MATH 595 - Lecture 1

Course Information:

- No formal evaluation: paper is voluntary
- Office Hours: TR 11.30-12.30 AM (or by appointment)

Course Contents:

1) Theory: · Lie Groupoids
   · Lie Algebroids
   · Actions and Representations

II) Applications:
   - Moduli / Singular spaces ($\simeq$ smooth stacks)
   - Noncommutative Geometry & index theory
   - Symplectic & Poisson Geometry
   - (Higher) Gauge Theory

Pioneers of Lie groupoid Theory:

- A. Grothendieck: Algebraic Geometry
- C. Ehresmann: Differential Geometry
- D. Spencer: Partial Differential Equations

} E. Cartan

- A. Haefliger: Topology and Foliation Theory
- A. Connes, Operator algebras & non-commutative Geometry
- A. Weinstein, Symplectic & Poisson Geometry

0) Why Groupoids?

Classical view:

symmetry = theory of groups & their actions

Example: Symmetries of $\Omega \subset \mathbb{R}^n$:

Euclidean Group:

$$E(n) = \{ \phi : \mathbb{R}^n \to \mathbb{R}^n \mid \phi \text{ preserves distance} \} \simeq O(n) \ltimes \mathbb{R}^n$$

$$(\phi(x) = Ax + b, \text{ w/ } A \in O(n) \; b \in \mathbb{R}^n)$$

Defining action:

$$E(n) \curvearrowright \mathbb{R}^n \qquad \phi \cdot x := \phi(x)$$

Symmetry Group of Ω:

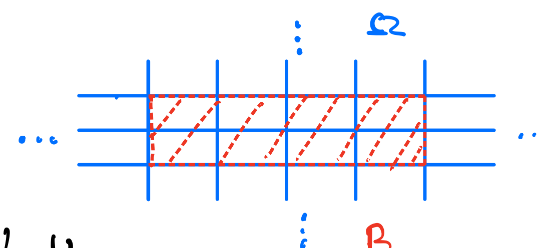
$$G_\Omega = \{ \phi \in E(n) \; : \; \phi(\Omega) = \Omega \}$$

Usual credo:

Ω is very symmetric $\iff$ G_Ω is large

Exercise:

$\Omega = (\mathbb{R} \times \mathbb{Z}) \sqcup (2\mathbb{Z} \times \mathbb{R}) \subset \mathbb{R}^2$



$G_\Omega = \{ \text{Translations by } b \in \Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \} \cup$

$\cup \{ \text{reflections through points in } \tfrac{1}{2}\Lambda \} \cup$

$\cup \{ \text{reflections through vertical \& horizontal lines through } \tfrac{1}{2}\Lambda \}$

If $B = [0,2m] \times [0,m]$ (finite rectangle)

$\tilde{\Omega}$ = finite tiling $= B \cap \Omega$

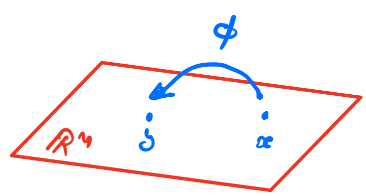
$G_{\tilde{\Omega}} = \mathbb{Z}_2 \times \mathbb{Z}_2$

<u>Conclusion:</u> Ω is very symmetric, but $\tilde{\Omega}$ is not (independent of number of Tiles!) Classical Theory does not always capture symmetries of an object.

Groupoids allow to fix this!

<u>Transformation Groupoid or action groupoid</u> associated with $G_\Omega \circlearrowright \mathbb{R}^n$

· $\mathcal{G}_\Omega := \{ (y, \phi, x) : \phi \in G_\Omega, x, y \in \mathbb{R}^n, y = \phi(x) \}$



Partially defined multiplication:

$$(z, \psi, y) \cdot (y, \phi, x) := (z, \psi \circ \phi, x)$$

Satisfying:

(1) <u>Composition:</u> If $g, h \in \mathcal{G}$, $g \cdot h$ is defined only if $s(g) = t(h)$

where: $s : \mathcal{G} \to \mathbb{R}^n$, $(y, \phi, x) \mapsto x$ (source map)

$t : \mathcal{G} \to \mathbb{R}^n$, $(y, \phi, x) \mapsto y$ (target map)

and then $s(gh) = s(h)$, $t(gh) = t(g)$.

(2) <u>Associative:</u> $(gh)k = g(hk)$ (if defined, i.e., $s(g) = t(h)$, $s(h) = t(k)$)

(3) Units: $1_x := (x, id, x)$ are left/right identities:

$$1_{t(g)} \cdot g = g = g \cdot 1_{s(g)}$$

(4) Inverses: Each $g = (y, \phi, x)$ has an inverse $g^{-1} = (x, \phi^{-1}, y)$:

$$g g^{-1} = 1_{t(g)}, \quad g^{-1} g = 1_{s(g)}$$

These are exactly the properties characterizing a groupoid.

Def: A groupoid over a set M is a set $\mathcal{G}$ together with maps:

- $s, t : \mathcal{G} \to M$
- $m : \{(g,h) \in \mathcal{G} \times \mathcal{G} : s(g) = t(h)\} \to \mathcal{G}, \quad (g,h) \mapsto gh$
- $u : M \to \mathcal{G}, \quad x \mapsto 1_x$
- $i : \mathcal{G} \to \mathcal{G}, \quad g \mapsto g^{-1}$

satisfying (1)-(4):

(1) If $z \xleftarrow{g} y \xleftarrow{h} x$ then $z \xleftarrow{gh} x$

(2) If $z \xleftarrow{g} y \xleftarrow{h} x \xleftarrow{k} u$ then $(gh)k = g(hk)$

(3) $\exists \; x \xleftarrow{1_x} x$ such that $\forall \; y \xleftarrow{g} x$, $1_y g = g = g 1_x$

(4) If $y \xleftarrow{g} x$ there exists $x \xleftarrow{g^{-1}} y$ such that $g g^{-1} = 1_y \; g^{-1} g = 1_x$

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Rmks:

- A groupoid is just a (small) category where every arrow is invertible
- Isotropy group of $x \in M$:

$$\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x) = \{ \overset{g}{\circlearrowleft}_{x} \}$$

- Orbit of $x \in M$:

$$\mathcal{O}_x = \{ y \in M \mid \exists\, g \in \mathcal{G} : y \xleftarrow{g} x \}$$

- Groupoids can be restricted to subsets:

$$\left.\begin{array}{l} \mathcal{G} \rightrightarrows M \\ N \subset M \end{array}\right\} \Rightarrow \mathcal{G}|_N \rightrightarrows N \qquad \mathcal{G}|_N = \{ g \in \mathcal{G} : s(g), t(g) \in N \}$$

Exercise

A bisection of $\mathcal{G} \rightrightarrows M$ is a map $b : M \to \mathcal{G}$ such that $s \circ b = \mathrm{id}_M$ and $t \circ b : M \to M$ is a bijection (e.g., the identity $u : M \to \mathcal{G}$ is a bisection).

Show that the set of bisections $\Gamma(\mathcal{G})$ has a natural group structure.

Symmetry groupoids of finite tiling:

$$\left.\begin{array}{l} \cdot\ \Omega \subset \mathbb{R}^2 \\ \cdot\ G_\Omega = \text{symmetry group} \end{array}\right\} \Rightarrow \begin{array}{c} \text{transformation groupoid} \\ \mathcal{G}_\Omega \rightrightarrows \mathbb{R}^2 \end{array}$$

- $B = [0, 2m] \times [0, m] \subset \mathbb{R}^2$, $\tilde{\Omega} = \Omega \cap B$

$$\Rightarrow \mathcal{G}_{\tilde{\Omega}} := \mathcal{G}_\Omega|_B \rightrightarrows B$$

This captures symmetry of finite tiling:

- $x, y \in B$ belong to same orbit iff they are similarly placed in their tilings
- $x \in B$ has trivial isotropy unless if $x \in \frac{1}{2}\Lambda \cap B$ for which isotropy group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Rmk: The construction of $\mathcal{G}_{\tilde{\Omega}}$ uses the infinite lattice Ω. One can also construct a local symmetry groupoid of $\tilde{\Omega}$ which does not use Ω:

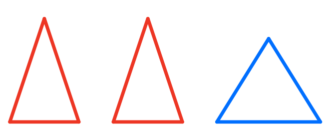
$$\mathcal{G}^{loc}_{\tilde{\Omega}} = \left\{ \begin{array}{ll} (y, \phi, x) \in B \times E(2) \times B \;\Big|\; y = \phi(x) \text{ and } x \text{ has neighborhood} & \\ \quad U \subset \mathbb{R}^2 \text{ such that: } & \phi(U \cap \tilde{\Omega}) \subset \tilde{\Omega} \\ & \phi(U \cap (B \setminus \tilde{\Omega})) \subset B \setminus \tilde{\Omega} \\ & \phi(U \cap (\mathbb{R}^2 \setminus B)) \subset \mathbb{R}^2 \setminus B \end{array} \right\}$$

Exercise: Find orbits and isotropy groups of $\mathcal{G}^{loc}_{\tilde{\Omega}}$. Why are the extra conditions necessary?

See more in: A. Weinstein, "Groupoids: unifying internal and external symmetry", Notices of AMS, Vol 43, N. 7.

- Symmetry Groupoid of a Family of Moduli Spaces

Family w/ 3 triangles:

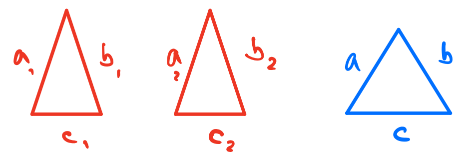
$\mathcal{F} = \{$  $\}$

Symmetries of $\mathcal{F}$ = similarity transformations between triangles
(Translations, scalings, rotations, reflections)

This is a Groupoid: $\mathcal{G} \rightrightarrows M$

- M = Objects = 3 triangles = $\left\{ \underset{T_1}{\circ} \quad \underset{T_2}{\circ} \quad \underset{T_3}{\circ} \right\}$
- $\mathcal{G}$ = Arrows = similarity transformations

There are 14 arrows:



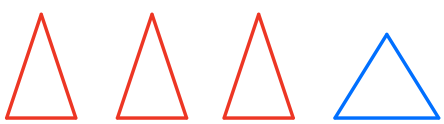
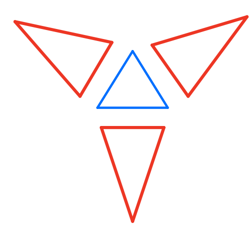
- $\mathcal{G}_{T_1} = s^{-1}(T_1) \cap t^{-1}(T_1) \simeq D_2$ (2 elements)
- $\mathcal{G}_{T_2} = s^{-1}(T_2) \cap t^{-1}(T_2) \simeq D_2$ (2 elements)
- $\mathcal{G}_{T_3} = s^{-1}(T_3) \cap t^{-1}(T_3) \simeq D_3$ (6 elements)

- $s^{-1}(T_1) \cap t^{-1}(T_2) = \left\{ \begin{matrix} a_1 \to a_2 \\ b_1 \to b_2 \\ c_1 \to c_2 \end{matrix} \,,\; \begin{matrix} a_1 \to b_2 \\ b_1 \to a_2 \\ c_1 \to c_2 \end{matrix} \right\}$ (2 elements)

- $t^{-1}(T_1) \cap s^{-1}(T_2) = \left\{ \begin{matrix} a_2 \to a_1 \\ b_2 \to b_1 \\ c_2 \to c_1 \end{matrix} \,,\; \begin{matrix} a_2 \to b_1 \\ b_2 \to a_1 \\ c_2 \to c_1 \end{matrix} \right\}$ (2 elements)

- No arrows between a red and a blue triangles

Another family:

$\mathcal{F}' = \{$  $\}$ $\simeq$ 

Symmetry Groupoid of $\mathcal{F}' \simeq$ Action Groupoid $D_3 \curvearrowright \{0, 1, 2, 3\}$

See more in: K. Behrend, "Introduction to Algebraic Stacks", in London Math. Society Lecture Notes Series vol. 411

Remark:

- We can replace triangles and similarities by other objects and their isomorphisms; e.g., Riemannian metrics on a manifold and isometries between them.
- Instead of finite (or discrete) families, one can consider "continuous" or "smooth" families of objects. Their symmetry groupoids are relevant to describe the moduli space of all such deformations.

Singular Spaces:

- $G \curvearrowright M$ smooth action of a Lie group on a manifold
  - Free: $g \cdot x = x$, for some $x \Rightarrow g = e$
  - proper: $G \times M \xrightarrow{\Phi} M \times M$, $(g, x) \mapsto (g \cdot x, x)$ is a proper map (i.e, K compact $\Rightarrow \Phi^{-1}(K)$ compact)

$$\left(\Leftrightarrow \begin{cases} x_m \to x \\ g_m x_m \to y \end{cases} \Rightarrow \exists\, g_{m_k} \to g \right)$$

Free + proper action $\Rightarrow$ M/G has unique smooth structure s.t. $\pi : M \to M/G$ is submersion

Example:

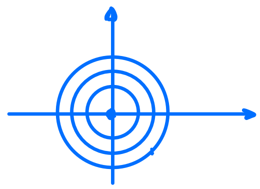
$G = \mathbb{Z}^k \curvearrowright M = \mathbb{R}^k$, $(m_1, .., m_k) \cdot (x^1, .., x^k) := (x^1 + m_1, .., x^k + m_k)$

$$\rightsquigarrow \mathbb{T}^n = \mathbb{R}^k / \mathbb{Z}^k$$

What if action is not free or proper? M/G is a "singular space"

Example:

- $SO(2) \circlearrowleft \mathbb{R}^2$



$\mathbb{R}^2/_{SO(2)}$ =  $= [0, +\infty[$

If we remove origin, action is free & proper:

$$(\mathbb{R}^2 - \{0\})/_{SO(2)} \simeq \mathbb{R}$$

- $SO(3) \circlearrowleft \mathbb{R}^3 \Rightarrow \mathbb{R}^3/_{SO(3)} = [0, +\infty[$

Should these two singular spaces be considered the same?

No! The singular point $\{0\}$ is of different type. The singular space should contain information about this hidden symmetry.

We will see this can be expressed in groupoid language: The action groupoids $SO(2) \circlearrowleft \mathbb{R}^2$, $SO(3) \circlearrowleft \mathbb{R}^3$ are not Morita equivalent.