MATH 595 - LECTURE 10

Characteristic class of 1-dim reps

Let $(\mathbb{L}_M, \nabla)$ be a rep of $A \to M$ of rank 1. We define a class

$$c_1(\mathbb{L}_M, \nabla) \in H^1(A)$$

For this consider the cases where $\mathbb{L}_M$ is orientable or not:

- $\underline{\mathbb{L}_M \text{ orientable}}$: Let $s \in \Gamma(\mathbb{L}_M)$ be nowhere vanishing

$$\nabla_\alpha s = \omega(\alpha) s, \quad \forall \alpha \in \Gamma(A)$$

We have $\omega \in \Omega^1(A)$. Moreover:

$$0 = \nabla_{[\alpha,\beta]} s - \nabla_\alpha \nabla_\beta s + \nabla_\beta \nabla_\alpha s$$
$$= \underbrace{\big(\omega([\alpha,\beta]) - \rho(\alpha)(\omega(\beta)) + \rho(\beta)(\omega(\alpha))\big)}_{-(d_A\omega)(\alpha,\beta)} s \quad \Rightarrow \quad d_A \omega = 0$$

If $s' \in \Gamma(\mathbb{L}_M)$ is another nowhere vanishing section, then

$$s' = f s \quad \text{w/ } f \text{ non-vanishing}$$

$$\Rightarrow \quad \omega'(\alpha) s' = \nabla_\alpha s' = \nabla_\alpha(f s) = f \nabla_\alpha s + \rho(\alpha)(f) s$$
$$= \omega(\alpha) f s + \frac{1}{f} \rho(\alpha)(f) f s$$
$$= \omega(\alpha) s' \pm \underbrace{\rho(\alpha)(\log|f|)}_{d_A \log|f|} s'$$

$$\Rightarrow \quad \omega' = \omega \pm d_A \log|f|$$

Hence, we can set:

$$c_1(\mathbb{L}_M, \nabla) := [\omega] \in H^1(A)$$

- $\underline{\mathbb{L}_M \text{ not orientable}}$: then
  - $\mathbb{L}_M^2 := \mathbb{L}_M \otimes \mathbb{L}_M, \quad \nabla^{\otimes}_\alpha (s_1 \otimes s_2) = \nabla_\alpha s_1 \otimes s_2 + s_1 \otimes \nabla_\alpha s_2$

Since $\mathbb{L}^2_M$ is orientable, fixing non vanishing section as before:

$$\nabla^{\otimes}_{\alpha} s = \omega(\alpha)\, s$$

and we now set:

$$c_1(\mathbb{L}_A, \nabla) := \frac{1}{2}[\omega] \in H^1(A)$$

The factor $1/2$ is included so that we got the same result as in the oriented case.

<u>Def.</u> The <u>modular class of A</u> is the characteristic class of the Rep $\Lambda^{top} A \otimes \Lambda^{top} T^*M$ and is denoted $\mathrm{mod}(A) \in H^1(A)$. If $\mathrm{mod}(A) = 0$, when calls A <u>unimodular</u>

<u>Examples</u>

1. If $A = \mathfrak{g} \to \{*\}$ then $\Lambda^{top} A \otimes \Lambda^{top} T^*M \simeq \Lambda^{top}\mathfrak{g} \simeq \Lambda^{top}\mathfrak{g}^*$

$\mathfrak{g}$ is unimodular iff any connected Lie group G with Lie algebra $\mathfrak{g}$ has a bi-invariant volume form. So $\mathrm{mod}(\mathfrak{g}) \in H^1(\mathfrak{g})$ is the obstruction to existence of a bi-invariant volume form on G.

2. If $A = T\mathcal{F}$ then $\Lambda^{top} A \otimes \Lambda^{top} T^*M \simeq \Lambda^{top} \nu^*(\mathcal{F})$ w/ $\nabla = \otimes^{Top} \nabla^{Bot}$

$\mathcal{F}$ is unimodular iff there exists a transverse $1/2$-density which is holonomy invariant. So $\mathrm{mod}(\mathcal{F}) \in H^1(\mathcal{F})$ is the obstruction to existence of such density

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<u>Rmks.</u>

- There is a similar interpretation for any Lie algebroid of $\mathrm{mod}(A)$ as an obstruction class to existence of invariant $1/2$-density.
- One can define higher characteristic classes for any Rep (E, ∇)

Lie Algebroid Connections

Defn. Given a Lie algebroid $A \to M$ and a v.b. $E \to M$ an <u>A-connection</u> is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, s) \mapsto \nabla_\alpha s$ satisfying:

$$\nabla_{f\alpha} s = f \nabla_\alpha s, \quad \nabla_\alpha (fs) = f \nabla_\alpha s + \rho(\alpha)(f) s$$

The <u>curvature</u> of ∇ is the $\mathrm{End}(E)$-valued A-form $R \in \Omega^2(A; \mathrm{End}\, E)$

$$R^\nabla(\alpha, \beta) := [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha, \beta]}$$

Given A-connection ∇ one defines a "differential" on E-valued A-forms:

$$d_A^\nabla : \Omega^\bullet(A; E) \to \Omega^{\bullet+1}(A; E)$$

$$d_A^\nabla \eta(\alpha_0, \dots, \alpha_k) = \sum_{i=0}^{k} (-1)^i \nabla_{\alpha_i}(\eta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k))$$
$$+ \sum_{i<j} (-1)^{i+j} \eta([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k)$$

The A-de Rham differential corresponds to the case of the trivial Rep $E = \mathbb{R}_M$.

Direct computation shows that for $\omega \in \Omega^\bullet(A)$, $\eta \in \Omega^\bullet(A; E)$:

(i) $d_A^\nabla(\omega \wedge \eta) = d_A \omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d_A^\nabla \eta$

(ii) $(d_A^\nabla)^2 \eta = R^\nabla \wedge \eta$ $\quad \left(R^\nabla \wedge \eta(\alpha_1, \dots, \alpha_n) = \frac{1}{k!} \sum_\sigma (-1)^{|\sigma|} R(\alpha_{\sigma(1)}, \alpha_{\sigma(2)})(\eta(\alpha_{\sigma(3)}, \dots, \alpha_{\sigma(n)}))\right)$

Exercise. Show that there is a 1-1 correspondence:

$$\left\{ \begin{array}{l} A\text{-connections} \\ \nabla \text{ on v.b. } E \to M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{R}\text{-linear operators} \\ d_A^\nabla : \Omega^\bullet(A; E) \to \Omega^{\bullet+1}(A; E) \\ \text{satisfying (i)} \end{array} \right\}$$

Hence, also a similar correspondence for $\mathrm{Rep}(A)$.

Rmk. When ∇ is flat, the cohomology of the complex $(\Omega^\bullet(A;E), d_A^\nabla)$ is denoted $H^\bullet(A;E)$ and is called Lie algebroid cohomology with coefficients in A.

Examples

1. A TM-connection on a v.b. $E \to M$ is just an ordinary connection. Given any Lie algebroid $A \to M$ and a TM-connection ∇ on $E \to M$ we obtain an A-connection $\tilde{\nabla}$ by setting:

$$\tilde{\nabla}_a s := \nabla_{\rho(a)} s$$

So A-connections always exist. The curvature of $\tilde{\nabla}$ & ∇ are related by:

$$R^{\tilde{\nabla}}(\alpha, \beta) = R^{\nabla}(\rho(\alpha), \rho(\beta)) \iff R^{\tilde{\nabla}} = \rho^* R^{\nabla}$$

2. Pullback Connections: Given a Lie algebroid morphism

$$\begin{array}{ccc} A_1 & \xrightarrow{\Phi} & A_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[\phi]{} & M_2 \end{array}$$

and a A_2-connection ∇ on $E \to M_2$ we obtain a pullback connection $\Phi^*\nabla$ on $\phi^* E \to M_1$ by setting:

$$(\Phi^*\nabla)_a(\phi^* s) = \nabla_{\Phi(a)} s \qquad (a \in A_1)$$

One finds that the curvatures are related by pull-back

$$R_{\Phi^*\nabla} = \Phi^* R_\nabla \qquad (\Phi^* : \Omega^\bullet(A_2; E) \to \Omega^\bullet(A_1; \phi^* E))$$

In the previous example $\tilde{\nabla} = \rho^* \nabla$. In the language of differentials the pullback connection is characterized by:

$$d^{\Phi^*\nabla} \Phi^* \eta := \Phi^* d^\nabla \eta$$

Def. Given an A-connection ∇ on $A \to M$ its **torsion** is the A-valued form $T^\nabla \in \Omega^2(A;A)$:

$$T^\nabla(\alpha,\beta) := \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha,\beta].$$

Exercise. Prove the Bianchi identity:

$$d^\nabla T^\nabla(\alpha,\beta,\gamma) = \underset{\alpha,\beta,\gamma}{\odot} R^\nabla(\alpha,\beta)\cdot\gamma$$

Examples

1. Levi-Civita Connection. Given a fiberwise metric g on A there exists a unique A-connection on A with $T^\nabla \equiv 0$ and compatible w/ metric:

$$\rho(\alpha)(g(\beta,\gamma)) = g(\nabla_\alpha\beta,\gamma) + g(\beta,\nabla_\alpha\gamma) \quad \forall \alpha,\beta,\gamma \in \Gamma(A)$$

2. Basic Connections. A pair of A-connections (∇^A, ∇^{TM}) is called basic if:

$$\rho(\nabla^A_\alpha \beta) = \nabla^{TM}_\alpha \rho(\beta), \qquad \forall\ \alpha,\beta \in \Gamma(A)$$

Given an ordinary TM-connection on the v.b. A, we obtain a pair of basic connections:

$$\overline{\nabla}_\alpha \beta := \nabla_{\rho(\beta)}\alpha + [\alpha,\beta], \qquad \overline{\nabla}_\alpha X := \rho(\nabla_X \alpha) + [\rho(\alpha), X]$$

3. Induced A-Connections. An A-connection ∇ on A induces connections on "associated" bundles:

- Dual A-connection on A^*: unique connection ∇ s.t.

$$\langle \nabla_\alpha \xi, \beta\rangle + \langle \xi, \nabla_\alpha \beta\rangle = \rho(\alpha)\langle \xi,\beta\rangle, \quad \forall \xi \in \Gamma(A^*), \alpha,\beta \in \Gamma(A)$$

- A-connection on $\wedge^k A$:

$$\nabla_\alpha(\alpha_1 \wedge \dots \wedge \alpha_n) = (\nabla_\alpha \alpha_1)\wedge \dots \wedge \alpha_n + \dots + \alpha_1 \wedge \dots \wedge \nabla_\alpha \alpha_k$$

- A-Connection on $\otimes^k A \otimes^\ell A^*$: (...)

Parallel Transport

Def. Let $p: A \to M$ be a Lie algebroid. A path $a: I \to A$ is called an A-path if:

$$\rho(a(t)) = \frac{d}{dt} p(a(t)), \quad \forall t \in I$$

We will denote by $\gamma_a(t) = p(a(t))$ the base path of the A-path.

Rmk This is the right notion of path in Lie algebroid theory:

- We can define //-transport only along A-paths
- We have observed that $x, y \in M$ belong to same orbit iff

$\exists$ A-path $a: I \to A$ with $\gamma_a(0) = x$ & $\gamma_a(1) = y$

- We will also introduce A-homotopies between A-paths and construct the A-homotopy groupoid.
- $A = TM$: A-paths $\leftrightarrow$ ordinary paths

Exercise. Show that:

$a: I \to A$ is A-path $\iff$

$$\begin{tikzcd} TI \arrow[r, "a\,dt"] \arrow[d] & A \arrow[d] \\ I \arrow[r, "\gamma_a"'] & M \end{tikzcd}$$

is Lie algebroid morphism

Assume:

- $\nabla \equiv$ A-connection on $\mu: E \to M$
- $a: I \to A \equiv$ A-path w/ base path $\gamma_a: I \to M$
- $u: I \to E \equiv$ path above γ_a

The A-derivative of u along a is the path $D_a u: I \to E$ above γ_a:

$$(*) \qquad (D_a u)(t) := \nabla_{a(t)} S_t(x) + \frac{d}{dt} S_t(x) \quad \text{at } x = \gamma_a(t)$$

with $S_t \in \Gamma(E)$ any time-dependent family of sections extending u:

$$S_t(\gamma_a(t)) = u(t), \quad t \in I$$

Remarks

1) One should verify that $(*)$ is independent of the choice of extension s_t of $u(t)$ (Exercise!)

2) Since $\dot{\gamma}_a(t)$ can vanish (e.g., $a(t) \equiv a_0 \in \ker \rho_{x_0}$ defines an A-path with $\gamma_a(t) = x_0$!), one needs to time-dependent sections.

3) Another point of view:

- A-path $a : I \to A$ $\Longleftrightarrow$ Lie algebroid morphism
$$\begin{tikzcd} TI \arrow[r, "a\,dt"] \arrow[d] & A \arrow[d] \\ I \arrow[r, "\gamma_a"] & M \end{tikzcd}$$
- A-connection ∇ on $E \to M$ $\Longrightarrow$ pullback connection ∇^* on $\gamma_a^* E$
- path $u : I \to E$ above γ_a $\Longleftrightarrow$ $u \in \Gamma(\gamma_a^* E)$
- $D_a u : I \to E$ $\Longleftrightarrow$ $\nabla^*_{\frac{d}{dt}} u \in \Gamma(\gamma_a^* E)$

Local Coordinate Expressions

- $\{\alpha_\ell\}$ - basis of local sections for $A \to M$
- $\{e_r\}$ - basis of local sections for $E \to M$
- $\nabla_{\alpha_\ell} e_r = \Gamma^s_{\ell r}\, e_s$, $\quad \Gamma^s_{lr} \equiv$ Christoffel symbols
- $a(t) = a^\ell(t)\, \alpha_\ell(\gamma_a(t))$, $\quad u(t) = u^r(t)\, e_r(\gamma_a(t))$

$$(D_a u)(t) = \left(\frac{d}{dt} u^s(t) + \Gamma^s_{\ell r}(\gamma_a(t))\, a^\ell(t)\, u^r(t) \right) e_s(\gamma_a(t))$$

Geometric Interpretation of Curvature

Consider a Lie algebroid morphism

$$\Phi : T(I \times I) \longrightarrow A , \qquad \Phi(t,\varepsilon) = \Phi_1(t,\varepsilon)\,dt + \Phi_2(t,\varepsilon)\,d\varepsilon$$

with base map $\gamma : I \times I \to M$ (an parametrized surface of A-paths)

For any $u : I \times I \to E$ above γ:

- Fixing $\varepsilon = \varepsilon_0$: $t \mapsto \Phi_1(t, \varepsilon_0)$ is A-path with base path $t \mapsto \gamma(t, \varepsilon_0)$

$\rightsquigarrow$ A-derivative $D_{\Phi_1} u : [0,1]\times[0,1] \to E$

w/ $(D_{\Phi_1} u)(t,\varepsilon_0) := \big(D_{\Phi_1(\cdot,\varepsilon_0)} u(\cdot,\varepsilon_0)\big)(t)$

- Fixing $t = t_0$: $\varepsilon \longmapsto \Phi_2(t_0,\varepsilon)$ is A-path with base path $\varepsilon \longmapsto \gamma(t_0,\varepsilon)$

$\rightsquigarrow$ A-derivative $D_{\Phi_2} u : [0,1]\times[0,1] \to E$

<u>Proposition</u> With this notation:

$$R^\nabla(\Phi_1,\Phi_2)\,u = D_{\Phi_1} D_{\Phi_2} u - D_{\Phi_2} D_{\Phi_1} u$$

<u>Proof</u>

We can pullback the connection via the algebroid map $\Phi : T(I\times I) \to A$. The curvatures are then related by pullback:

$$R^{\nabla^*}(\Phi^*\alpha, \Phi^*\beta)(\gamma^* s) = \gamma^*\big(R^\nabla(\alpha,\beta)(s)\big)$$

Then the classical interpretation of R yields the proposition

$\boxtimes$

Geometric Interpretation of Torsion

Consider a vector bundle map

$$\begin{tikzcd} T(I\times I) \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\ I\times I \arrow[r, "\gamma"'] & M \end{tikzcd} \qquad \Phi = \Phi_1\, dt + \Phi_2\, d\varepsilon$$

Assume that Φ preserves anchors:

$$\rho \circ \Phi = d\gamma$$

We still have the A-paths obtained by freezing ε / t.

Proposition

Given an A-connection ∇ on A, a v.b. map $\Phi : T(I\times I) \to A$ compatible w/ anchors is a Lie algebroid morphism iff

$$T^{\nabla}(\Phi_1, \Phi_2) = D_{\Phi_1}\Phi_2 - D_{\Phi_2}\Phi_1$$

<u>Proof.</u> Pullback ∇ along Φ to obtain classical connection on $T(I\times I)$ and apply classical result. $\boxtimes$

We will come back to this result and its proof later.