GABOR DUAL SPLINE WINDOWS

R. S. LAUGESEN

Abstract. An algorithm is presented for constructing dual Gabor window functions that are splines. The spline windows are supported in $[-1,1]$, with a knot at $x=0$, and can be taken C^m smooth and symmetric. The translation and modulation parameters satisfy $0 < ab \leq 1/2$. The full range $0 < ab < 1$ is handled by introducing an additional knot.

Many explicit examples are found.

1. Introduction

A Gabor or Weyl–Heisenberg system

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$$

arises by translating and modulating the *window* or *generator* function $g \in L^2(\mathbb{R})$. The translation and modulation parameters $a, b > 0$ are fixed. A *dual window* $h \in L^2(\mathbb{R})$ is one for which analysis by g followed by synthesis with h yields perfect reconstruction, meaning

$$f = ab \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}h, \qquad f \in L^2(\mathbb{R}), \tag{1}$$

or equivalently the same expression with g and h interchanged. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on L^2.

The richness of Gabor theory [1, 7, 8] has not been matched by a similarly rich collection of examples. To help fill this gap, the current paper constructs explicit dual windows using splines.

These Gabor dual spline windows are supported on the interval $[-1,1]$, with knots at $x = 0, \pm1$. The attractive examples in Table 1 are drawn from tables later in the paper, ordered roughly by increasing smoothness and by the degrees of the splines. These examples illustrate an intuitive principle: the more smoothness or symmetry one wants, or the fewer knots, then the higher must be the degrees of the windows. For instance, the second, fourth and sixth examples in Table 1 are symmetric, as their accompanying figures reveal, and they have generally higher degrees than the first, third and fifth examples, which have the same smoothness but are not symmetric. Further, in the second example g has knots only at ±1, not at 0.

Date: June 23, 2008.

2000 *Mathematics Subject Classification.* Primary 42C15. Secondary 41A15.

Key words and phrases. Gabor system, dual frame.

m	$\begin{cases} g(x), & -1 \le x \le 0 \\ g(x), & 0 \le x \le 1 \end{cases}$	$\begin{cases} h(x), & -1 \le x \le 0 \\ h(x), & 0 \le x \le 1 \end{cases}$	Figure	Table
0	$\begin{cases} 1+x \\ 1+x-2x^2 \end{cases}$	$\begin{cases} 1-x-2x^2 \\ 1-x \end{cases}$	5	3
0	$\begin{cases} 1-x^2 \\ 1-x^2 \end{cases}$	$\begin{cases} 1-(5/3)x^2-(2/3)x^3 \\ 1-(5/3)x^2+(2/3)x^3 \end{cases}$	1	2
1	$\begin{cases} 1+2x+x^2 \\ 1+2x+3x^2+4x^3-30x^4+20x^5 \end{cases}$	$\begin{cases} 1-2x+3x^2-4x^3-30x^4-20x^5 \\ 1-2x+x^2 \end{cases}$	6	3
1	$\begin{cases} 1-3x^2-2x^3 \\ 1-3x^2+2x^3 \end{cases}$	$\begin{cases} 1+3x^2+2x^3-18x^4-24x^5-8x^6 \\ 1+3x^2-2x^3-18x^4+24x^5-8x^6 \end{cases}$	9	4
2	$\begin{cases} 1+3x+3x^2+x^3 \\ \\ 1+3x+3x^2+x^3+6x^4+18x^5 \\ \qquad -236x^6+342x^7-138x^8 \end{cases}$	$\begin{cases} 1-3x+6x^2-10x^3+15x^4-21x^5-1064x^6 \\ \qquad -2934x^7-3384x^8-1872x^9-414x^{10} \\ 1-3x+6x^2-10x^3+9x^4-3x^5 \\ \end{cases}$	8	3
2	$\begin{cases} 1-6x^2-8x^3-3x^4 \\ \\ 1-6x^2+8x^3-3x^4 \\ \end{cases}$	$\begin{cases} 1+6x^2+8x^3+39x^4+96x^5-(244/5)x^6 \\ \qquad -(1452/5)x^7-(1242/5)x^8-(324/5)x^9 \\ 1+6x^2-8x^3+39x^4-96x^5-(244/5)x^6 \\ \qquad +(1452/5)x^7-(1242/5)x^8+(324/5)x^9 \end{cases}$	10	4

Table 1: Dual window pair examples supported on $[-1,1]$, with C^m-smoothness on $\mathbb{R}$. Here $a=1, 0<b\le\frac{1}{2}$.

The algorithm for constructing C^m-smooth dual windows is summarized in Section 2. The construction is implemented in Sections 3–5 for windows supported in $[-1,1]$ with 2 or 3 knots, when $a=1$ and $0<b\le 1/2$. The remaining range $1/2<b<1$ is treated in Sections 6 and 7, by rescaling the earlier examples and inserting flat segments in the graphs. The range $0<b\le 1/4$ is reconsidered in Section 8, where the frequency concentration of the windows is improved by dilating their support in the time domain. Finally, the situation where one window is smoother than the other is treated in Section 9.

Some known dual window constructions are described in Section 10, including the work of Christensen and Kim [3, 5], who follow a different approach from this paper, the square-root construction of tight frames from partitions of unity, and the standard construction of the canonical dual.

Acknowledgment. I am grateful to Ole Christensen for sending me an advance copy of his book [2], which inspired the current paper.

2. **Overview of the method, and symmetries**

We fix the translation step

$$a = 1$$

for the remainder of the paper. We are free to do so because every pair of dual windows rescales to a pair with $a = 1$, by the transformations $g \mapsto \sqrt{a}g(ax), h \mapsto \sqrt{a}h(ax)$ and $a \mapsto 1, b \mapsto ab$. We further assume

$$0 < b \leq \frac{1}{2}.$$

Only in Section 7 will we consider $\frac{1}{2} < b < 1$. Remember that $ab \leq 1$ is necessary for existence of dual windows [1, Theorem 8.3.1].

We need a result of Janssen characterizing dual windows. Assume $g, h \in L^2(\mathbb{R})$ are bounded with compact support. Then the Gabor analysis operator using g is bounded from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z} \times \mathbb{Z})$, and the Gabor synthesis operator using h is bounded from $\ell^2(\mathbb{Z} \times \mathbb{Z})$ to $L^2(\mathbb{R})$, with unconditional convergence [1, Theorem 8.4.4], [2, Corollary 9.1.7]. Hence the Gabor series on the right side of (1) converges unconditionally in $L^2(\mathbb{R})$.

Janssen's result says g and h are dual windows if

$$\sum_{n \in \mathbb{Z}} g(x-n)\overline{h(x-n)} = 1, \qquad x \in \mathbb{R}, \tag{2}$$

$$\sum_{n \in \mathbb{Z}} g(x-n-k/b)\overline{h(x-n)} = 0, \qquad x \in \mathbb{R}, \quad k \neq 0. \tag{3}$$

(See [9, §1.3.2] or [2, Theorem 9.3.5].) The converse holds too, though we will not need it.

Suppose g and h are continuous and supported in the interval $[-1, 1]$ of length 2. Then Janssen's second condition (3) holds automatically, because $|k/b| \geq 2$ by the assumption $b \leq 1/2$. The first condition (2) says

$$g(x-1)\overline{h(x-1)} + g(x)\overline{h(x)} = 1, \qquad 0 \leq x \leq 1. \tag{4}$$

This window condition lies at the heart of the paper.

In addition to (4), all our examples will satisfy the normalization

$$g(0) = 1, \qquad h(0) = 1.$$

No generality is lost by imposing this normalization, because if g and h satisfy the window condition (4), then so do $g/g(0)$ and $h/h(0)$ (using that $g(0)\overline{h(0)} = 1$ by (4)).

Now we summarize the method. We seek splines g and h that are supported on $[-1, 1]$ and satisfy the following conditions:

- C^m smoothness of g and h at $x = \pm 1$ (the boundary conditions)
- C^m-smoothness of g at $x = 0$ (smoothness at the central knot)
- the normalization $g(0) = h(0) = 1$
- symmetry of g and h (if desired; see below)
- the window condition (4).

We do *not* impose smoothness on the dual window h at the knot point $x = 0$. That smoothness follows automatically from the window condition, as we show in Lemma 1.

The algorithm consists of counting the number of conditions imposed, and determining which spline degrees provide at least that many coefficients in g and h. Dual windows are then computed by employing symbolic computation software (such as Mathematica) to solve the coefficient equations resulting from the above conditions.

The method does not always work, due to the nonlinear nature of the coefficient restrictions imposed on g and h by the window condition (4). Nor are the expected spline degrees always minimal: cancellations can occur in the highest order coefficients, as happens for example in the Pauli conjugate symmetric case in Section 5. Nonetheless, the method provides a robust, practical framework for the construction of Gabor spline dual windows.

Symmetries. Symmetry of the window functions can be important in applications. A pair of dual windows is called:

$$\textit{symmetric} \text{ if } g(-x) = g(x) \text{ and } h(-x) = h(x), \tag{5}$$

$$\textit{conjugate symmetric} \text{ if } g(-x) = \overline{g(x)} \text{ and } h(-x) = \overline{h(x)}, \tag{6}$$

$$\textit{Pauli symmetric} \text{ if } g(-x) = h(x) \text{ and } h(-x) = g(x), \tag{7}$$

$$\textit{Pauli conjugate symmetric} \text{ if } g(-x) = \overline{h(x)} \text{ and } h(-x) = \overline{g(x)}. \tag{8}$$

The most appealing examples in the paper turn out to be symmetric or Pauli conjugate symmetric.

The "Pauli" terminology is motivated by rewriting the defining condition (7) as $\mathcal{T}\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right)$ where $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the first Pauli matrix and $\mathcal{T}$ denotes the time-reversal or symmetry operator.

Aside. No genuinely new examples can arise from using the second Pauli matrix $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ or the third one $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, as one concludes after some thought.

3. **2-knot windows are impossible**

We start with a negative result: dual windows with only 2-knots cannot exist. For suppose $g = g_* \mathbb{1}_{[-1,1]}$ and $h = h_* \mathbb{1}_{[-1,1]}$ for some polynomials g_* and h_* of degrees $p, q \geq 1$ respectively. Then the window condition (4) says

$$g_*(x-1)\overline{h_*(x-1)} + g_*(x)\overline{h_*(x)} = 1, \qquad 0 \leq x \leq 1,$$

which is impossible because the left hand side is a polynomial of degree $p + q > 0$ while the right hand side is a constant.

A stronger negative result is due to Christensen and Kim [5, Proposition 2.4]. They allow g and h to have different supports.

4. **2-knot/3-knot windows**

Next suppose g is a 2-knot spline and h is a 3-knot spline, meaning $g = g_* \mathbb{1}_{[-1,1]}$ and $h = h_l \mathbb{1}_{[-1,0)} + h_r \mathbb{1}_{[0,1]}$ for some polynomials g_*, h_l, h_r of degrees $p, q, q \geq 1$ respectively. (Here we assume h_l and h_r have the same degree, which is necessary for the window condition (9) to hold, below.)

General case. We claim that C^m-smooth dual windows of this form can generally be constructed when

$$\begin{aligned} p &\geq 2m+2, \\ q &\geq 4m+3. \end{aligned}$$

To justify this claim, first observe the C^m-smoothness of g and h at $x = \pm 1$ imposes $4(m+1)$ conditions, which we call *boundary conditions.* The normalizations $g(0) = h(0) = 1$ impose a further two conditions. The conditions on g can all be satisfied provided $p \geq 2m+2$. Next, the window condition (4) says for $x \in [0,1]$ that

$$g_*(x-1)\overline{h_l(x-1)} + g_*(x)\overline{h_r(x)} = 1. \tag{9}$$

This polynomial equation of degree $p+q$ already holds at $x = 0$ (by our normalizations and boundary conditions) and thus imposes $p+q$ further conditions.

The number of coefficients available to us in g and h is $(p+1)+2(q+1) = p+2q+3$, and so to ensure that the number of parameters exceeds or equals the number of conditions, we require

$$p + 2q + 3 \geq 4(m+1) + 2 + (p+q),$$

which simplifies to $q \geq 4m+3$.

The C^m-smoothness of h at the knot point $x = 0$ follows automatically from our construction (assuming g and h can be found as above), in view of Lemma 1 below with g_l and g_r taken to equal g_*.

Lemma 1. *Let m be a nonnegative integer and $c \in \mathbb{C}$. Assume the smooth functions g_l, g_r, h_l, h_r satisfy the boundary conditions*

$$g_l(-1) = g_l'(-1) = \cdots = g_l^{(m)}(-1) = 0, \quad g_r(1) = g_r'(1) = \cdots = g_r^{(m)}(1) = 0,$$

as well as the normalization $g_r(0) = 1$ and the C^m-joining conditions

$$g_l(0) = g_r(0),\ g_l'(0) = g_r'(0),\ \ldots\ ,\ g_l^{(m)}(0) = g_r^{(m)}(0),$$

on g_l and g_r at $x = 0$, and the window condition

$$g_l(x-1)\overline{h_l(x-1)} + g_r(x)\overline{h_r(x)} = c, \qquad 0 \leq x \leq 1. \tag{10}$$

Then h_l and h_r satisfy the C^m-joining conditions at $x = 0$:

$$h_l(0) = h_r(0),\ h_l'(0) = h_r'(0),\ \ldots\ ,\ h_l^{(m)}(0) = h_r^{(m)}(0).$$

Proof of Lemma 1. We induct on m. When $m = 0$, the hypotheses of the lemma say $g_l(-1) = 0, g_r(1) = 0$ and $g_l(0) = g_r(0) = 1$. Thus evaluating the window condition (10) at $x = 1$ gives $c = g_l(0)\overline{h_l(0)} + g_r(1)\overline{h_r(1)} = \overline{h_l(0)}$, while evaluating at $x = 0$ gives similarly that $c = \overline{h_r(0)}$. Hence $h_l(0) = h_r(0)$, proving the induction base.

For the induction step, let $m \geq 1$ and suppose the lemma holds with $m-1$ in place of m. To prove the lemma we need only show $h_l^{(m)}(0) = h_r^{(m)}(0)$.

Differentiating the window condition (10) m times gives

$$\sum_{k=0}^{m} \binom{m}{k} \left[g_l^{(m-k)}(x-1)\overline{h_l^{(k)}(x-1)} + g_r^{(m-k)}(x)\overline{h_r^{(k)}(x)} \right] = 0, \qquad 0 \leq x \leq 1.$$

Evaluating at $x = 1$ yields

$$\sum_{k=0}^{m} \binom{m}{k} g_l^{(m-k)}(0)\overline{h_l^{(k)}(0)} = 0$$

by the boundary conditions on g_r, while evaluating at $x = 0$ gives similarly

$$\sum_{k=0}^{m} \binom{m}{k} g_r^{(m-k)}(0)\overline{h_r^{(k)}(0)} = 0.$$

By comparing these last two expressions and using the joining conditions on g_l and g_r at $x = 0$ and the induction hypothesis that $h_l^{(k)}(0) = h_r^{(k)}(0)$ for $k = 0, \ldots, m-1$, along with the normalization $g_l(0) = g_r(0) = 1$, we conclude that $h_l^{(m)}(0) = h_r^{(m)}(0)$ as desired. □

Symmetric case. The 2-knot/3-knot windows constructed above can (in principle) be chosen symmetric if in addition p is even and q is odd, as we now explain.

Assume g is symmetric about $x = 0$, so that p is even and all the odd order coefficients in g_* must vanish. Suppose q is odd, and take $h_l(x) = h_r(-x)$ to enforce symmetry of h. Let us count the conditions and parameters. There are $2(m + 1)$ boundary conditions at $x = 1$, on g and h. (The boundary conditions at $x = -1$ then follow by symmetry.) The normalization $g(0) = h(0) = 1$ imposes 2 further conditions. The window condition (4) says

$$g_*(1-x)\overline{h_r(1-x)} + g_*(x)\overline{h_r(x)} = 1, \tag{11}$$

in view of the symmetry of g and h. This window condition holds already at $x = 0$, by our normalizations, and so to ensure it holds for all x we want the derivative of the left side of (11) to vanish identically. That is, we want the polynomial $(g_*\overline{h_r})'$ of degree $p + q - 1$ to be even about $x = 1/2$. This evenness imposes a further $(p + q - 1)/2$ conditions, to annihilate the odd order derivatives of the polynomial at $x = 1/2$; here we use that $p + q - 1$ is an even number.

m	(p,q)	$g(x), x \in [-1,1]$	$h(x), x \in [0,1]$	Figure
0	$(2,3)$	$1-x^2$	$1-(5/3)x^2+(2/3)x^3$	1
1	$(4,7)$	$(1-x^2)^2$	$1+2x^2-(647/27)x^4+(1016/27)x^5$ $-(550/27)x^6+(100/27)x^7$	3
2	$(6,11)$	$(1-x^2)^3$	$1+3x^2+6x^4-(24742/81)x^6$ $+(70840/81)x^7-(83533/81)x^8$ $+(49570/81)x^9-(14671/81)x^{10}$ $+(1726/81)x^{11}$	—

Table 2: 2-knot/3-knot symmetric C^m-smooth windows from Section 4. *Notation:* $p = \deg g = 2m+2$, and $q = \deg h = 4m+3$ on $[0,1]$, with $h(x) = h(-x)$ on $[-1,0)$.

Meanwhile, we have $(p/2)+q+2$ parameters in g_* and h_r. To guarantee that the number of parameters is at least as large as the number of conditions, we require

$$(p/2)+q+2 \geq 2(m+1)+2+(p+q-1)/2,$$

which simplifies to $q \geq 4m+3$.

The C^m-smoothness of h at $x=0$ follows once more from Lemma 1.

Notice the oddness assumption on q is necessary in order for the highest degree term in the window condition (11) to vanish, since p is even.

Lastly, we observe the even polynomial g_* has $(p/2)+1$ coefficients and must satisfy the $m+1$ boundary conditions at $x=1$ as well as the normalization $g_*(0)=1$. Hence we require $(p/2)+1 \geq m+2$, or $p \geq 2m+2$.

Symmetric examples. Symmetric dual window examples of minimal degree are given in Table 2, where formulas are stated for g on $[-1,1]$ and h on $x \in [0,1]$. Since h is even about $x=0$, one obtains h on $[-1,0)$ from $h(x)=h(-x)$. For all other x-values, g and h equal 0.

Other examples could be constructed by choosing larger values of p or q. Indeed, raising the degree in any construction in this paper should allow additional constraints to be imposed on the window functions.

Other symmetries. The examples in Table 2 are all real valued and symmetric, and so they are also conjugate symmetric. No other conjugate symmetric examples seem to exist when $m=0,1,2$.

Pauli symmetric windows cannot arise, in the 2-knot/3-knot situation, because if $h(x)=g(-x)$ then h as well as g would be a polynomial on the whole interval $[-1,1]$, contradicting the non-existence of 2-knot dual windows shown in Section 3. Pauli conjugate symmetric windows cannot occur either, for the same reason.

Comparison with the canonical dual. Let us compute the canonical dual window for the example in Table 2 with $(m, p, q) = (0, 2, 3)$. The canonical dual to g is $H(x) = g(x)/\sum_{n\in\mathbb{Z}} |g(x-n)|^2$ by [2, Corollary 9.1.8]. Explicitly,

$$H(x) = \begin{cases} \frac{1-x^2}{1+2x^2+4x^3+2x^4} & \text{when } -1 \le x \le 0, \\ \frac{1-x^2}{1+2x^2-4x^3+2x^4} & \text{when } 0 \le x \le 1, \\ 0 & \text{when } |x| \ge 1, \end{cases} \tag{12}$$

as plotted in Figure 2. The non-canonical dual h in Figure 1 is very similar, and has the advantage of being piecewise polynomial, whereas the canonical dual H is piecewise rational.

5. **3-knot windows**

Now take both g and h to be 3-knot splines, that is, $g = g_l\mathbb{1}_{[-1,0)} + g_r\mathbb{1}_{[0,1]}$ and $h = h_l\mathbb{1}_{[-1,0)} + h_r\mathbb{1}_{[0,1]}$ for some polynomials g_l, g_r, h_l, h_r of degrees $p_l, p_r, q_l, q_r \ge 1$ respectively.

The extra knot in g enables us to reduce the degrees, compared to the previous section, while preserving the same smoothness or symmetry.

General case. It appears that C^m-smooth dual windows with 3 knots can generally be constructed when

$$\begin{aligned} p_l + q_l = p_r + q_r &\ge 5m + 3, \\ q_l + q_r, p_l + p_r &\ge 3m + 2, \\ p_l, p_r, q_l, q_r &\ge m + 1. \end{aligned} \tag{13}$$

Caution. These conditions are not always sufficient, and not always minimal, as we will show below. All the same, they seem a good place to start, in practice.

To see why the conditions ought to work, begin with the C^m-smoothness of g at $x = 0, \pm 1$, which imposes $3(m + 1)$ conditions. The C^m-smoothness of h at $x = \pm 1$ imposes $2(m+1)$ more. Our normalizations $g(0) = h(0) = 1$ add two further conditions. The window condition (4) says for $x \in [0, 1]$ that

$$g_l(x-1)\overline{h_l(x-1)} + g_r(x)\overline{h_r(x)} = 1. \tag{14}$$

For this polynomial equation to hold, the highest degree terms on the left side must necessarily cancel, and so the degrees $p_l + q_l$ and $p_r + q_r$ must agree. Note equation (14) already holds at $x = 0$, by our normalizations and boundary conditions. Thus (14) imposes an additional $p_l + q_l$ conditions.

The number of parameters in g and h is $p_l + p_r + q_l + q_r + 4$, and so to make sure we have an equal or greater number of parameters than conditions, we require

$$p_l + p_r + q_l + q_r + 4 \ge 5(m + 1) + 2 + (p_l + q_l),$$

which simplifies to $p_r + q_r \ge 5m + 3$.

The C^m-smoothness of h at the knot point $x = 0$ is then automatic, by Lemma 1.

Lastly, to allow g to be C^m smooth at $x = 0, \pm 1$ and not identically zero, it is sufficient to assume $p_l + p_r \geq 3m+2$, and so we include this assumption in our method. Further, for g_l to satisfy $m+1$ boundary conditions at $x = -1$ and not be identically zero one needs $p_l \geq m+1$; similarly $p_r \geq m+1$. The same arguments apply to q_l and q_r.

Examples. Two further reductions help simplify the construction of examples. First, we can assume

$$q_l + q_r \geq p_l + p_r$$

by swapping g and h, if necessary. Second, we may take

$$p_r \geq p_l$$

by changing $x \mapsto -x$ and using windows $g(-x), h(-x)$, instead of $g(x), h(x)$.

Table 3 presents examples where the windows are reasonably simple and the total degree $p_r + q_r$ is minimal. Formulas are stated there for $x \in [-1, 1]$. For all other x-values, g and h equal 0.

Table 3 omits some examples in which g and h seem too complicated to be interesting, such as when $m = 1$ and $p_l = q_l = p_r = q_r = 4$.

Remark. Caution is required when applying the degree conditions (13). First, they are not always sufficient for existence of an example. For instance, dual windows do not exist when $m = 2, p_l = 3, p_r = 10, q_l = 10, q_r = 3$. More precisely, when I use Mathematica to construct dual windows, it says that no such solution exists. Second, the conditions (13) do not always give the correct minimal degrees. For example, when $m = 1$ they say $p_l + q_l \geq 5m + 3 = 8$. But $p_l + q_l = 7$ in the example in Table 3 with $m = 1, p_l = 2, p_r = 5, q_l = 5, q_r = 2$: the highest order coefficients in h_l and h_r just "happen" to be zero here. We explain this miracle later in the section, in terms of Pauli conjugate symmetry.

Third, our degree conditions can lead to multiple window pairs, such as when $m = 1, p_l = 3, p_r = 3, q_l = 5, q_r = 5$. These examples are omitted from Table 3 because the coefficients in the windows are messy (and complex valued).

Symmetric case. Examples of symmetric windows are in Table 4. Formulas are given there for $x \in [0, 1]$, with the understanding that the windows are even and supported in $[-1, 1]$.

These symmetric 3-knot windows fit the pattern that $p + q$ is odd and

$$p + q \geq \begin{cases} 5m+3 & \text{if } m \text{ is even,} \\ 5m+4 & \text{if } m \text{ is odd,} \end{cases}$$

$$q > p \geq \begin{cases} \frac{3}{2}m+1 & \text{if } m \text{ is even,} \\ \frac{3}{2}(m+1) & \text{if } m \text{ is odd.} \end{cases}$$

Let us investigate why symmetric 3-knot examples should satisfy these relations.

m	$\begin{cases} p_l, q_l \\ p_r, q_r \end{cases}$	$\begin{cases} g(x), & -1 \le x \le 0 \\ g(x), & 0 \le x \le 1 \end{cases}$	$\begin{cases} h(x), & -1 \le x \le 0 \\ h(x), & 0 \le x \le 1 \end{cases}$	Figure
0	$\begin{cases} 1, 2 \\ 1, 2 \end{cases}$	$\begin{cases} 1 + x \\ 1 - x \end{cases}$	$\begin{cases} 1 - x - 2x^2 \\ 1 + x - 2x^2 \end{cases}$	4
0	$\begin{cases} 1, 2 \\ 2, 1 \end{cases}$	$\begin{cases} 1 + x \\ 1 + x - 2x^2 \end{cases}$	$\begin{cases} 1 - x - 2x^2 \\ 1 - x \end{cases}$	5
1	$\begin{cases} 2, 6 \\ 3, 5 \end{cases}$	$\begin{cases} 1 + 2x + x^2 \\ 1 + 2x - 7x^2 + 4x^3 \end{cases}$	$\begin{cases} 1 - 2x + 3x^2 - 4x^3 + 50x^4 + 140x^5 + 80x^6 \\ 1 - 2x + 11x^2 - 40x^3 + 50x^4 - 20x^5 \end{cases}$	—
1	$\begin{cases} 2, 6 \\ 4, 4 \end{cases}$	$\begin{cases} 1 + 2x + x^2 \\ 1 + 2x - 2x^2 - 6x^3 + 5x^4 \end{cases}$	$\begin{cases} 1 - 2x + 3x^2 - 4x^3 - 55x^4 - 70x^5 - 25x^6 \\ 1 - 2x + 6x^2 - 10x^3 + 5x^4 \end{cases}$	—
1	$\begin{cases} 2, 5 \\ 5, 2 \end{cases}$	$\begin{cases} 1 + 2x + x^2 \\ 1 + 2x + 3x^2 + 4x^3 - 30x^4 + 20x^5 \end{cases}$	$\begin{cases} 1 - 2x + 3x^2 - 4x^3 - 30x^4 - 20x^5 \\ 1 - 2x + x^2 \end{cases}$	6
2	$\begin{cases} 3, 10 \\ 5, 8 \end{cases}$	$\begin{cases} 1 + 3x + 3x^2 + x^3 \\ 1 + 3x + 3x^2 - 37x^3 + 48x^4 - 18x^5 \end{cases}$	$\begin{cases} 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 - 9722x^6 - 37566x^7 - 55332x^8 - 36504x^9 - 9072x^{10} \\ 1 - 3x + 6x^2 + 28x^3 - 261x^4 + 1083x^5 - 2018x^6 + 1668x^7 - 504x^8 \end{cases}$	7
2	$\begin{cases} 3, 10 \\ 6, 7 \end{cases}$	$\begin{cases} 1 + 3x + 3x^2 + x^3 \\ 1 + 3x + 3x^2 - 13x^3 - 24x^4 + 54x^5 - 24x^6 \end{cases}$	$\begin{cases} 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 - 266x^6 + 162x^7 + 1116x^8 + 1032x^9 + 288x^{10} \\ 1 - 3x + 6x^2 + 4x^3 - 45x^4 + 75x^5 - 50x^6 + 12x^7 \end{cases}$	—
2	$\begin{cases} 3, 10 \\ 7, 6 \end{cases}$	$\begin{cases} 1 + 3x + 3x^2 + x^3 \\ 1 + 3x + 3x^2 + (-41x^3 - 108x^4 + 90x^5 + 108x^6 - 84x^7)/5 \end{cases}$	$\begin{cases} 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + (-22598x^6 - 54858x^7 - 53136x^8 - 23340x^9 - 3864x^{10})/25 \\ 1 - 3x + 6x^2 + (-4x^3 - 93x^4 + 123x^5 - 46x^6)/5 \end{cases}$	—
2	$\begin{cases} 3, 10 \\ 8, 5 \end{cases}$	$\begin{cases} 1 + 3x + 3x^2 + x^3 \\ 1 + 3x + 3x^2 + x^3 + 6x^4 + 18x^5 - 236x^6 + 342x^7 - 138x^8 \end{cases}$	$\begin{cases} 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 - 1064x^6 - 2934x^7 - 3384x^8 - 1872x^9 - 414x^{10} \\ 1 - 3x + 6x^2 - 10x^3 + 9x^4 - 3x^5 \end{cases}$	8

$$\textit{Notation:} \qquad \deg g = \begin{cases} p_l & \text{on } [-1,0] \\ p_r & \text{on } [0,1] \end{cases} \qquad \deg h = \begin{cases} q_l & \text{on } [-1,0] \\ q_r & \text{on } [0,1] \end{cases}$$

TABLE 3. 3-knot C^m-smooth windows from Section 5. Here $p_l + q_l = p_r + q_r = 5m+3$ (with one exception), $q_l + q_r \ge p_l + p_r \ge 3m+2, p_r \ge p_l \ge m+1$ and $q_l, q_r \ge m+1$.

m	(p,q)	$g(x), x \in [0,1]$	$h(x), x \in [0,1]$	Figure
0	$(1,2)$	$1-x$	$1+x-2x^2$	4
1	$(3,6)$	$1-3x^2+2x^3$	$1+3x^2-2x^3-18x^4+24x^5-8x^6$	9
1	$(4,5)$	$1-(\sqrt{41}-2)x^2+(2\sqrt{41}-8)x^3$	$1+(\sqrt{41}-2)x^2-(2\sqrt{41}-8)x^3$	—
		$-(\sqrt{41}-5)x^4$	$-(15-\sqrt{41})x^4+8x^5$	
1	$(4,5)$	$1+(\sqrt{41}+2)x^2-(2\sqrt{41}+8)x^3$	$1-(\sqrt{41}+2)x^2+(2\sqrt{41}+8)x^3$	—
		$+(\sqrt{41}+5)x^4$	$-(15+\sqrt{41})x^4+8x^5$	
2	$(4,9)$	$1-6x^2+8x^3-3x^4$	$1+6x^2-8x^3+39x^4-96x^5$	10
			$-(244/5)x^6+(1452/5)x^7$	
			$-(1242/5)x^8+(324/5)x^9$	
2	$(5,8)$	multiple solutions	multiple solutions	11

Table 4: 3-knot symmetric C^m-smooth windows from Section 5. *Notation:* $p = \deg g$ and $q = \deg h$ on $[0,1]$, with $g(x) = g(-x)$ and $h(x) = h(-x)$ on $[-1,0)$. If m is even then $p+q = 5m+3$ and $q > p \geq \frac{3}{2}m+1$, while if m is odd then $p+q = 5m+4$ and $q > p \geq \frac{3}{2}(m+1)$.

For symmetry we insist $g_l(x) = g_r(-x)$ and $h_l(x) = h_r(-x)$. Suppose $p+q$ is odd, and that $q > p$ (by swapping g and h, if necessary). Let us count the conditions and parameters. There are $2(m+1)$ boundary conditions at $x = 1$, for g and h. The normalization $g(0) = h(0) = 1$ imposes 2 further conditions. To ensure g is C^m-smooth at $x = 0$ we require the odd order derivatives of g_r to equal zero, at $x = 0$, which imposes $\lceil m/2 \rceil$ conditions. The window condition (4) says

$$g_r(1-x)\overline{h_r(1-x)} + g_r(x)\overline{h_r(x)} = 1, \tag{15}$$

which imposes $(p+q-1)/2$ conditions by arguing like we did for (11).

Meanwhile, we have $p+q+2$ parameters in g and h, and so we require

$$p+q+2 \geq 2(m+1)+2+\lceil m/2 \rceil + (p+q-1)/2,$$

which simplifies to $p+q \geq 5m+3$ if m is even, and $p+q \geq 5m+4$ if m is odd.

The C^m-smoothness of h at $x = 0$ follows from Lemma 1.

The oddness assumption on $p+q$ is forced upon us in order for the highest degree term in the window condition to vanish.

Lastly, observe g_r has $p+1$ coefficients, and must satisfy the $m+1$ boundary conditions at $x = 1$ as well as the normalization $g_r(0) = 1$. The C^m-smoothness at $x = 0$ imposes another $\lceil m/2 \rceil$ conditions, as explained above. Hence $p+1 \geq m+1+1+\lceil m/2 \rceil$, so that $p \geq (3/2)m+1$ if m is even, and $p \geq (3/2)(m+1)$ if m is odd.

m	(p,q)	$g(x), x \in [0,1]$	$h(x), x \in [0,1]$	Figure
0	$(1,2)$	$1-x$	$1+x-2x^2$	5 $(g \leftrightarrow h)$
1	$(2,5)$	$1-2x+x^2$	$1+2x+3x^2+4x^3-30x^4+20x^5$	6 $(g \leftrightarrow h)$
1	$(3,4)$	complicated coefficients	complicated coefficients	12

Table 5: 3-knot Pauli conjugate symmetric C^m-smooth windows from Section 5. *Notation:* $p = \deg g$ and $q = \deg h$ on $[0,1]$, with $g(x) = \overline{h(-x)}$ and $h(x) = \overline{g(-x)}$ on $[-1,0)$. Here $q > p \geq m+1$, and $p+q = 5m+3$ if m is even while $p+q = 5m+2$ if m is odd. In the first two examples, g and h should be swapped in the corresponding figure.

Conjugate symmetric case. The examples in Table 4 are real valued and symmetric, and hence are conjugate symmetric too. Complex valued conjugate symmetric examples of lower degree do exist when m is odd (for example, $(m,p,q) = (1,3,5)$), but these examples seem unlikely to be useful, since they have complicated coefficients involving roots of quartic equations.

Pauli conjugate symmetric case. Surprisingly, Pauli conjugate symmetry yields examples of lower degree than predicted by the general case.

Table 5 gives some Pauli conjugate symmetric duals g and h. The formulas are given for $x \in [0,1]$, with the understanding that $g(x) = \overline{h(-x)}$ and $h(x) = \overline{g(-x)}$ when $x \in [-1,0)$. For all other x-values, g and h equal 0.

The general rule is that $p+q$ is odd and

$$\begin{aligned} p+q &\geq \begin{cases} 5m+3 & \text{if } m \text{ is even,} \\ 5m+2 & \text{if } m \text{ is odd,} \end{cases} \\ q &> p \geq m+1, \end{aligned}$$

as we justify below. Note the reduction of 1 degree when m is odd: the general case (13) has $p+q \geq 5m+3$ whereas here $p+q \geq 5m+2$.

Assume g and h are Pauli conjugate symmetric, so that $g_l(x) = \overline{h_r(-x)}$ and $h_l(x) = \overline{g_r(-x)}$. Suppose $p+q$ is odd, and that $q > p$ (by swapping g and h, if necessary). Once more we count the conditions and parameters. There are $2(m+1)$ boundary conditions at $x=1$, for g and h. The normalization $g_r(0) = h_r(0) = 1$ imposes 2 further conditions. The window condition (4) again reduces to (15), which imposes $(p+q-1)/2$ conditions. We further impose the $\lfloor m/2 \rfloor$ conditions $g_r^{(k)}(0) = \overline{h_r^{(k)}(0)}$ for k even, $1 \leq k \leq m$, which imply $g_r^{(k)}(0) = g_l^{(k)}(0)$ for all $0 \leq k \leq m$ by Lemma 2 below, so that g is C^m-smooth at $x=0$.

Since we have $p+q+2$ parameters in g and h, we require

$$p+q+2 \geq 2(m+1) + 2 + (p+q-1)/2 + \lfloor m/2 \rfloor,$$

which simplifies to $p+q \geq 5m+3$ if m is even, and $p+q \geq 5m+2$ if m is odd.

The C^m-smoothness of $h(x) = \overline{g(-x)}$ follows immediately from smoothness of g. Note the oddness of $p+q$ is necessary for the highest degree term in the window condition (15) to vanish.

Lastly, $p \geq m+1$ is needed because g_r must satisfy the $m+1$ boundary conditions at $x = 1$ as well as the normalization $g_r(0) = 1$.

Regarding C^2-smooth windows, which are not covered by Table 5, I have found a complicated example with $m = 2$ and $(p, q) = (5, 8)$, but no examples with $(p, q) = (3, 10), (4, 9), (6, 7)$, even though each of these choices would satisfy the requirements $p + q \geq 5m + 3 = 13$ and $q > p \geq m + 1 = 3$. These missing examples serve as a reminder that the criteria in this paper are sometimes not sufficient.

Some complex valued solutions have been omitted from Table 5, too, because their coefficients are too complicated.

We must still prove:

Lemma 2. *Let m be a nonnegative integer and $c \in \mathbb{C}$. Assume the smooth functions g_l, g_r, h_l, h_r satisfy the boundary conditions*

$$g_l(-1) = g_l'(-1) = \cdots = g_l^{(m)}(-1) = 0$$

at $x = -1$, as well as the normalization $g_r(0) = 1$ and the window condition

$$g_l(x-1)\overline{h_l(x-1)} + g_r(x)\overline{h_r(x)} = c, \qquad 0 \leq x \leq 1. \tag{16}$$

Assume $g_r^{(k)}(0) = \overline{h_r^{(k)}(0)}$ for all even k with $0 \leq k \leq m$.

Then $g_r^{(k)}(0) = (-1)^k \overline{h_r^{(k)}(0)}$ for all $0 \leq k \leq m$.

Proof of Lemma 2. When $m = 0$, there is nothing to prove. Let $m \geq 1$ and suppose the lemma holds with $m-1$ in place of m. To prove the lemma we must show $g_r^{(m)}(0) = (-1)^m \overline{h_r^{(m)}(0)}$.

If m is even, then the desired conclusion is already one of the hypotheses.

So suppose m is odd. Differentiate the window condition (16) m times and put $x = 0$. Then using the boundary conditions on g_l at $x = -1$ gives

$$\sum_{k=0}^{m} \binom{m}{k} g_r^{(m-k)}(0)\overline{h_r^{(k)}(0)} = 0.$$

The terms with $k \leq m-1$ can be handled by the induction hypothesis, yielding

$$g_r(0)\overline{h_r^{(m)}(0)} + \sum_{k=0}^{m-1} \binom{m}{k} g_r^{(m-k)}(0)(-1)^k g_r^{(k)}(0) = 0.$$

Rearranging, we find

$$\begin{aligned} g_r(0)\left[\overline{h_r^{(m)}(0)} - (-1)^m g_r^{(m)}(0)\right] &= -\sum_{k=0}^{m} \binom{m}{k} g_r^{(m-k)}(0)(-1)^k g_r^{(k)}(0) \\ &= -\left.\frac{d^m}{dx^m}\, g_r(x)g_r(-x)\right|_{x=0} \\ &= 0 \end{aligned}$$

because $g_r(x)g_r(-x)$ is an even function of x, while m is odd. Now the normalization $g_r(0) = 1$ implies $\overline{h_r^{(m)}(0)} - (-1)^m g_r^{(m)}(0) = 0$, as we wanted. □

Pauli symmetric case. The examples in Table 5 are Pauli symmetric too, since they are real valued and Pauli conjugate symmetric. I would not expect to find complex valued Pauli symmetric examples with any lower degree.

6. **3-knot flat windows**

Next we seek *flat* spline windows, meaning windows whose derivatives vanish to order m at all knot points, in particular at $x = 0$:

$$\begin{aligned} g_l'(0) = g_r'(0) = \cdots = g_l^{(m)}(0) = g_r^{(m)}(0) = 0, \\ h_l'(0) = h_r'(0) = \cdots = h_l^{(m)}(0) = h_r^{(m)}(0) = 0. \end{aligned}$$

These flat windows are needed in the next section, for treating translation parameters in the range $\frac{1}{2} < b < 1$.

General case. We claim that C^m-smooth flat dual windows can generally be constructed when

$$\begin{aligned} p_l + q_l = p_r + q_r \geq 6m + 3, \\ p_l, p_r, q_l, q_r \geq 2m + 1. \end{aligned}$$

The justification is similar to the non-flat general case in Section 5, except here we impose an additional m flatness conditions on g at $x = 0$. The flatness of h at $x = 0$ follows from flatness of g, by the proof of Lemma 1.

Symmetric case. Symmetric flat windows should exist if in addition $p_l = p_r, q_l = q_r$ and $p_r + q_r$ is odd, as one sees by modifying the non-flat symmetric case in Section 5 (changing $\lceil m/2 \rceil$ to m, because now we require all derivatives of g_r of order $\leq m$ to equal zero at $x = 0$, not just the odd order derivatives).

Table 6 shows examples of symmetric flat windows.

Other symmetries. The other flat window types (conjugate symmetric, Pauli symmetric and Pauli conjugate symmetric) all generate the same conditions as in the symmetric case.

m	(p,q)	$g(x), x \in [0,1]$	$h(x), x \in [0,1]$	Figure
0	$(1,2)$	$1-x$	$1+x-2x^2$	4
1	$(3,6)$	$1-3x^2+2x^3$	$1+3x^2-2x^3-18x^4+24x^5-8x^6$	9
1	$(4,5)$	$1-(\sqrt{41}-2)x^2+(2\sqrt{41}-8)x^3$ $-(\sqrt{41}-5)x^4$	$1+(\sqrt{41}-2)x^2-(2\sqrt{41}-8)x^3$ $-(15-\sqrt{41})x^4+8x^5$	—
1	$(4,5)$	$1+(\sqrt{41}+2)x^2-(2\sqrt{41}+8)x^3$ $+(\sqrt{41}+5)x^4$	$1-(\sqrt{41}+2)x^2+(2\sqrt{41}+8)x^3$ $-(15+\sqrt{41})x^4+8x^5$	—
2	$(5,10)$	$1-10x^3+15x^4-6x^5$	$1+10x^3-15x^4+6x^5-200x^6$ $+600x^7-690x^8+360x^9-72x^{10}$	—

Table 6: 3-knot symmetric flat C^m-smooth windows from Section 6. *Notation:* $p = \deg g$ and $q = \deg h$ on $[0,1]$, with $g(-x) = g(x)$ and $h(-x) = h(x)$ on $[-1,0)$. Here $p+q = 6m+3$ and $q > p \geq 2m+1$.

7. Windows when $1/2 < ab < 1$

Dual windows for $1/2 < b < 1$ will be constructed by "horizontally compressing" the two halves of a 3-knot flat window and then inserting a constant graph between them. Continuous and C^1 examples are shown in Figures 13 and 14, respectively, for $b = 3/4$.

The construction can be expressed concisely in terms of a time-rescaling function

$$N(x) = \begin{cases} \big(x+1-\frac{1}{2b}\big)/\big(\frac{1}{b}-1\big), & x \leq \frac{1}{2b}-1, \\ 0, & |x| \leq 1-\frac{1}{2b}, \\ \big(x-(1-\frac{1}{2b})\big)/\big(\frac{1}{b}-1\big), & x \geq 1-\frac{1}{2b}. \end{cases}$$

Clearly N is continuous and odd, and is linear and increasing except on the interval $[\frac{1}{2b}-1, 1-\frac{1}{2b}]$, where N is constantly zero. See Figure 15. In the extreme case $b = 1/2$, $N(x) = x$ is the identity function.

The windows g and h will be rescaled to give new windows G and H:

Proposition 3. *Assume g and h are continuous and supported on $[-1,1]$, and*

$$g(x-1)\overline{h(x-1)} + g(x)\overline{h(x)} = 1, \qquad x \in [0,1]. \tag{17}$$

If $\frac{1}{2} < b < 1$ then

$$G = g \circ N \qquad \text{and} \qquad H = h \circ N$$

are continuous dual windows supported on $[-\frac{1}{2b}, \frac{1}{2b}]$.

Proof of Proposition 3. First, G is continuous, and has compact support because if $|x| > \frac{1}{2b}$ then $|N(x)| > 1$ and so $G(x) = g(N(x)) = 0$. The same holds for H. Hence G and H are dual windows provided the window conditions (2) and (3) hold.

Condition (3) is immediate, since G and H are supported in the interval $[-\frac{1}{2b}, \frac{1}{2b}]$ of length $1/b$.

For condition (2), it suffices to consider x in some interval of length 1, by periodicity. Specifically, it is enough to show

$$G(x-1)\overline{H(x-1)} + G(x)\overline{H(x)} = 1 \tag{18}$$

for x in the interval $[\frac{1}{2b}-1, \frac{1}{2b}]$, because $\frac{1}{2b}$ is the right endpoint of the support of G and that support has length $\frac{1}{b} < 2$ (so that $G(x-n) = 0$ whenever $n \neq 0, 1$).

Formula (18) holds when $x \in [1-\frac{1}{2b}, \frac{1}{2b}]$, because the easily verified identity $N(x-1) = N(x) - 1 \in [-1, 0]$ implies

$$\begin{aligned}
G(x-1)&\overline{H(x-1)} + G(x)\overline{H(x)} \\
&= g\big(N(x)-1\big)\overline{h\big(N(x)-1\big)} + g\big(N(x)\big)\overline{h\big(N(x)\big)} \\
&= 1
\end{aligned}$$

by the assumption (17) on g and h.

In the remaining range $x \in [\frac{1}{2b}-1, 1-\frac{1}{2b})$, we have $N(x) = 0$ and so $G(x)\overline{H(x)} = g(0)\overline{h(0)} = 1$ by (17), and also $N(x-1) < -1$ because $x - 1 < -\frac{1}{2b}$, so that $G(x-1) = 0$. Now (18) follows. □

Smoothness of the new windows is given by the next lemma.

Lemma 4. *Let m be a positive integer. If g and h in Proposition 3 are C^m-smooth and flat, then G and H are C^m-smooth.*

Recall g and h are called *flat* (to order m) if $g^{(k)}$ and $h^{(k)}$ vanish at $x = 0, \pm 1$, for $k = 1, \dots, m$.

Proof of Lemma 4. The only points where $G = g \circ N$ might not be C^m-smooth are $x = \pm(1 - \frac{1}{2b})$, where N has corner points.

Clearly G is constant immediately to the left of $x = 1 - \frac{1}{2b}$, since N is constant there; hence the derivatives of G from the left are all zero at $x = 1 - \frac{1}{2b}$. The derivatives from the right are zero too, up to order m, because on the right N is linear and $g^{(k)}(0) = 0$ for $k = 1, \dots, m$ by the flatness assumption. Hence G is C^m-smooth at $x = 1 - \frac{1}{2b}$. Argue similarly at the other corner point $x = \frac{1}{2b} - 1$. The proof is the same for H. □

Examples. Combining Proposition 3 and Lemma 4, we see that the C^m-smooth symmetric flat 3-knot windows g and h in Table 6 generate C^m-smooth symmetric 4-knot windows G and H, when $\frac{1}{2} < b < 1$. Plots of G and H for the choices $(m, p, q) = (0, 1, 2)$ and $(1, 3, 6)$ are shown in Figures 13 and 14, respectively, for $b = 3/4$.

Remark on $b = 1$. The indicator functions $g = h = \mathbb{1}_{[-1/2,1/2)}$ are well known dual windows when $b = 1$. Indicator functions are typically unsuitable for applications, though, because jumps in the time domain lead to the Gibbs phenomenon and poor concentration in the frequency domain. That is why we consider continuous windows, in this paper.

8. **Wider windows when** $ab \leq 1/4$

Our windows become more concentrated in the frequency domain when they are dilated in the time domain. We begin by showing how the window condition and modulation parameter are affected by dilation.

Proposition 5. *Assume* $g, h \in L^2(\mathbb{R})$ *satisfy Janssen's window conditions* (2) *and* (3). *Let* $B \in \mathbb{N}$.

Then the functions $g_B(x) = g(x/B)/\sqrt{B}$ *and* $h_B(x) = h(x/B)/\sqrt{B}$ *satisfy* (2) *and* (3) *with* b *replaced by* b/B. *That is,*

$$\sum_{n\in\mathbb{Z}} g_B(x-n)\overline{h_B(x-n)} = 1, \qquad x \in \mathbb{R},$$

$$\sum_{n\in\mathbb{Z}} g_B(x-n-kB/b)\overline{h_B(x-n)} = 0, \qquad x \in \mathbb{R}, \quad k \neq 0.$$

Proof of Proposition 5. To verify the second equation, we substitute in the definitions of g_B and h_B and then write $n = m + n'B$ where $m = 1, \ldots, B$ and $n' \in \mathbb{Z}$, obtaining

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} g_B(x-n-kB/b)\overline{h_B(x-n)} \\
&= \frac{1}{B}\sum_{m=1}^{B}\sum_{n'\in\mathbb{Z}} g\big((x-ma)/B - n'a - k/b\big)\overline{h\big((x-ma)/B - n'a\big)},
\end{aligned}$$

which equals 0 by applying (3) to the sum over n'.

The first equation follows similarly, by taking $k = 0$ and applying (2). □

Example. Suppose g and h are spline dual windows for $b = 1/2$ that are supported in the interval $[-1, 1]$, such as any of the examples found in Sections 4–6. Then g_B and h_B are dual windows for $b = 1/2B$ that are supported in the wider interval $[-B, B]$, by Proposition 5.

Specifically, consider the C^1-smooth 3-knot symmetric example in Table 4 with $(m, p, q) = (1, 3, 6)$, shown in Figure 9. Dilating with $B = 2$ yields C^1-smooth symmetric dual windows for $b = 1/4$ that are supported on $[-2, 2]$, with

$$\begin{aligned}
g(x) &= \frac{1}{\sqrt{2}}\left(1 - \frac{3}{4}x^2 + \frac{1}{4}x^3\right), \\
h(x) &= \frac{1}{\sqrt{2}}\left(1 + \frac{3}{4}x^2 - \frac{1}{4}x^3 - \frac{9}{8}x^4 + \frac{3}{4}x^5 - \frac{1}{8}x^6\right),
\end{aligned}$$

for $x \in [0, 2]$. These windows are shown in Figure 16.

9. **Windows of unequal smoothness**

The analysis window need not always be as smooth as the synthesis window. Suppose we want the analysis window to be C^m-smooth and the synthesis window to be C^{m-1}-smooth. Then we need only drop the highest order boundary conditions in our earlier construction.

Example. Consider the 3-knot symmetric case, with $0 < b \leq 1/2$. Dropping the boundary condition $h^{(m)}(1) = 0$ leads to the requirements that $p + q$ be odd with

$$p + q \geq \begin{cases} 5m + 1 & \text{if } m \text{ is even,} \\ 5m + 2 & \text{if } m \text{ is odd,} \end{cases}$$

$$p \geq \begin{cases} \frac{3}{2}m + 1 & \text{if } m \text{ is even,} \\ \frac{3}{2}(m + 1) & \text{if } m \text{ is odd,} \end{cases}$$

$$q \geq \begin{cases} \frac{3}{2}m & \text{if } m \text{ is even,} \\ \frac{1}{2}(3m + 1) & \text{if } m \text{ is odd.} \end{cases}$$

When $m = 1$, these requirements are satisfied by $(p, q) = (3, 4), (4, 3)$ and $(5, 2)$. The last of these three choices yields the most pleasing windows (see Figure 17), with

$$g(x) = 1 + x^2 - \frac{26}{3}x^3 + \frac{28}{3}x^4 - \frac{8}{3}x^5 \qquad \text{and} \qquad h(x) = 1 - x^2$$

for $x \in [0, 1]$, and $g(x) = g(-x)$ and $h(x) = h(-x)$ for $x \in [-1, 0)$ (symmetry), and $g, h = 0$ outside $[-1, 1]$. Obviously g is C^1-smooth and h is continuous.

10. **Dual window constructions in the literature**

A substantial class of spline windows has been constructed by Christensen [3]. The main result of [3] says that if g is bounded, real valued and supported in an interval of length $L \in \mathbb{N}$, and if g has constant periodization $\sum_{n \in \mathbb{Z}} g(x - n) = 1$, then a dual window is given by

$$h(x) = g(x) + 2 \sum_{n=1}^{L-1} g(x + n)$$

whenever $0 < b \leq 1/(2L - 1)$. Christensen chooses g to be a B-spline, and hence obtains some pleasingly explicit dual window pairs. Unfortunately, the upper bound on b gets smaller as the support length L increases, and L necessarily increases as the order of smoothness of the window g (the B-spline) is increased. Thus smoother windows can be obtained only at the cost of a smaller b-value.

In contrast, the constructions in this paper can handle any order of smoothness, for any $0 < b \leq 1/2$ (and even $\frac{1}{2} < b < 1$, in Section 7), with windows that are supported in the fixed interval $[-1,1]$. On the other hand, we do not *prove* that the algorithm works in general.

The recent work of Christensen and Kim [5] is also similar in spirit to this paper. Their Corollary 2.7 constructs 2-knot/$(L+1)$-knot spline duals on a support interval of width L. The 2-knot spline g has degree $L-1$, and $b \leq 1/L$. The width again increases with the degree, and hence with the smoothness. One can compare with Section 4, where we construct 2-knot/3-knot windows on an interval of fixed width 2 with no restrictions on b or on the smoothness.

Other developments include Lemvig's analogous work on wavelets [10], and Christensen and Kim's construction of Gabor systems in higher dimensions [4].

A further construction in the literature is the square-root method of Daubechies *et al.* [6, §IIE]. Assume $s(x)$ is nonnegative, bounded and supported on an interval of length L, and that s has constant periodization $\sum_{n\in\mathbb{Z}} s(x-n) = 1$. Then defining $g = h = \sqrt{s}$ gives a pair of dual windows satisfying (2) and (3), provided $0 < b \leq 1/L$. (These frames are tight, since $g = h$.) More generally, one can take $g = s^p$ and $h = s^{1-p}$ whenever $0 < p < 1$. Two aspects of this root method deserve comment. First, spline examples need only addition and multiplication in their definition, and so are more elementary than examples using roots. Second, a root reduces smoothness at the edge of the support. To compensate for this loss of smoothness, s must be constructed to have a higher order of vanishing at the endpoints, which increases its complexity.

Lastly we recall that the canonical dual is $g(x)/\sum_{n\in\mathbb{Z}} |g(x-n)|^2$, provided g is supported in an interval of length $1/b$ and the denominator is bounded above and is bounded below away from zero [2, Corollary 9.1.8]. This canonical dual is clearly more complicated than the original window g. For example, if the original window is piecewise polynomial then the canonical dual is piecewise rational, like in example (12). For that reason, this paper seeks non-canonical dual windows that have the same spline form as the original window.

References

[1] O. Christensen. An Introduction to Frames and Riesz Bases. Birkhäuser, Boston, 2003.

[2] O. Christensen. Frames and Bases: An Introductory Course. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, to appear, 2008.

[3] O. Christensen. *Pairs of dual Gabor frame generators with compact support and desired frequency localization.* Appl. Comput. Harmon. Anal. 20:403–410, 2006.

[4] O. Christensen and R. Y. Kim. *Pairs of explicitly given dual Gabor frames in* $L^2(\mathbb{R}^d)$. J. Fourier Anal. Appl. 12:243–255, 2006.

[5] O. Christensen and R. Y. Kim. *On dual Gabor frame pairs generated by polynomials.* Preprint, 2007.

[6] I. Daubechies, A. Grossmann and Y. Meyer. *Painless nonorthogonal expansions.* J. Math. Phys. 27:1271–1283, 1986.

[7] H. G. Feichtinger and T. Strohmer, editors. Gabor Analysis and Algorithms. Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 1998.

[8] K. Gröchenig. Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Boston, MA, 2001.

[9] A. J. E. M. Janssen. *The duality condition for Weyl–Heisenberg frames.* In: Gabor Analysis and Algorithms, pp. 33–84, Appl. Numer. Harmon. Anal. Editors: H. G. Feichtinger and T. Strohmer. Birkhäuser Boston, Boston, MA, 1998.

[10] J. Lemvig. *Constructing pairs of dual bandlimited framelets with desired time localization.* Adv. Comput. Math., appeared online, 2008.

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.
E-mail address: Laugesen@illinois.edu

List of Figures

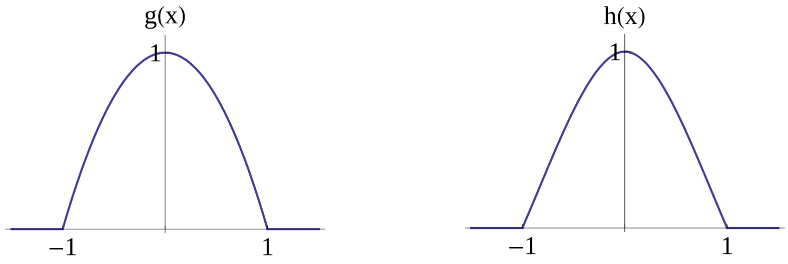

FIGURE 1. 2-knot/3-knot symmetric: $(m, p, q) = (0, 2, 3)$ [Table 2]

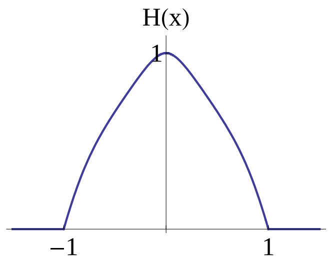

FIGURE 2. Canonical dual H for the 2-knot symmetric window g shown in Figure 1 [see formula (12)]

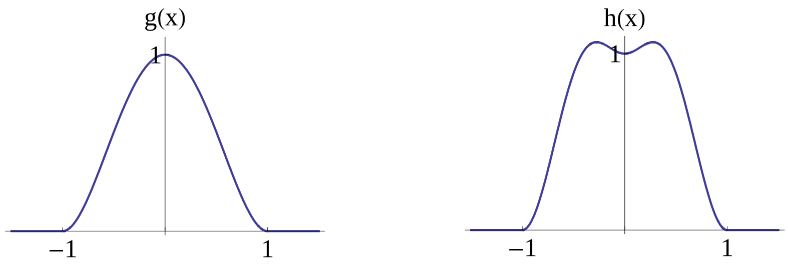

FIGURE 3. 2-knot/3-knot symmetric: $(m, p, q) = (1, 4, 7)$ [Table 2]

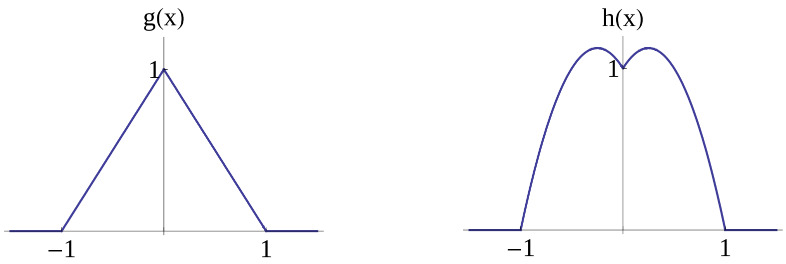

Figure 4. 3-knot: $(m, p_l, p_r, q_l, q_r) = (0, 1, 1, 2, 2)$ [Table 3]. Same as 3-knot symmetric: $(m, p, q) = (0, 1, 2)$ [Table 4]

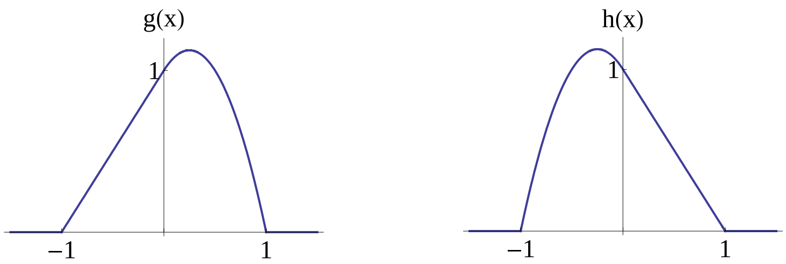

Figure 5. 3-knot: $(m, p_l, p_r, q_l, q_r) = (0, 1, 2, 2, 1)$ [Table 3]

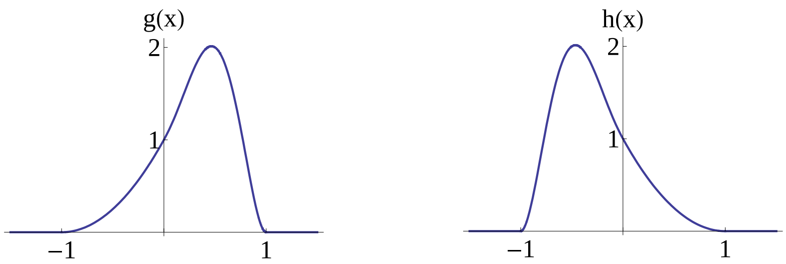

Figure 6. 3-knot: $(m, p_l, p_r, q_l, q_r) = (1, 2, 5, 5, 2)$ [Table 3]

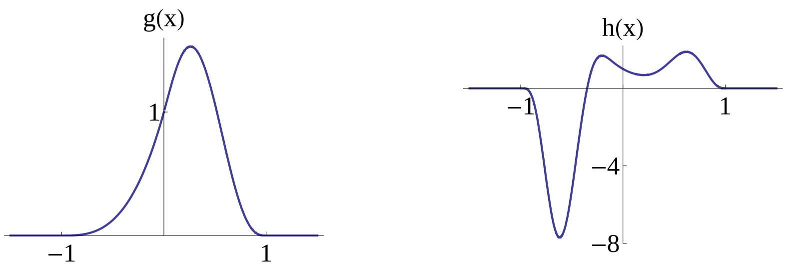

Figure 7. 3-knot: $(m, p_l, p_r, q_l, q_r) = (2, 3, 5, 10, 8)$ [Table 3]

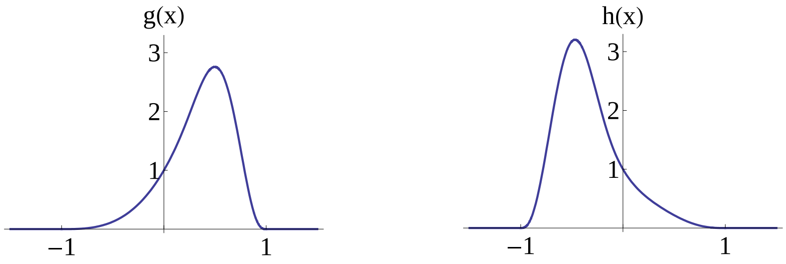

FIGURE 8. 3-knot: $(m, p_l, p_r, q_l, q_r) = (2, 3, 8, 10, 5)$ [Table 3]

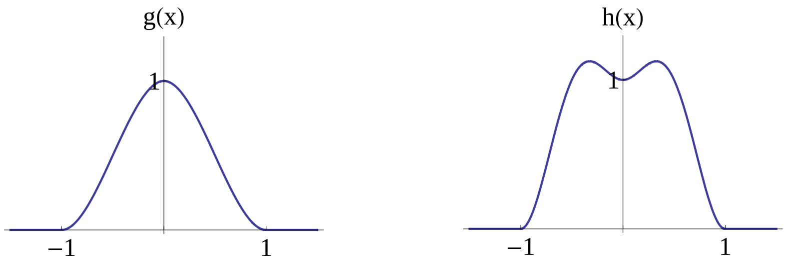

FIGURE 9. 3-knot symmetric: $(m, p, q) = (1, 3, 6)$ [Table 4]

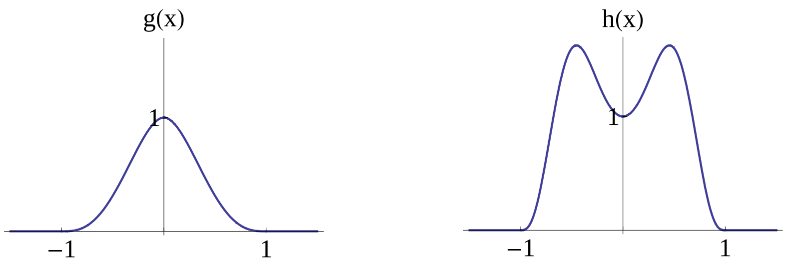

FIGURE 10. 3-knot symmetric: $(m, p, q) = (2, 4, 9)$ [Table 4]

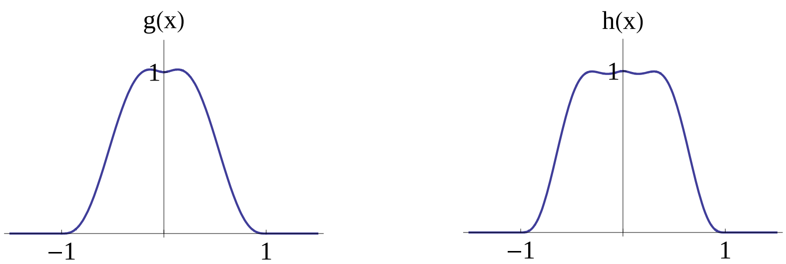

FIGURE 11. 3-knot symmetric: $(m, p, q) = (2, 5, 8)$ [Table 4]. Two other window pairs exist also.

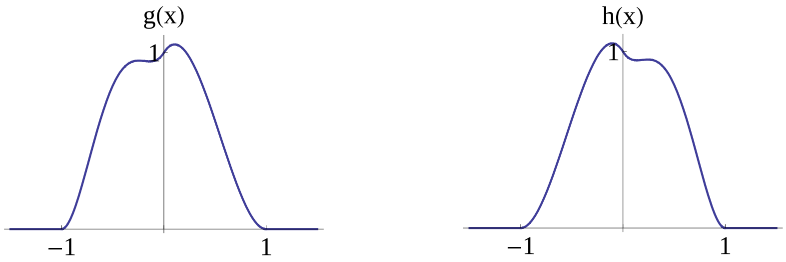

FIGURE 12. 3-knot Pauli (conjugate) symmetric: $(m, p, q) = (1, 3, 4)$ [Table 5]

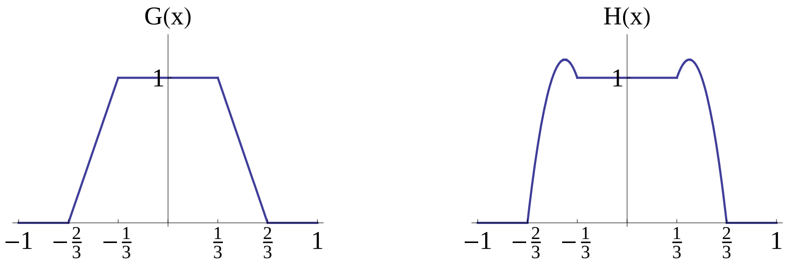

FIGURE 13. 4-knot symmetric with $b = 3/4$: $(m, p, q) = (0, 1, 2)$ [Section 7]

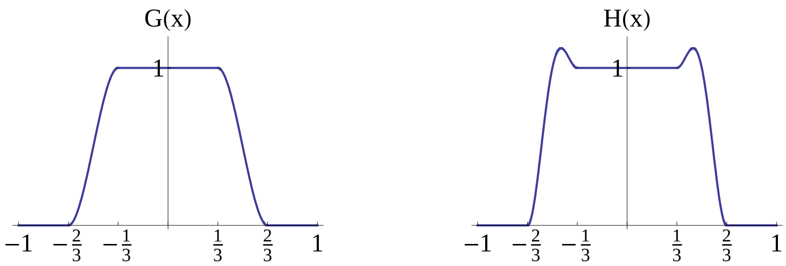

FIGURE 14. 4-knot symmetric with $b = 3/4$: $(m, p, q) = (1, 3, 6)$ [Section 7]

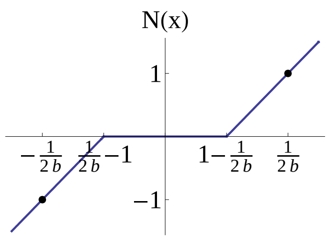

FIGURE 15. The time-rescaling function $N(x)$ from Section 7.

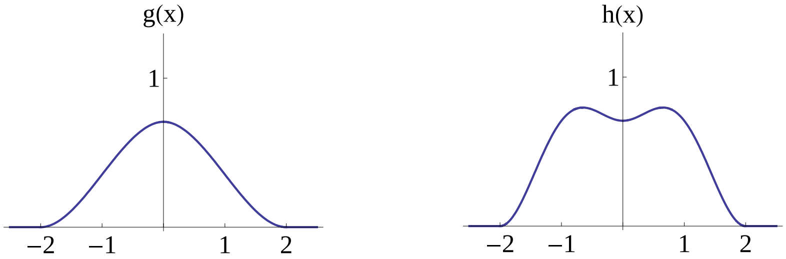

FIGURE 16. 3-knot symmetric on $[-2, 2]$, with $0 < b \leq 1/4$: $(m, p, q) = (1, 3, 6)$ [Section 8]

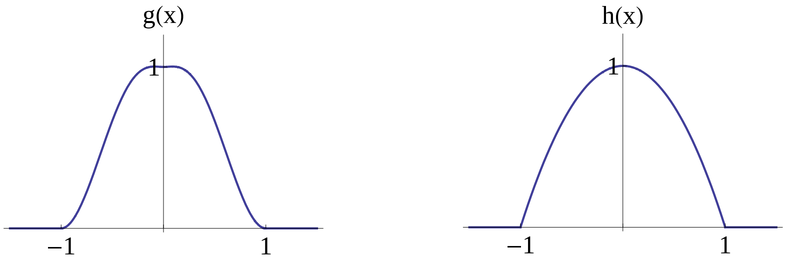

FIGURE 17. 3-knot symmetric with unequal smoothness: g is C^1 and h is continuous, with degrees $(p, q) = (5, 2)$ [Section 9]