$$

Proof of claim:

At $t=0$, boundary conditions for A-path homotopy Φ give:

$$D_{\Phi_2} c\,(0,\varepsilon) = \frac{d}{d\varepsilon} c(0,\varepsilon) = 0$$

Since ∇ is flat, geometric interpretation of curvature gives:

$$D_{\Phi_1} D_{\Phi_2} c = D_{\Phi_2} \underbrace{D_{\Phi_1} c}_{=0} = 0$$

So $D_{\Phi_2} c$ is parallel along $t \mapsto \Phi_1(t,\varepsilon)$. By uniqueness of //-paths, we must have $D_{\Phi_2} c = 0$ $\blacksquare$