AFFINE SYNTHESIS AND COEFFICIENT NORMS FOR LEBESGUE, HARDY AND SOBOLEV SPACES

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ABSTRACT. The affine synthesis operator $Sc = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k}$ is shown to map the mixed-norm sequence space $\ell^1(\ell^p)$ surjectively onto $L^p(\mathbb{R}^d), 1 \le p < \infty$, under mild conditions on the synthesizer $\psi \in L^p(\mathbb{R}^d)$ (say, having a radially decreasing L^1 majorant near infinity) and assuming $\int_{\mathbb{R}^d} \psi\, dx = 1$. Here $\psi_{j,k}(x) = |\det a_j|^{1/p}\psi(a_j x - k)$, for some dilation matrices a_j that expand.

Therefore the standard norm on $f \in L^p(\mathbb{R}^d)$ is equivalent to the minimal coefficient norm of realizations of f in terms of the affine system:

$$\|f\|_p \approx \inf\left\{ \sum_{j>0} \Big(\sum_{k\in\mathbb{Z}^d} |c_{j,k}|^p \Big)^{1/p} : f = \sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} \right\}.$$

We hence show the synthesis operator maps a discrete Hardy space onto $H^1(\mathbb{R}^d)$, which yields a norm equivalence involving convolution with a discrete Riesz kernel sequence $\{z_\ell\}$:

$$\|f\|_{H^1} \approx \inf\left\{ \sum_{j>0}\sum_{k\in\mathbb{Z}^d} \Big(|c_{j,k}| + \Big| \sum_{\ell\in\mathbb{Z}^d} z_\ell c_{j,k-\ell} \Big| \Big) : f = \sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} \right\}.$$

Coefficient norm equivalences are established also for the Sobolev spaces $W^{m,p}(\mathbb{R}^d)$, by applying difference operators to the coefficient sequence $c_{j,k}$.

1. Introduction

This paper studies mapping properties of the *affine synthesis* operator

$$c = \{c_{j,k}\} \mapsto \sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} = Sc$$

where

$$\psi_{j,k}(x) = |\det a_j|^{1/p}\psi(a_j x - bk), \qquad x \in \mathbb{R}^d,$$

assuming $\int_{\mathbb{R}^d} \psi\, dx = 1$. Affine synthesis arises naturally in harmonic analysis and approximation theory, as a discretization of convolution.

We first explain our notation, and then our goals.

- The *dimension* $d \in \mathbb{N}$ is fixed throughout the paper, as is the exponent p.

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- The *dilation matrices* a_j are invertible $d \times d$ real matrices that are *expanding*:
$$\|a_j^{-1}\| \to 0 \qquad \text{as } j \to \infty.$$
(Here $\|\cdot\|$ denotes the norm of a matrix as an operator from the column vector space $\mathbb{R}^d$ to itself.) For example, one could take $a_j = 2^j I$.
- The *translation matrix* b is an invertible $d \times d$ real matrix, for example the identity matrix.

Our goal is to synthesize surjectively onto the classic function spaces of analysis, while assuming as little as possible about the synthesizer ψ. This will demonstrate that the ability to decompose arbitrary functions into linear combinations of the translates and dilates $\psi_{j,k}$ does not require any special properties of ψ, even though the efficiency of decomposition naturally does depend on such properties.

Lebesgue space. In [6] we showed by explicit construction that

$$S : \{\text{finite sequences}\} \to L^p \quad \text{has dense range}$$

for $1 \le p < \infty$, assuming only that ψ has periodization locally in L^p (meaning $\sum_{k\in\mathbb{Z}^d} |\psi(x - bk)| \in L^p_{loc}$) and has nonzero integral ($\int_{\mathbb{R}^d} \psi\, dx \neq 0$). This density of the range of S means that the small-scale affine system $\{\psi_{j,k} : j > 0, k \in \mathbb{Z}^d\}$ *spans* L^p. Notice Strang–Fix conditions are not imposed: the translates $\{\psi(\cdot - bk)\}_{k\in\mathbb{Z}^d}$ need not form a partition of unity. And recall from [6] that the periodization assumption on ψ is easily satisfied, holding for example if $\psi \in L^p$ has a radially decreasing L^1 majorant near infinity.

Two natural questions arise from this L^p density result: what is the right *domain* for S? and does S map this domain *onto* L^p? We answer these questions in Section 3, where Theorems 1 and 2 show that

$$S : \ell^1(\ell^p) \to L^p \quad \text{is linear, bounded and onto;}$$

here the domain $\ell^1(\ell^p)$ is the mixed-norm space of coefficients satisfying

$$\sum_{j>0} \Big(\sum_{k\in\mathbb{Z}^d} |c_{j,k}|^p \Big)^{1/p} < \infty.$$

This surjectivity was proved earlier with non-constructive (duality) methods by Terekhin [40, 41].

From the surjectivity of S onto L^p one deduces a coefficient norm equivalence of the form

$$\|f\|_p \approx \inf\{\|c\|_{\ell^1(\ell^p)} : f = Sc\}.$$

Bruna [5, Theorem 4] earlier proved the case $p = 1$. The constants for this norm equivalence are evaluated in Corollary 3, and Corollary 4 shows that in fact equality holds under suitable conditions.

Corollary 5 proves a *localized* norm equivalence, on a domain $\Omega \subset \mathbb{R}^d$.

Similar results for L^p when $0 < p < 1$ are developed in [33], where it is shown $S : \ell^p(\ell^p) \to L^p$ is surjective.

Hardy space. Section 4 considers the Hardy space $H^1 = H^1(\mathbb{R}^d) = \{f \in L^1 : f * (x/|x|^{d+1}) \in L^1\}$. Again we ask what the domain of the synthesis operator

should be, and whether S maps this domain onto H^1. Theorems 7 and 8 answer the question by showing

$$S : \ell^1(h^1) \to H^1 \quad \text{is linear, bounded and onto,}$$

provided $\psi \in L^1$ has nonzero integral and is somewhat "nice" (being for example an L^p function with compact support for some $p > 1$, or a Schwartz function). The discrete Hardy space h^1 here is defined in Section 4 by convolution of sequences against a "singular" kernel in the k-variable (which is the discrete analogue of convolution against $x/|x|^{d+1}$ in the definition of the continuous Hardy space H^1).

Notice the synthesizer ψ cannot belong to H^1, because it has nonzero integral, and so to ensure the linear combination Sc belongs to H^1 we must invoke cancellation properties of the coefficient space h^1. This approach to synthesis in H^1 seems natural to us because the analysis operator takes H^1 to h^1 (see below), and one would like to reconstruct H^1 using only the data obtained from analysis.

Corollary 9 deduces a coefficient norm equivalence of the form

$$\|f\|_{H^1} \approx \inf\{\|c\|_{\ell^1(h^1)} : f = Sc\}.$$

Sobolev space. For synthesis into the Sobolev space $W^{m,p}(\mathbb{R}^d)$, we establish boundedness and surjectivity in Theorems 11 and 12 of Section 5. The sequence space from which we synthesize involves difference operators with respect to the k-index, which act as discrete analogues of differentiation.

Discussion. Surjectivity of the synthesis operator onto Hardy and Sobolev space is a qualitatively new result. Our proofs rely on a method of "scale-averaged convergence" that we developed only recently for L^p in [6]. Surjectivity onto L^p has been proved before, admittedly, but even there our constructive method is new. Boundedness of the synthesis operator has of course been studied previously.

We also consider boundedness of the affine *analysis operator* at each scale j, in other words, boundedness of $f \mapsto \{\langle f, \phi_{j,k}\rangle\}_{k\in\mathbb{Z}^d}$ from L^p to ℓ^p, from H^1 to h^1 and from $W^{m,p}$ to $w^{m,p}$. See Propositions 16, 21 and 23 respectively. We show the full analysis operator (over all scales) maps isomorphically onto its range in Corollaries 6, 10 and 15, thus giving coefficient norms in terms of the analysis operator.

If one works at a fixed scale j, rather than considering all scales $j > 0$ as we do in this paper, then synthesis yields a shift invariant subspace of L^p. Aldroubi, Sun and Tang's p-frame work for such shift invariant spaces [2] is described at the end of Section 3.

Topics we do not pursue in this paper include Gabor systems (modulations and translations) and wavepacket decompositions (modulations, translations and dilations). For some recent work in these areas one can consult [13, 18, 20, 28, 32].

The paper is structured as follows. Section 2 establishes notation and definitions. Sections 3–5 present our synthesis results on Lebesgue, Hardy and Sobolev spaces. The proofs are in Sections 8–10. Appendices treat discrete Hardy spaces, and Banach frames.

Parallel results on Triebel–Lizorkin spaces are discussed in Section 6. Open problems when ψ has zero integral ($\int_{\mathbb{R}^d} \psi \, dx = 0$) are treated in Section 7, including Meyer's Mexican hat spanning problem for L^p and its counterpart for the Hardy space. We hope this paper helps contribute towards an eventual resolution of these fascinating open problems.

2. **Further definitions and notation**

1. We use doubly-indexed sequences $c = \{c_{j,k}\}_{j>0,k\in\mathbb{Z}^d}$ of complex numbers, with the norm

$$\|c\|_{\ell^1(\ell^p)} := \sum_{j>0} \left(\sum_{k\in\mathbb{Z}^d} |c_{j,k}|^p \right)^{1/p}$$

when $1 \le p < \infty$. When $p = \infty$, define $\|c\|_{\ell^1(\ell^\infty)} := \sum_{j>0} \sup_{k\in\mathbb{Z}^d} |c_{j,k}|$. Then

$$\ell^1(\ell^p) := \left\{ c : \|c\|_{\ell^1(\ell^p)} < \infty \right\}$$

is a Banach space.

2. Write $L^p = L^p(\mathbb{R}^d)$ for the class of complex valued functions with finite L^p-norm. Given $\psi \in L^p$ and $\phi \in L^q$, with

$$\frac{1}{p} + \frac{1}{q} = 1$$

by notational convention, we define rescalings

$$\psi_{j,k}(x) = |\det a_j|^{1/p} \psi(a_j x - bk), \quad \phi_{j,k}(x) = |\det a_j|^{1/q} \phi(a_j x - bk).$$

These rescalings preserve the L^p-norm $\|\psi_{j,k}\|_p = \|\psi\|_p$ and the L^q-norm $\|\phi_{j,k}\|_q = \|\phi\|_q$, respectively. *Alert:* the notation $\psi_{j,k}$ conceals its dependence on p.

3. The *synthesis operator* is

$$Sc = S_{\psi,b} c = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k} \psi_{j,k}.$$

Our theorems will specify acceptable domains for this operator, and will explain the sense in which the sums over j and k converge. Occasionally we will synthesize at a fixed scale j by writing

$$S_j s = \sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k},$$

for sequences $s = \{s_k\}_{k\in\mathbb{Z}^d}$.

4. The *analysis operator at scale j* is

$$T_j f = T_{j,\phi} f = \{ |\det b| \langle f, \phi_{j,k} \rangle \}_{k\in\mathbb{Z}^d}.$$

That is, T_j maps a function f to its sequence of sampled ϕ-averages at scale j. The full *analysis operator* simply combines these sequences as

$$Tf = T_\phi f = \{ |\det b| \langle f, \phi_{j,k} \rangle \}_{j>0,k\in\mathbb{Z}^d}.$$

Our analysis and synthesis operators depend implicitly on the exponent p, through the definitions of $\phi_{j,k}$ and $\psi_{j,k}$.

5. The *periodization* of a function f is

$$Pf(x) = |\det b| \sum_{k\in\mathbb{Z}^d} f(x - bk) \qquad \text{for } x \in \mathbb{R}^d.$$

6. Write $\mathcal{C} = [0,1)^d$ for the unit cube in $\mathbb{R}^d$, and $\mathcal{C}_0 = (-1/2, 1/2)^d$ for the centered open unit cube. We regard $\mathcal{C}$ as consisting of column vectors and $\mathcal{C}_0$ as consisting of row vectors, as the context will always make clear.

3. L^p results

First we obtain boundedness of the synthesis operator, when the periodization of the synthesizer belongs locally to L^p. This was already observed by Aldroubi, Sun and Tang [2, formula (2.3)], but we give a proof in Section 8.1 anyway, to keep the paper self-contained.

Theorem 1 (Synthesis into L^p)**.** *Assume $1 \le p \le \infty$ and $\psi \in L^p$ with $P|\psi| \in L^p_{loc}$.*

Then $S : \ell^1(\ell^p) \to L^p$ is bounded. More precisely, if $c \in \ell^1(\ell^p)$ then the series

$$Sc = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} \tag{1}$$

converges in L^p in the sense that

the sum over k in (1) *converges pointwise absolutely a.e. to a function in L^p, and the sum over j converges absolutely in L^p,* (2)

and furthermore

$$\|Sc\|_p \le |\det b|^{-1} \| P|\psi| \|_{L^p(b\mathcal{C})} \|c\|_{\ell^1(\ell^p)}. \tag{3}$$

After proving the theorem in Section 8.1, we also give examples to show $\ell^1(\ell^p)$ is the "correct" domain for S, when the synthesizer ψ has nonzero integral. When $p = 2$ and ψ has zero integral, we point out that synthesis can be bounded on the larger domain $\ell^2(\ell^2)$, for wavelet and affine frame generators.

Remember that Sc in (1) depends implicitly on the value of p, through the renormalization factor $|\det a_j|^{1/p}$ in the definition of $\psi_{j,k}$. This dependence would be problematic if we synthesized into more than one L^p-space at a time, but we will not.

The hypothesis that the periodization of $|\psi|$ be locally in L^p is rather weak, and is easily verified in many cases. For example when $p = 1$ it holds for all $\psi \in L^1$. And for $p > 1$ it holds when $\psi \in L^p$ has compact support or when ψ has a bounded, radially decreasing L^1-majorant, or when ψ equals a sum of such functions. On the other hand, $P|\psi| \in L^p_{loc}$ can hold even when ψ does not decay at infinity. See [6, §3.1] for all these observations. Thus Theorem 1 improves somewhat on earlier boundedness results (which go back as far as [29, 38, 39]) because ψ need have neither compact support nor decay at infinity.

In the case $p = 2$, the periodization hypothesis on ψ can be weakened to just $\psi \in L^2$ with $P(|\widehat{\psi}|^2) \in L^\infty$. Specifically, one has bounded synthesis with $\|Sc\|_2 \le \|P(|\widehat{\psi}|^2)\|_\infty^{1/2} \|c\|_{\ell^1(\ell^2)}$ where this last periodization is taken with respect to the

lattice $\mathbb{Z}^d b^{-1}$. This estimate is proved in [10, Theorem 7.2.3], by showing that $P(|\widehat{\psi}|^2) \in L^\infty$ is exactly the condition for the integer translates of ψ to form a Bessel sequence (that is, to satisfy an upper frame bound).

For all p one might ask whether the periodization assumption on $|\psi|$ in Theorem 1 can be weakened to just $\psi \in L^1 \cap L^p$. We do not know.

Next we show the synthesis operator is surjective, when $1 \le p < \infty$. In other words, we show every $f \in L^p$ can be expressed by a series of the form (1).

Theorem 2 (Synthesis onto L^p)**.** *Assume $1 \le p < \infty$ and $\psi \in L^p$ with $P|\psi| \in L^p_{loc}$ and $\int_{\mathbb{R}^d} \psi \, dx = 1$.*

Then $S : \ell^1(\ell^p) \to L^p$ is open, and surjective. Indeed, if $f \in L^p$ and $\varepsilon > 0$ then a sequence $c \in \ell^1(\ell^p)$ exists such that $Sc = f$ with convergence as in (2)*, and such that*

$$\|c\|_{\ell^1(\ell^p)} \le |\det b|^{1/q} \|f\|_p + \varepsilon.$$

Section 8.4 has the proof, which proceeds by explicit construction. The integral of ψ is well defined, in the statement of Theorem 2, because the assumption $P|\psi| \in L^p_{loc}$ implies $P|\psi| \in L^1_{loc}$ and hence $\psi \in L^1$.

Terekhin [40, Theorem 2] proved the surjectivity in Theorem 2 by nonconstructive duality methods (based on his proof of Corollary 6 below), for ψ supported in the unit cube with dyadic dilations. Surjectivity for the more general ψ of Theorem 2 was stated in [41].

Earlier, Filippov and Oswald had constructed "representation systems" [21, Theorems 1 and 3], [22] by which every $f \in L^p$ can be written as a convergent series $Sc = f$. This looks like surjectivity, but the drawback is that their result yields no control over the size of coefficients in the sequence c, and thus it is unclear what the domain of S really is. Further, Filippov and Oswald worked only with isotropic dilation matrices.

We also mention the density results coming from Strang–Fix theory (discussed in [6, §3], although note Theorem 2 holds without needing the Strang–Fix hypotheses). When ψ has zero integral, the wavelet theory [10, 14, 30, 34] and related phi-transform theory (described in Section 6) provide sufficient conditions for obtaining frames, orthonormal bases, and unconditional bases, provided the large scales $j \le 0$ are included in the synthesis. These conditions all ensure surjectivity of S on suitable domains. But the zero-integral case also raises intriguing open problems, discussed in Section 7.

For the special case $p = 2$, we remarked above that the condition $P(|\widehat{\psi}|^2) \in L^\infty$ implies bounded synthesis. We suspect it also implies surjectivity onto L^2, provided $|\widehat{\psi}|$ is continuous near the origin and $\widehat{\psi}(0) = 1$. These conditions certainly guarantee the $\psi_{j,k}$ span L^2, by a result of Daubechies [14, Proposition 5.3.2], and thus S has dense range. One would like to improve this to full range.

Remark on non-injectivity. The synthesis operator is not injective, and indeed has a very large kernel. For example, we could discard the dilation a_1 (in other words, discard all terms with $j = 1$ in the sum defining Sc) and still show S maps onto L^p, by applying Theorem 2 with the remaining dilations $\{a_2, a_3, \ldots\}$.

Equivalence of the L^p and $\ell^1(\ell^p)$ norms follows immediately from Theorems 1 and 2:

Corollary 3 (Synthesis norm for L^p)**.** *Assume $1 \le p < \infty$ and $\psi \in L^p$ with $P|\psi| \in L^p_{loc}$ and $\int_{\mathbb{R}^d} \psi\, dx = 1$. Then*

$$\|f\|_p \approx \inf\left\{\|c\|_{\ell^1(\ell^p)} : f = Sc \ \text{ as in (2)}\right\}$$

for all $f \in L^p$. Explicitly,

$$\begin{aligned} |\det b|\||P|\psi|\|^{-1}_{L^p(b\mathcal{C})}\|f\|_p &\le \inf\left\{\|c\|_{\ell^1(\ell^p)} : f = Sc \ \text{ as in (2)}\right\} \\ &\le |\det b|^{1/q}\|f\|_p. \end{aligned}$$

For $p = 1$, the corollary was proved by Bruna [5, Theorem 4]. His duality methods apply without needing our assumption that the translations be restricted to a lattice. Recall when $p = 1$ that the periodization condition $P|\psi| \in L^p_{loc}$ is superfluous, holding automatically for all $\psi \in L^1$.

Remark on norm equivalence. As soon as S maps *onto* L^p, we know the map $\widetilde{S} : \ell^1(\ell^p)/\ker S \to L^p$ is a bounded linear bijection. (Here we use the canonical norm on the quotient space: $\inf_{c' \in \ker S} \|c + c'\|_{\ell^1(\ell^p)}$.) Then the inverse map $\widetilde{S}^{-1}$ is bounded by the closed graph theorem, giving equivalence of the $\ell^1(\ell^p)$ and L^p norms like in Corollary 3. But Corollary 3 goes further, for it provides an *explicit* upper bound for the norm equivalence, based on explicitly estimating the norm of $\widetilde{S}^{-1}$ in Theorem 2.

Corollary 4 goes further still, giving actual equality of the norms when ψ is nonnegative.

Corollary 4 (Synthesis norm equality)**.** *Assume $1 \le p < \infty$ and that $\psi \in L^p$ is nonnegative. When $p = 1$ assume $\int_{\mathbb{R}^d} \psi\, dx = 1$, and when $1 < p < \infty$ assume $P\psi \equiv 1$ (which implies $\int_{\mathbb{R}^d} \psi\, dx = 1$).*

Then for all $f \in L^p$,

$$\|f\|_p = |\det b|^{-1/q} \inf\left\{\|c\|_{\ell^1(\ell^p)} : f = Sc \ \text{ as in (2)}\right\}.$$

The constant periodization condition $P\psi \equiv 1$ in this corollary says that the collection $\{|\det b|\psi(x - bk) : k \in \mathbb{Z}^d\}$ of translates of ψ is a partition of unity. Examples of such ψ (when $b = I$) include the indicator function $\mathbb{1}_{\mathcal{C}}$ and convolutions of this indicator function with any nonnegative function having integral 1.

We next "localize" Corollary 3 to an open set $\Omega \subset \mathbb{R}^d$.

Definition. Say that a sequence $c = \{c_{j,k}\}_{j>0, k \in \mathbb{Z}^d}$ is *adapted to* Ω *and* ψ if $\operatorname{spt}(\psi_{j,k}) \subset \Omega$ whenever $c_{j,k} \ne 0$, or in other words if $c_{j,k} = 0$ whenever $\operatorname{spt}(\psi_{j,k}) \cap \Omega^c \ne \emptyset$.

The purpose of the definition is to ensure $Sc = 0$ on the complement of Ω.

Corollary 5 (Synthesis norm for $L^p(\Omega)$)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and nonempty, take $1 \le p < \infty$, and suppose $\psi \in L^p(\mathbb{R}^d)$ is compactly supported with $\int_{\mathbb{R}^d} \psi\, dx = 1$. Then for all $f \in L^p(\Omega)$,*

$$\|f\|_{L^p(\Omega)} \approx \inf\left\{\|c\|_{\ell^1(\ell^p)} : f = Sc \ \text{ as in (2), and } c \text{ is adapted to } \Omega \text{ and } \psi\right\}.$$

The constants in this norm equivalence are the same as in Corollary 3, and so they depend on ψ and b but are independent of Ω. The corollary is proved in Section 8.5.

We turn now to the analysis operator, which also yields a coefficient norm.

Corollary 6 (Analysis norm for L^p)**.** *Assume $1 \le p < \infty$, and take an analyzer $\phi \in L^q$ with $P|\phi| \in L^\infty$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$.*

Then for all $f \in L^p$,

$$\|f\|_p \approx \|Tf\|_{\ell^\infty(\ell^p)} = \sup_{j>0} \left(\sum_{k \in \mathbb{Z}^d} |\langle f, \phi_{j,k} \rangle|^p \right)^{1/p} |\det b|.$$

The result is due to Terekhin [40, Corollary 1],[41], who saw how to reduce to the case where the analyzer ϕ is the indicator function of a cube.

The corollary says the analysis operator is linear, bounded and injective from L^p onto its range in the mixed norm sequence space $\ell^\infty(\ell^p)$. To explain the appearance of $\ell^\infty(\ell^p)$ in the corollary, note the analysis and synthesis operators are adjoints, with $T_\phi : L^p \to \ell^\infty(\ell^p)$ being the adjoint of $S_\phi : \ell^1(\ell^q) \to L^q$, at least when $1 < p \le \infty$. Thus the injectivity of analysis in Corollary 6 is equivalent to the surjectivity of synthesis in Theorem 2. But we prove the corollary in Section 8.6 anyway, for the sake of concreteness and to handle $p = 1$.

We close the section by describing Aldroubi, Sun and Tang's work on p-frames [2], which is close in subject matter to this paper but has little direct overlap. They study the shift invariant range space $V_j = S_j(\ell^p)$ of the synthesis operator at a *single* scale j (whereas our results combine all scales $j > 0$). Roughly, they showed that the reconstruction formula $S_j T_j =$ identity on V_j (which says functions in V_j can be synthesized from their sampled average values at scale j) holds if and only if V_j is closed in L^p, if and only if T_j is injective on V_j when $\phi = \psi$, if and only if $\widehat{\psi}$ satisfies a certain "bracket product" condition. This is all carried out in the multiply generated case, with synthesizers $\psi_1, \ldots, \psi_r$. They observe that the range V_j need not be closed, for example when $\psi = \mathbb{1}_{[0,1)} - \mathbb{1}_{[1,2)}$ in one dimension with $1 < p < \infty$ [2, page 7].

4. **Hardy space results**

Our Hardy space results assume the dilation matrices are isotropic and expanding. To be precise, all the results in this section assume that

$$a_j = \alpha_j I$$

for some "dilation" sequence $\alpha = \{\alpha_j\}_{j>0}$ of nonzero real numbers with $|\alpha_j| \to \infty$ as $j \to \infty$.

We will recall the Hardy space H^1, and then construct a discrete Hardy space on which the synthesis operator can act. Then we state our synthesis results.

Hardy space H^1. Define the Fourier transform with 2π in the exponent:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi x}\, dx,$$

for row vectors $\xi \in \mathbb{R}^d$. Write $C_d = \Gamma((d+1)/2)\pi^{-(d+1)/2}$ and

$$Z(x) = C_d \begin{cases} x/|x|^{d+1}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

for the Riesz kernel, so that the Riesz transform of $f \in L^1$ is

$$Rf(x) = (f * Z)(x) = \text{p.v.} \int_{\mathbb{R}^d} f(x-y) Z(y)\, dy.$$

Then Rf is finite a.e., and is a measurable vector-valued function of $x \in \mathbb{R}^d$. Notice

$$\widehat{Rf}(\xi) = -i \frac{\xi}{|\xi|} \widehat{f}(\xi).$$

Recall the *Hardy space* is

$$H^1 = H^1(\mathbb{R}^d) = \{f \in L^1 : Rf \in L^1\}, \quad \text{with the norm } \|f\|_{H^1} := \|f\|_1 + \|Rf\|_1.$$

Functions in the Hardy space have vanishing integral: if $f \in H^1$ then $Rf \in L^1$ and so $\widehat{Rf}$ is continuous, which implies

$$\widehat{f}(0) = \int_{\mathbb{R}^d} f(x)\, dx = 0 \quad \text{and} \quad \widehat{Rf}(0) = \int_{\mathbb{R}^d} Rf(x)\, dx = 0. \tag{4}$$

The Riesz transform commutes with dilations and translations, meaning: $R(f(\alpha x - x_0)) = \text{sign}(\alpha)(Rf)(\alpha x - x_0)$ when $\alpha \in \mathbb{R}\backslash\{0\}, x_0 \in \mathbb{R}^d$. But dilation invariance fails when α is an arbitrary matrix, which is why we restrict to isotropic dilations in this section.

All these facts about Riesz transforms and H^1 can be found in [36, 37].

Discrete Hardy space h^1. Take a smooth, compactly supported cut-off function ν supported in the centered unit cube $\mathcal{C}_0$, with $\nu \equiv 1$ near the origin. Then define a "discrete Riesz kernel" sequence $z = \{z_k\}_{k \in \mathbb{Z}^d} \in \ell^2$ by specifying its Fourier series:

$$\zeta(\xi) = \sum_{k \in \mathbb{Z}^d} z_k e^{-2\pi i \xi k} := -i \frac{\xi b^{-1}}{|\xi b^{-1}|} \nu(\xi), \qquad \xi \in \mathcal{C}_0, \tag{5}$$

where ξk denotes the dot product (recall ξ is a row and k is a column vector) and where for later convenience we use $-\xi$ rather than $+\xi$ in the exponent of the Fourier series.

The sequence z is vector-valued (since $z_k \in \mathbb{C}^d$), and belongs to ℓ^2 because $(-i\xi b^{-1}/|\xi b^{-1}|)\nu(\xi)$ is bounded and hence belongs to $L^2(\mathcal{C}_0)$. Thus the series for the periodic function ζ converges unconditionally in $L^2(\mathcal{C}_0)$. When $b = I$, observe from (5) that ζ is simply a cut-off version of the Fourier transform of the Riesz kernel.

Naturally $s * z \in \ell^2$ whenever $s \in \ell^1$. We define a "discrete Hardy space" by requiring that $s * z$ belong to the smaller space ℓ^1:

$$h^1 = \left\{ s \in \ell^1 : s * z \in \ell^1 \right\},$$

with a norm

$$\|s\|_{h^1} := \|s\|_{\ell^1} + \|s * z\|_{\ell^1}$$

that makes h^1 a Banach space.

Appendix B investigates some properties of h^1, including its independence from the cut-off function ν, and its relation to the atomic sequence space $H^1(\mathbb{Z}^d)$ studied by several authors. The appendix also points out that sequences in h^1 have vanishing mean: $\sum_{k\in\mathbb{Z}^d} s_k = 0$.

The mixed-norm Banach space we will need is

$$\begin{aligned} \ell^1(h^1) &= \left\{ c \in \ell^1(\ell^1) : c * z \in \ell^1(\ell^1) \right\} \\ &= \{ c \in \ell^1(\ell^1) : \|c\|_{\ell^1(h^1)} < \infty \}, \end{aligned}$$

where the convolution is taken with respect to the k-index and the norm is

$$\begin{aligned} \|c\|_{\ell^1(h^1)} &:= \|c\|_{\ell^1(\ell^1)} + \|c * z\|_{\ell^1(\ell^1)} \\ &= \sum_{j>0} \|c_j\|_{h^1} = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} \left(|c_{j,k}| + |(c * z)_{j,k}| \right) \end{aligned}$$

with the notation $c_j = \{c_{j,k}\}_{k\in\mathbb{Z}^d}$.

Properties of the synthesis operator. We will prove boundedness and surjectivity of the synthesis operator. Note

$$\psi_{j,k}(x) = |\det a_j| \psi(a_j x - bk)$$

in the next two theorems, because we implicitly take $p = 1$ when working with $\psi \in L^1$.

First we show boundedness.

Theorem 7 (Synthesis into H^1)**.** *Assume $\psi \in L^1$ and*

$$\sup_{|y|\le 1} \|\psi - \psi(\cdot - y)\|_{H^1} < \infty. \tag{6}$$

Then $S : \ell^1(h^1) \to H^1$ is bounded. More precisely, if $c \in \ell^1(h^1)$ then the series $Sc = \sum_{j>0} \sum_{k\in\mathbb{Z}^d} c_{j,k} \psi_{j,k}$ converges in H^1 in the sense that

the sum over k converges absolutely in L^1 to a function belonging to H^1 and the sum over j converges absolutely in H^1, (7)

and furthermore $\|Sc\|_{H^1} \le C \|c\|_{\ell^1(h^1)}$ for some constant $C = C(\psi, b)$.

See Section 9.1 for the proof. To understand why we so carefully describe the convergence of Sc in H^1, in (7), just recall that $\psi_{j,k} \notin H^1$ when $\int_{\mathbb{R}^d} \psi \, dx \neq 0$.

Many functions ψ satisfy the finite supremum assumption in (6), for example if $\psi \in L^p$ for some $p > 1$ and ψ has compact support, or if ψ is a Schwartz function; cf. [7, §3.3-3.4]. Incidentally, the supremum in (6) can equivalently be taken over any other ball of y-values, by [7, Lemma 6].

Theorem 7 was proved for synthesizers $\psi \in L^2$ having compact support by S. Boza and M. J. Carro in [3, Proposition 3.11], [4, Theorem 3.1]. Their methods are very different from ours, involving a maximal characterization of $H^p(\mathbb{Z}^d), 0 < p \leq 1$. Theorem 7 was also proved earlier in one dimension for synthesizers $\psi \in L^1(\mathbb{R})$ having compact support and locally integrable Hilbert transform $R\psi \in L^1_{loc}(\mathbb{R})$, by Q. Sun [39, Theorem 13]. Note that Sun's assumptions on ψ imply the condition (6). Interestingly, Sun proved a converse theorem for compactly supported ψ, saying that bounded synthesis implies $R\psi$ must be locally integrable.

Now we show surjectivity of synthesis from $\ell^1(h^1)$ to H^1, a result that seems to be qualitatively new.

Theorem 8 (Synthesis onto H^1)**.** *Assume $\psi \in L^1$ with $\int_{\mathbb{R}^d} \psi \, dx = 1$ and*

$$\|\psi - \psi(\cdot - y)\|_{H^1} \to 0 \qquad \text{as } y \to 0. \tag{8}$$

Then $S : \ell^1(h^1) \to H^1$ is open, and surjective. Indeed if $f \in H^1$ and $\varepsilon > 0$ then a sequence $c \in \ell^1(h^1)$ exists such that $Sc = f$ (with convergence as in (7)*) and*

$$\|c\|_{\ell^1(h^1)} \leq C\|f\|_{H^1} + \varepsilon,$$

for some constant $C = C(b)$.

See Section 9.4 for the proof. The constant C can be evaluated, if desired.

Assumption (8) holds, for example, if $\psi \in L^p$ for some $p > 1$ and ψ has compact support, or if ψ is a Schwartz function (cf. [7, §3.3]).

By combining the last two theorems, we obtain a norm for H^1 in terms just of coefficients in affine expansions.

Corollary 9 (Synthesis norm for H^1)**.** *Assume $\psi \in L^1$ with $\int_{\mathbb{R}^d} \psi \, dx = 1$, and suppose* (8) *holds. Then for all $f \in H^1$,*

$$\|f\|_{H^1} \approx \inf \left\{ \|c\|_{\ell^1(h^1)} : c \in \ell^1(h^1) \text{ and } f = Sc \text{ as in } (7) \right\}.$$

The proof is easy: assumption (8) implies (6) by [7, §3.4], and so Theorems 7 and 8 both apply here, giving the corollary.

The analysis operator also provides a coefficient norm for H^1:

Corollary 10 (Analysis norm for H^1)**.** *Take an analyzer $\phi \in L^1$ with $\int_{\mathbb{R}^d} \phi \, dx = 1$, and suppose ϕ is a Schwartz function with $\widehat{\phi}$ supported in $\mathcal{C}_0 b^{-1}$.*

Then for all $f \in H^1$,

$$\|f\|_{H^1} \approx \|Tf\|_{\ell^\infty(h^1)} = \sup_{j>0} \|\{\langle f, \phi_{j,k} \rangle\}\|_{h^1} \, |\det b|.$$

The corollary says the analysis operator T is linear, bounded and injective from H^1 onto its range in $\ell^\infty(h^1)$. See Section 9.5 for the proof.

5. Sobolev space results

Write $W^{m,p} = W^{m,p}(\mathbb{R}^d)$ for the class of Sobolev functions with m derivatives in L^p, normed by

$$\|f\|_{W^{m,p}} = \sum_{|\rho| \leq m} \|D^\rho f\|_p.$$

Here $\rho = (\rho_1, \ldots, \rho_d)$ is a multiindex of order $|\rho| = \rho_1 + \cdots + \rho_d$.

We continue to assume the dilation matrices are isotropic and expanding, with $a_j = \alpha_j I$, like in the previous section.

We first construct a sequence space on which the synthesis operator will act, and then construct a class of synthesizers, before stating our Sobolev synthesis results.

Discrete Sobolev space. Define difference operators on sequences $s = \{s_k\}_{k\in\mathbb{Z}^d}$ by

$$\Delta_t s = \{s_k - s_{k-e_t}\}_{k\in\mathbb{Z}^d},$$

for $t = 1, \ldots, d$, where e_t is the unit vector in the t-th coordinate direction. Define higher difference operators by

$$\Delta^\rho s = \Delta_1^{\rho_1} \cdots \Delta_d^{\rho_d} s.$$

Then we can define a discrete Sobolev space by

$$w^{m,p} = \left\{ s \in \ell^p : \Delta^\rho s \in \ell^p \text{ for each multiindex } \rho \text{ of order } |\rho| \le m \right\},$$

with norm

$$\|s\|_{w^{m,p}} = \sum_{|\rho|\le m} \|\Delta^\rho s\|_{\ell^p}.$$

This space is just ℓ^p with a new norm, but we proceed to weight the norm by appropriate powers of the dilation sequence, as follows.

Recall sequences are multiplied term-by-term, meaning $(c\tilde{c})_{j,k} = c_{j,k}\tilde{c}_{j,k}$ and so on. In particular we can define powers $\alpha^r = \{\alpha_j^r\}_{j>0}$ of the dilation sequence, whenever r is a nonnegative integer. With these conventions, we define a dilation-weighted discrete Sobolev space by

$$\ell^1(w^{m,p}, \alpha) = \left\{ c \in \ell^1(\ell^p) : \alpha^{|\rho|}\Delta^\rho c \in \ell^1(\ell^p) \text{ for each multiindex } \rho \text{ of order } |\rho| \le m \right\},$$

with norm

$$\begin{aligned}\|c\|_{\ell^1(w^{m,p},\alpha)} &:= \sum_{|\rho|\le m} \|\alpha^{|\rho|}\Delta^\rho c\|_{\ell^1(\ell^p)} \\ &= \sum_{j>0}\sum_{|\rho|\le m} |\alpha_j|^{|\rho|} \left(\sum_{k\in\mathbb{Z}^d} |(\Delta^\rho c)_{j,k}|^p \right)^{1/p}.\end{aligned}$$

(The difference operators here should be understood as acting on the k-index of the sequence $c_{j,k}$.) One can check $\ell^1(w^{m,p}, \alpha)$ is a Banach space.

The class of synthesizers. Our synthesis results will hold when ψ has the special convolution form

$$\psi = \overbrace{\beta * \cdots * \beta}^{m \text{ factors}} * \eta, \tag{9}$$

where

$$\beta = |b\mathcal{C}|^{-1} \mathbb{1}_{b\mathcal{C}}$$

is the normalized indicator function of the period box in the lattice $b\mathbb{Z}^d$. (One can show that the convolution form (9) is equivalent in one dimension to a Strang–Fix

condition on ψ, but it is definitely stronger in higher dimensions, as explained in [8, *Notes on Theorem 1*].) The point of this convolution form (9) is that derivatives of ψ turn into differences, in formula (23) later on, and these differences transfer to the coefficient sequence in formula (27).

Now we can state the boundedness of the synthesis operator on Sobolev space. Our statement involves the matrix rescaling operator

$$(M_b f)(x) = |\det b| f(bx).$$

Choosing $b = I$ gives the simplest results in what follows, of course.

Theorem 11 (Synthesis into $W^{m,p}$)**.** *Assume $1 \le p \le \infty$ and $\eta \in L^p$ with $P|\eta| \in L^p_{loc}$. Let $m \in \mathbb{N}$ and define ψ by* (9)*.*

Then $\psi \in W^{m,p}$, and $S : \ell^1(w^{m,p}, \alpha) \to W^{m,p}$ is bounded. More precisely, if $c \in \ell^1(w^{m,p}, \alpha)$ then the series $Sc = \sum_{j>0} \sum_{k \in \mathbb{Z}^d} c_{j,k} \psi_{j,k}$ converges in $W^{m,p}$ in the sense that

$$\begin{aligned} &\textit{the sum over } k \textit{ converges pointwise absolutely a.e. to a function in } W^{m,p} \\ &\textit{and the sum over } j \textit{ converges absolutely in } W^{m,p}, \end{aligned} \tag{10}$$

and furthermore

$$\|M_b Sc\|_{W^{m,p}} \le |\det b|^{-1/p} \| P|\eta| \|_{L^p(b\mathcal{C})} \|c\|_{\ell^1(w^{m,p},\alpha)}.$$

See Section 10.1 for the proof. Prior results on bounded synthesis include [29, 39], for compactly supported ψ. Here ψ need not be compactly supported.

Regarding the appearance of the rescaling operator M_b in the theorem, note the analogous L^p inequality (3) in Theorem 1 can be put in the same form as above, namely

$$\|M_b Sc\|_p \le |\det b|^{-1/p} \| P|\psi| \|_{L^p(b\mathcal{C})} \|c\|_{\ell^1(\ell^p)},$$

by applying Theorem 1 and then making a change of variable on the lefthand side.

Now we obtain surjectivity of S.

Theorem 12 (Synthesis onto $W^{m,p}$)**.** *Assume $1 \le p < \infty$ and $\eta \in L^p$ with $P|\eta| \in L^p_{loc}$ and $\int_{\mathbb{R}^d} \eta \, dx = 1$. Let $m \in \mathbb{N}$ and define ψ by* (9)*.*

Then $S : \ell^1(w^{m,p}, \alpha) \to W^{m,p}$ is open, and surjective. Indeed if $f \in W^{m,p}$ and $\varepsilon > 0$ then a sequence $c \in \ell^1(w^{m,p}, \alpha)$ exists such that $Sc = M_b^{-1} f$ with convergence as in (10)*, and such that*

$$\|c\|_{\ell^1(w^{m,p},\alpha)} \le \|f\|_{W^{m,p}} + \varepsilon.$$

The theorem is proved in Section 10.4. Note the conclusion of the L^p result Theorem 2 can be rephrased to look like Theorem 12, by applying it to $M_b^{-1} f$ instead of to f.

There seem to be no direct predecessors in the literature for the surjectivity in Theorem 12. Indirect predecessors include the density results from Strang–Fix theory, which we discuss in [8, §3.5] and after Theorem 24 below. To learn about wavelet and affine frame expansions in Sobolev space, using all scales $j \in \mathbb{Z}$, consult [30, 34] and the recent work in [11].

Norm equivalence now follows from Theorems 11 and 12:

Corollary 13 (Synthesis norm for $W^{m,p}$)**.** *Assume $1 \le p < \infty$ and $\eta \in L^p$ with $P|\eta| \in L^p_{loc}$ and $\int_{\mathbb{R}^d} \eta\, dx = 1$. Let $m \in \mathbb{N}$ and define ψ by* (9)*.*

Then

$$\|f\|_{W^{m,p}} \approx \inf\left\{\|c\|_{\ell^1(w^{m,p},\alpha)} : M_b^{-1} f = Sc \ \text{ as in (10)}\right\}$$

for all $f \in W^{m,p}$. Explicitly,

$$\begin{aligned} |\det b|^{1/p} \|P|\eta|\|^{-1}_{L^p(b\mathcal{C})} \|f\|_{W^{m,p}} &\le \inf\left\{\|c\|_{\ell^1(w^{m,p},\alpha)} : M_b^{-1} f = Sc \ \text{ as in (10)}\right\} \\ &\le \|f\|_{W^{m,p}}. \end{aligned}$$

The coefficient norm can equal the standard Sobolev norm:

Corollary 14 (Synthesis norm equality)**.** *Assume $1 \le p < \infty$ and that $\eta \in L^p$ is nonnegative. When $p = 1$ assume $\int_{\mathbb{R}^d} \eta\, dx = 1$, and when $1 < p < \infty$ assume $P\eta \equiv 1$ (which implies $\int_{\mathbb{R}^d} \eta\, dx = 1$). Let $m \in \mathbb{N}$ and define ψ by* (9)*.*

Then for all $f \in W^{m,p}$,

$$\|f\|_{W^{m,p}} = \inf\left\{\|c\|_{\ell^1(w^{m,p},\alpha)} : M_b^{-1} f = Sc \ \text{ as in (10)}\right\}.$$

The analysis operator gives a norm to Sobolev space also:

Corollary 15 (Analysis norm for $W^{m,p}$)**.** *Assume $1 \le p < \infty$ and $m \in \mathbb{N}$. Take an analyzer $\phi \in L^q$ with $P|\phi| \in L^\infty$ and $\int_{\mathbb{R}^d} \phi\, dx = 1$, and assume $b = I$.*

Then for all $f \in W^{m,p}$,

$$\begin{aligned} \|f\|_{W^{m,p}} \approx \|Tf\|_{\ell^\infty(w^{m,p},\alpha)} &= \sup_{j>0} \sum_{|\rho|\le m} |\alpha_j|^{|\rho|} \|\Delta^\rho T_j f\|_{\ell^p} \\ &= \sup_{j>0} \sum_{|\rho|\le m} |\alpha_j|^{|\rho|} \|\{\Delta^\rho \langle f, \phi_{j,k}\rangle\}\|_{\ell^p}. \end{aligned}$$

Thus the analysis operator is linear, bounded and injective from $W^{m,p}$ onto its range. Section 10.5 has the proof.

6. Connection to phi-transforms and Triebel–Lizorkin spaces

The decomposition or representation of function spaces by means of an affine system generated by a single function (or collection of functions) is a well-established technique in harmonic analysis. Important examples include wavelet expansions and the phi-transforms. For wavelet theory we refer to [10, 14, 30, 34] and the references therein. The connection between phi-transform theory and our results is briefly explained below.

Triebel–Lizorkin spaces include L^p, H^1 and $W^{m,p}$, $1 < p < \infty$, but neither L^1 nor $W^{m,1}$. Phi transform theory implies the following results about these spaces; see [9, 23, 24, 25] for a complete account.

(i) That the Triebel–Lizorkin space norm of an element f is equivalent to the infimum of the corresponding sequence space norm of c, where the infimum is taken over all representations $f = Sc$ and the synthesizing ψ satisfies some moment condition (in particular $\int_{\mathbb{R}^d} \psi\, dx = 0$), and a decay condition and a "Tauberian" condition. Note that the dilations are assumed isotropic, the translation matrix b

must be "sufficiently small", and that the sequence space norm in this theory also involves an integration with respect to the continuous variable on $\mathbb{R}^d$.

(ii) That there exists an analyzing function ϕ (or a collection of functions $\phi^{(j,k)}$) satisfying similar conditions to ψ such that the Triebel–Lizorkin space norm of f is equivalent to the sequence space norm of $Tf = \{\langle f, \phi_{j,k}\rangle\}$, and f is represented by $f = \sum_{j,k} S_j T_j f$.

The large scales $j \le 0$ must be included in the synthesis (i) and the analysis (ii).

Versions of these results have also been proved for affine systems generated by interesting families of functions (instead of one function ψ); see [42] and the references there for "Gausslet" and "Quarkonial" analysis.

Our main results in Sections 3–5 can be viewed as analogues of the above results when both the analyzer ϕ and the synthesizer ψ have non-zero integral (that is, no vanishing moments). Further, our sequence space norm is simpler, involving only discrete sums and no integration with respect to the continuous variable. And the norm equivalence statements in our Corollaries 6, 10 and 15 take an especially simple form. These corollaries are in the spirit of norm equivalences in frame theory, but in our case the frame decomposition $f = \sum_{j,k} S_j T_j f$ cannot hold, because if $f \mapsto \sum_{j,k} S_j T_j f$ is bounded on L^2, then either ϕ or ψ must have a vanishing moment by [26, Theorem B]. Nevertheless, our corollaries give rise to Banach frames in the sense of K. Gröchenig [27]. Details are given in Appendix C.

7. **Open problems**

Throughout this paper we have assumed the synthesizer ψ has nonvanishing integral. When it has vanishing integral, $\int_{\mathbb{R}^d} \psi \, dx = 0$, we do not have a comprehensive understanding of conditions under which one can synthesize surjectively onto L^p or Hardy or Sobolev space. Of course there are substantial classes of synthesizers ψ for which surjectivity and even injectivity is known, for example the class of wavelets [14, 30, 34] (provided one includes large scales $j \le 0$ in the affine systems). But wavelets seem rather restricted objects to us. In accordance with the goal expressed in the Introduction, we conjecture that some much more general surjectivity result should hold when ψ has vanishing integral. Recent L^2 work of Gilbert *et al.* [26, Theorem G] is a step in the right direction, for it assumes only that $\widehat{\psi}(0) = 0$ and $\widehat{\psi}$ has some cancellation properties. Unfortunately the result suffers from oversampling of both dilations and translations, and ways to remove that oversampling remain a mystery.

Problems with vanishing integrals can be more challenging than they appear. For instance, it is an open problem of Y. Meyer [34, p. 137] to determine whether the affine system $\{\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ spans $L^p(\mathbb{R})$ for each $1 < p < \infty$, when $\psi(x) = (1 - x^2)e^{-x^2/2}$ is the Mexican hat function (the second derivative of the Gaussian). This is known to be true when $p = 2$, since the system forms a frame, but it remains open for all other p-values. It is apparently also open to determine whether the system spans $H^1(\mathbb{R})$. To express this in terms of the synthesis operator, first notice S_ψ is bounded from $\ell^1(\ell^1)$ to H^1 because the Mexican hat ψ belongs

to H^1, and then ask: is $S_\psi : \ell^1(\ell^1) \to H^1(\mathbb{R})$ surjective? Surjectivity would give an atomic decomposition of H^1 in terms of the Mexican hat affine system.

There are two partial results in the literature dealing with the Mexican hat problem. In [25] the authors proved that there exist (sufficiently small) $r > 1$ and $s > 0$ such that the affine system $\{\psi(r^j x - sk) : j, k \in \mathbb{Z}\}$ spans $H^p(\mathbb{R})$, $1 \le p < \infty$, while in [9] the authors proved the same result for the affine system $\{\psi(2^j x - bk) : j, k \in \mathbb{Z}\}$, where $b > 0$ is sufficiently small and $1/2 < p < \infty$. Since $H^p = L^p$ for $1 < p < \infty$, these results show that the spanning property holds for the Mexican hat function provided we accept some degree of oversampling.

But note some caution is due, because (for example) the Haar system does not span Hardy space. Indeed, the closed span of the Haar system in $H^1(\mathbb{R})$ is the proper subspace $\{f \in H^1(\mathbb{R}) : \int_0^\infty f\,dx = 0\}$, by [1, Theorem 2.1].

We conclude by pointing out a gap in understanding of our Hardy space synthesis results. We do not know whether h^1 is the "natural" domain for the synthesis operator in Theorem 7. Can one prove it is natural, in the sense that

$$\sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k} \in H^1 \quad \Longrightarrow \quad s \in h^1$$

whenever $\psi \in L^1$ satisfies hypothesis (6)?

8. L^p proofs

8.1. Proof of Theorem 1 — synthesis $\ell^1(\ell^p) \to L^p$.

First assume $1 \le p < \infty$ and $\psi \in L^p$ with $P|\psi| \in L^p_{loc}$. We will synthesize at a fixed scale $j > 0$, by taking $s \in \ell^p$ and defining

$$f = S_j s = \sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k}.$$

The task is to show $f \in L^p$ with $\|f\|_p \le \|s\|_{\ell^p} \|P|\psi|\|_{L^p(b\mathcal{C})}/|\det b|$. Then Theorem 1 follows easily, by summing over the dilation scales.

We have

$$\begin{aligned} |\sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k}(x)|^p &\le \left(\sum_{k\in\mathbb{Z}^d} |s_k||\psi(a_j x - bk)|\right)^p |\det a_j| \\ &\le \sum_{k\in\mathbb{Z}^d} |s_k|^p |\psi(a_j x - bk)| \left(\sum_{k\in\mathbb{Z}^d} |\psi(a_j x - bk)|\right)^{p-1} |\det a_j| \end{aligned}$$

by Hölder's inequality on the sum. Integrating with respect to x yields that

$$\begin{aligned} \|\sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k}\|_p^p &\le \sum_{k\in\mathbb{Z}^d} |s_k|^p \int_{\mathbb{R}^d} |\psi(x - bk)| (P|\psi|(x))^{p-1}\,dx/|\det b|^{p-1} \\ &= \sum_{k\in\mathbb{Z}^d} |s_k|^p \|P|\psi|\|^p_{L^p(b\mathcal{C})}/|\det b|^p, \end{aligned} \tag{11}$$

by changing $x \mapsto x + bk$ and then periodizing the integral. We conclude that the sum over k in (1) converges pointwise absolutely a.e. to an L^p function, and that

$$\|f\|_p = \| \sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k}\|_p \le \|s\|_{\ell^p} \|P|\psi|\|_{L^p(b\mathcal{C})}/|\det b|.$$

It remains to prove the theorem when $p = \infty$. So assume $p = \infty$ and $\psi \in L^\infty$ with $P|\psi| \in L^\infty_{loc}$. Then $P|\psi|$ is bounded, since it is locally bounded and periodic. If $c \in \ell^1(\ell^\infty)$ then

$$\begin{aligned} |\sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k}(x)| &\le \sum_{j>0}(\sup_{k\in\mathbb{Z}^d}|c_{j,k}|)\sum_{k\in\mathbb{Z}^d}|\psi(a_j x - bk)| \\ &\le \|c\|_{\ell^{1,\infty}} \|P|\psi|\|_\infty/|\det b| < \infty \end{aligned}$$

for almost every x, from which the theorem follows.

Is $\ell^1(\ell^p)$ the correct domain for synthesis?

We have proved S is bounded from $\ell^1(\ell^p)$ into L^p. Could S be bounded on an even larger domain? The natural candidate would be $\ell^r(\ell^{p'})$ with $r \ge 1, p' \ge p$, but we will show by example that S need not be bounded on this domain unless $r = 1$ and $p' = p$.

Work in one dimension with $b = 1$ and dyadic dilations $a_j = 2^j$, and choose ψ to be supported in the unit interval $[0, 1)$. Then for any sequence $s \in \ell^{p'}$ we have $S_j s(x) = \sum_{k\in\mathbb{Z}} s_k 2^{j/p}\psi(2^j x - k)$, which has L^p norm $\|S_j s\|_p = \|\psi\|_p \|s\|_{\ell^p}$. Hence if $S_j s \in L^p$ then $s \in \ell^p$, so that for S to be bounded on $\ell^r(\ell^{p'})$ it is necessary that $p' = p$.

Further, if $t = \{t_j\}_{j>0}$ is any nonnegative sequence in ℓ^r then the sequence

$$c_{j,k} = \begin{cases} t_j 2^{-j/p}, & k = 0, 1, \ldots, 2^j - 1, \\ 0, & \text{otherwise,} \end{cases}$$

belongs to $\ell^r(\ell^p)$, and if $\psi = \mathbb{1}_{[0,1)}$ is the indicator function of the unit interval then $Sc = \|t\|_{\ell^1}\mathbb{1}_{[0,1)}$. Thus for Sc to belong to L^p it is necessary that $t \in \ell^1$. Hence $r = 1$, as claimed.

So we cannot expect to enlarge the domain $\ell^1(\ell^p)$, in general, when $\int_{\mathbb{R}} \psi\, dx \ne 0$. But there is a loophole relevant to wavelets, because if $\int_{\mathbb{R}} \psi\, dx = 0$ then the $\psi_{j,k}$ might exhibit some cancellation between different j-scales that invalidates our "t" example above and allows us to take $r > 1$. For example, if $p = 2$ and ψ is a wavelet (meaning the functions $\psi_{j,k}$ are orthonormal and complete in L^2) then S is not only bounded but is an isometry from $\ell^2(\mathbb{Z}\times\mathbb{Z})$ to L^2. Thus the natural domain in the wavelet case has $r = 2$. Recall integrable wavelets satisfy $\int_{\mathbb{R}} \psi\, dx = 0$ by [30, p. 348].

Continuing in the special case $p = 2$, the $\psi_{j,k}$ form a frame (which is more general than an orthonormal basis) if and only if the synthesis operator is bounded and surjective from $\ell^2(\mathbb{Z}\times\mathbb{Z})$ to L^2. This is a special case of Christensen's Hilbert space result [10, Theorem 5.5.1]. Thus the natural domain for frame synthesis has

$r = 2$. This agrees with our earlier remarks, because integrable frame generators are known to satisfy $\int_{\mathbb{R}} \psi \, dx = 0$.

8.2. **Analysis $L^p \to \ell^p$.**

The proof of Theorem 2 relies on the following estimate for analyzers ϕ with bounded periodization. Recall the analysis operator T_j at scale j was defined in Section 2.

Proposition 16 (Analysis into ℓ^p)**.** *Assume $1 \le p \le \infty$ and $\phi \in L^q$ with $P|\phi| \in L^\infty$. Then for each j,*

$$T_j : L^p \to \ell^p \qquad \textit{with norm } \|T_j\| \le |\det b|^{1/q} \|P|\phi|\|_\infty.$$

The proposition is known from Aldroubi, Sun and Tang [2, formula (2.2)].

The hypothesis that $|\phi|$ have bounded periodization means ϕ is bounded and its integer translates "do not overlap too often".

Proof of Proposition 16. Let $f \in L^p$. When $1 \le p < \infty$ we have

$$\begin{aligned}
&\|T_j f\|_{\ell^p} \\
&= |\det b| \left(\sum_{k \in \mathbb{Z}^d} |\langle f, \phi_{j,k} \rangle|^p \right)^{1/p} \\
&\le |\det b| \left(\sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} |f(y)|^p |\phi(a_j y - bk)| \, dy \left(\int_{\mathbb{R}^d} |\phi(a_j y - bk)| |\det a_j| \, dy \right)^{p/q} \right)^{1/p} \\
&\qquad\qquad \text{by Hölder's inequality on the inner product} \\
&= |\det b|^{1/q} \left(\int_{\mathbb{R}^d} |f(y)|^p P|\phi|(a_j y) \, dy \right)^{1/p} \|\phi\|_1^{1/q} \\
&\le |\det b|^{1/q} \|P|\phi|\|_\infty \|f\|_p,
\end{aligned}$$

using that $\|\phi\|_1 = \int_{b\mathcal{C}} \sum_{k \in \mathbb{Z}^d} |\phi(x - bk)| \, dx \le \|P|\phi|\|_\infty$.

When $p = \infty$, the proof is straightforward. □

Aside. For the special case $p = 2$, Proposition 16 is known [10, Theorem 7.2.3] for all $\phi \in L^2$ with $P(|\widehat{\phi}|^2) \in L^\infty$, and one can show this condition is weaker than $P|\phi| \in L^\infty$. That is, analysis is bounded from L^2 to ℓ^2 with norm $\|T_j\| \le \|P(|\widehat{\phi}|^2)\|_\infty^{1/2}$, where the periodization is with respect to the lattice $\mathbb{Z}^d b^{-1}$. This is another way of saying the translates of ϕ form a Bessel sequence, or satisfy an upper frame bound.

8.3. **Scale-averaged approximation in L^p.**

The following approximation result will be used in proving Theorem 2 (surjectivity of the synthesis operator). The result is interesting in its own right too, due to its explicit nature: we simply analyze with ϕ, then synthesize with ψ, and then average over all dilation scales to recover f.

Say that the dilations *expand exponentially* if

$$\|a_j a_{j+1}^{-1}\| \le \delta \qquad \text{for all } j > 0,$$

for some $0 < \delta < 1$. In one dimension, this means $|a_j|$ is a lacunary sequence.

Theorem 17 ([6, Theorem 1 and Lemma 2])**.** *Assume $1 \le p < \infty$ and $\psi \in L^p, P|\psi| \in L^p_{loc}, \phi \in L^q, P|\phi| \in L^\infty, f \in L^p$. Assume $\int_{\mathbb{R}^d} \phi\, dx = 1$, and write $\gamma = \int_{\mathbb{R}^d} \psi\, dx$.*
(a) [Constant periodization] *If $P\psi = \gamma$ a.e. then*

$$S_j T_j f \to \gamma f \qquad \text{in } L^p \text{ as } j \to \infty.$$

(b) [Scale-averaged approximation] *If the dilations a_j expand exponentially, then*

$$\frac{1}{J}\sum_{j=1}^{J} S_j T_j f \to \gamma f \qquad \text{in } L^p \text{ as } J \to \infty.$$

The scale averaged approximation in part (b) can be written in full as

$$\frac{1}{J}\sum_{j=1}^{J} |\det b| \left(\sum_{k\in\mathbb{Z}^d} \langle f, \phi_{j,k}\rangle \psi_{j,k} \right) \to \gamma f \quad \text{in } L^p, \text{ as } J \to \infty.$$

Aside. Part (a) of the theorem has a long history, summarized in [6, §3].

8.4. **Proof of Theorem 2 — synthesis onto L^p.**

We can assume the dilations expand exponentially, as follows. For each j we have $\|a_j a_{j+r}^{-1}\| \le \|a_j\|\|a_{j+r}^{-1}\| \to 0$ as $r \to \infty$ because the dilations are expanding. Thus $\|a_j a_{j+r}^{-1}\| \le 1/2$ provided we choose r sufficiently large. By iterating this argument we arrive at a subsequence of dilations that expands exponentially (with $\delta = 1/2$). It is enough to use only this subsequence of dilations, when proving openness and surjectivity of the synthesis operator.

Consider $f \in L^p$ and $J \in \mathbb{N}$, and define a sequence $c_J = \{c_{J;j,k}\}_{j>0,k\in\mathbb{Z}^d}$ by

$$c_{J;j,k} = \frac{1}{J}|\det b| \begin{cases} \langle f, \phi_{j,k}\rangle & \text{for } j = 1, \dots, J, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $\phi = |b\mathcal{C}|^{-1}\mathbb{1}_{b\mathcal{C}}$ is a normalized indicator function. Note $P|\phi| \equiv 1$. Then $c_J \in \ell^1(\ell^p)$ because applying Proposition 16 for each $j = 1, \dots, J$ gives that

$$\|c_J\|_{\ell^1(\ell^p)} \le \frac{1}{J}\sum_{j=1}^{J} |\det b|^{1/q}\|f\|_p = |\det b|^{1/q}\|f\|_p.$$

And clearly

$$\begin{aligned} Sc_J &= \frac{1}{J}\sum_{j=1}^{J} S_j T_j f \\ &\to f \qquad\qquad \text{in } L^p \text{ as } J \to \infty, \end{aligned}$$

by Theorem 17(b). (Here we use that the a_j expand exponentially.) Thus the open mapping theorem in Appendix A says $S : \ell^1(\ell^p) \to L^p$ is open and surjective, and that for each $f \in L^p$ and $\varepsilon > 0$ there exists $c \in \ell^1(\ell^p)$ with $Sc = f$ and $\|c\|_{\ell^1(\ell^p)} \le |\det b|^{1/q}\|f\|_p + \varepsilon$.

8.5. **Proof of Corollary 5 — Synthesis norm for $L^p(\Omega)$.**

The "$\le$" direction of the norm equivalence follows immediately from Corollary 3, since we are restricting the collection of sequences c that can be used to represent f.

For the "$\ge$" direction, we will modify the proof of Theorem 2. Like in that theorem we can assume the dilations expand exponentially, by passing to a subsequence of j-values.

Suppose $f \in L^p(\Omega)$ is supported on a compact subset of Ω. We claim there exists $j_0 \ge 0$ such that

$$\operatorname{spt}(\psi_{j,k}) \subset \Omega \text{ whenever } j > j_0 \text{ and } \langle f, \phi_{j,k}\rangle \neq 0, \tag{13}$$

where $\phi = |b\mathcal{C}|^{-1}\mathbb{1}_{b\mathcal{C}}$ as previously. The existence of j_0 should be clear intuitively, since the support of f lies at some positive distance from the boundary of Ω while ψ and ϕ have compact support and $\|a_j^{-1}\| \to 0$. We leave the detailed proof of (13) to the reader.

Next, consider $J \in \mathbb{N}$ and define

$$c_{J;j,k} = \frac{1}{J}|\det b| \begin{cases} \langle f, \phi_{j,k}\rangle & \text{for } j = j_0+1, \dots, j_0+J, \\ 0 & \text{otherwise,} \end{cases}$$

so that $c_J \in \ell^1(\ell^p)$ by Proposition 16, with

$$\|c_J\|_{\ell^1(\ell^p)} \le \frac{1}{J}\sum_{j=j_0+1}^{j_0+J} |\det b|^{1/q}\|f\|_p = |\det b|^{1/q}\|f\|_p. \tag{14}$$

We know $Sc_J \to f$ in $L^p(\mathbb{R}^d)$ as $J \to \infty$, by Theorem 17(b) with an index shift on the dilations.

Define

$$\mathcal{L} = \{c \in \ell^1(\ell^p) : c \text{ is adapted to } \Omega \text{ and } \psi\}.$$

Clearly $\mathcal{L}$ is a closed subspace of $\ell^1(\ell^p)$, and hence is a Banach space. Note $c_J \in \mathcal{L}$ by (13), so that $Sc = 0$ on Ω^c.

We have verified the hypotheses of the open mapping theorem in Appendix A for $S : \mathcal{L} \to L^p(\Omega)$, with the constant $A = |\det b|^{1/q}$ by (14). Admittedly we have verified the hypotheses only for the dense class of f having compact support in Ω, but a dense class suffices, by the comment at the end of Appendix A. The open mapping theorem tells us $S : \mathcal{L} \to L^p(\Omega)$ is surjective, and that for each $f \in L^p(\Omega)$ and $\varepsilon > 0$ there exists $c \in \mathcal{L}$ with $Sc = f$ and $\|c\|_{\ell^1(\ell^p)} \le |\det b|^{1/q}\|f\|_p + \varepsilon$. This proves the "$\ge$" direction of the corollary.

8.6. **Proof of Corollary 6 — analysis norm for L^p.**

By boundedness of the analysis operator in Proposition 16,

$$\sup_j \|T_j f\|_{\ell^p} \le |\det b|^{1/q} \| P|\phi|\|_\infty \|f\|_p.$$

To prove the other direction of the norm equivalence, choose a function ψ that satisfies the hypotheses of Theorem 17(a) with $\gamma = 1$. Then by that theorem, $S_j T_j f \to f$ in L^p as $j \to \infty$. Therefore it follows from Theorem 1 (bounded synthesis) that

$$\|f\|_p \le \sup_j \|S_j T_j f\|_p \le |\det b|^{-1} \| P|\psi|\|_{L^p(b\mathcal{C})} \sup_j \|T_j f\|_{\ell^p},$$

which proves the corollary.

9. Hardy space proofs

In this section we assume the dilations are isotropic and expanding, so that $a_j = \alpha_j I$ for some nonzero real numbers α_j with $|\alpha_j| \to \infty$.

9.1. **Proof of Theorem 7 — synthesis $\ell^1(h^1) \to H^1$.**

We begin by showing that the Riesz transform "almost" commutes with affine synthesis, which will be the key step in proving boundedness of the synthesis operator, in Theorem 7.

Lemma 18 (Riesz transforms commute with synthesis)**.** *Suppose $\psi \in L^1$. Take ν to be the smooth, compactly supported cut-off function used to define z in formula* (5)*. Let μ be the Schwartz function with $\widehat{\mu}(\xi) = \nu(\xi b)$, and let λ be a Schwartz function with $\widehat{\lambda}$ supported in $\mathcal{C}_0 b^{-1}$ and with $\widehat{\lambda}(0) = 1$.*

If $s \in h^1$ then

$$Z * \Big(\sum_{k\in\mathbb{Z}^d} s_k(\psi * \lambda * \mu)(x - bk)\Big) = \sum_{k\in\mathbb{Z}^d} (z * s)_k (\psi * \lambda)(x - bk) \in L^1.$$

We can rephrase the lemma (after rescaling $x \mapsto \alpha_j x$ and using the dilation invariance of the Riesz transform) as saying

$$R S_{j,\psi*\lambda*\mu} s = \operatorname{sign}(\alpha_j) S_{j,\psi*\lambda} R s, \qquad s \in h^1,$$

where on the lefthand side R denotes the continuous Riesz transform (convolution with the Riesz kernel Z) and on the righthand side R denotes the discrete Riesz transform (convolution with the discrete Riesz kernel z). Thus we see Lemma 18 is a discrete analogue of the formula $R(\psi * f) = \psi * (Rf)$, once we remember that affine synthesis is a discrete analogue of convolution with a synthesizer.

Proof of Lemma 18. Define $f(x) = \sum_{k\in\mathbb{Z}^d} s_k(\psi * \lambda * \mu)(x - bk)$. This sum converges absolutely in L^1, because

$$\|f\|_1 \le \sum_{k\in\mathbb{Z}^d} |s_k| \|\psi * \lambda * \mu(\cdot - bk)\|_1 = \|s\|_{\ell^1} \|\psi * \lambda * \mu\|_1.$$

Our task is to show $f \in H^1$, with its Riesz transform being as stated in the lemma.

Consider the periodic functions

$$\sigma(\xi) = \sum_{k\in\mathbb{Z}^d} s_k e^{-2\pi i \xi k}, \qquad \zeta(\xi) = \sum_{k\in\mathbb{Z}^d} z_k e^{-2\pi i \xi k},$$

which are well defined since $s \in \ell^1$ and $z \in \ell^2$. We have

$$\begin{aligned} -i\frac{\xi}{|\xi|}\widehat{f}(\xi) &= -i\frac{\xi}{|\xi|}\sigma(\xi b)\widehat{\psi}(\xi)\widehat{\lambda}(\xi)\widehat{\mu}(\xi) \\ &= \zeta(\xi b)\sigma(\xi b)\widehat{\psi}(\xi)\widehat{\lambda}(\xi) \end{aligned} \tag{15}$$

by definition of the Riesz kernel sequence z in (5), using here that $\widehat{\lambda}(\xi) = 0$ when $\xi b \notin \mathcal{C}_0$.

Of course $\zeta(\xi)\sigma(\xi) = \sum_{k\in\mathbb{Z}^d}(z*s)_k e^{-2\pi i\xi k}$ in $L^2(\mathcal{C}_0)$, by computing Fourier coefficients of the two sides in $L^2(\mathcal{C}_0)$ and using $z \in \ell^2, s \in \ell^1, z*s \in \ell^2$. Therefore (15) says

$$-i\frac{\xi}{|\xi|}\widehat{f}(\xi) = \text{Fourier transform of } \sum_{k\in\mathbb{Z}^d}(z*s)_k(\psi*\lambda)(x-bk),$$

where we note that $z*s \in \ell^1$ by the hypothesis $s \in h^1$. Thus $(Z*f)(x) = \sum_{k\in\mathbb{Z}^d}(z*s)_k(\psi*\lambda)(x-bk) \in L^1$. This proves the lemma. □

Proof of Theorem 7. Fix $j > 0$ and take $s \in h^1$. Define

$$f = \sum_{k\in\mathbb{Z}^d} s_k \psi_{j,k}.$$

We will show that $f \in H^1$ with $\|f\|_{H^1} \le C\|s\|_{h^1}$, where $C = C(\psi, b)$ is independent of j. Then Theorem 7 follows by summing over j.

It is enough to show that the function

$$g(x) = (M_{a_j}^{-1} f)(x) = \sum_{k\in\mathbb{Z}^d} s_k \psi(x - bk)$$

belongs to H^1 with $\|g\|_{H^1} \le C\|s\|_{h^1}$, because $\|f\|_{H^1} = \|g\|_{H^1}$ (using that $a_j = \alpha_j I$ is isotropic). Obviously $\|g\|_1 \le \|s\|_{\ell^1}\|\psi\|_1$, and so our task is to show $\|Rg\|_1 \le C\|s\|_{h^1}$.

To understand Rg, take ν, μ and λ as in Lemma 18, and decompose

$$\begin{aligned} g &= g_1 + g_2 \qquad \text{where} \\ g_1(x) &= \sum_{k\in\mathbb{Z}^d} s_k(\psi*\lambda*\mu)(x-bk), \\ g_2(x) &= \sum_{k\in\mathbb{Z}^d} s_k(\psi - \psi*\lambda*\mu)(x-bk). \end{aligned}$$

Lemma 18 implies

$$\|Rg_1\|_1 \le \|z*s\|_{\ell^1}\|\psi*\lambda\|_1 \le \|s\|_{h^1}\|\psi*\lambda\|_1.$$

(Thus we see Lemma 18 is used to push the Riesz transform onto the *coefficient* sequence s, which belongs to h^1, rather than onto the synthesizer $\psi * \lambda * \mu$, which does not belong to H^1.)

The sum defining g_2 converges absolutely in H^1, because $s \in h^1 \subset \ell^1$ by assumption and $\psi - \psi * \lambda * \mu \in H^1$ by Lemma 19 below (which is where we use the hypothesis (6)). Hence

$$\|Rg_2\|_1 \le \|s\|_{\ell^1} \|R(\psi - \psi * \lambda * \mu)\|_1.$$

These bounds prove Theorem 7.

We must still prove Lemma 19, needed to treat g_2 in the "smoothing" step of the proof above. We use the translation operator $\tau_y \psi = \psi(\cdot - y)$.

Lemma 19. *Suppose $\psi \in L^1$ and $\|R(\psi - \tau_y \psi)\|_1 \le C < \infty$ for all $|y| \le 1$, and $\lambda \in L^1$ with $|y|\lambda(y) \in L^1$ and $\int_{\mathbb{R}^d} \lambda(y)\, dy = 1$. Then*

$$\|R(\psi - \psi * \lambda)\|_1 \le C \int_{\mathbb{R}^d} (|y| + 1)|\lambda(y)|\, dy.$$

The assumption that $\|R(\psi - \tau_y \psi)\|_1 \le C$ for $|y| \le 1$ is a restatement of hypothesis (6).

Proof of Lemma 19. The first step is to show the Hardy norm of a difference grows at most linearly with the difference step, that is

$$\|R(\psi - \tau_y \psi)\|_{L^1} \le C(|y| + 1) \qquad \text{for all } y \in \mathbb{R}^d. \tag{16}$$

For this, let m be the integer satisfying $|y| < m \le |y| + 1$. After writing y as a sum of m vectors each having norm less than 1, we can prove (16) with Cm on the righthand side by telescoping the differences and using the triangle inequality, noting the Riesz transform is translation invariant.

Now observe the function

$$\psi(x) - (\psi * \lambda)(x) = \int_{\mathbb{R}^d} (\psi(x) - \tau_y \psi(x))\lambda(y)\, dy$$

belongs to H^1 and has Riesz transform

$$R(\psi - \psi * \lambda)(x) = \int_{\mathbb{R}^d} R(\psi - \tau_y \psi)(x)\lambda(y)\, dy,$$

by [7, Lemma 10]. That is, one can take the Riesz transform through the integral. Thus

$$\begin{aligned} \|R(\psi - \psi * \lambda)\|_1 &\le \int_{\mathbb{R}^d} \|R(\psi - \tau_y \psi)\|_1 |\lambda(y)|\, dy \\ &\le \int_{\mathbb{R}^d} C(|y| + 1)|\lambda(y)|\, dy \end{aligned}$$

by (16), as desired. □

9.2. **Analysis $H^1 \to h^1$.**

For the proof of Theorem 8 we want boundedness of the analysis operator T_j from H^1 to h^1, which we state in Proposition 21 below. First we show:

Lemma 20 (Riesz transforms commute with analysis)**.** *Assume the analyzer ϕ is a Schwartz function with $\widehat{\phi}$ supported in $\mathcal{C}_0 b^{-1}$. Fix $j > 0$. Choose ν as in the definition of h^1 in Section 4, and let μ_j be the Schwartz function with $\widehat{\mu_j}(\xi) = \nu(\xi a_j^{-1} b)$.*

If $f \in H^1$ then

$$RT_j f = \operatorname{sign}(\alpha_j) T_j R(\mu_j * f),$$

where on the lefthand side R denotes the discrete Riesz transform (convolution with the discrete Riesz kernel z) and on the righthand side R denotes the continuous Riesz transform (convolution with the Riesz kernel Z).

Remember α_j denotes the isotropic dilation factor in $a_j = \alpha_j I$.

Proof of Lemma 20. Observe $\phi_{j,k}(x) = \phi(a_j x - bk)$ in what follows, because we implicitly assume $p = 1, q = \infty$, wherever we deal with the Hardy space.

The k-th term of the sequence $RT_j f$ is

$$\begin{aligned}(z * T_j f)_k &= |\det b| \sum_{\ell \in \mathbb{Z}^d} z_\ell \langle f, \phi_{j,k-\ell} \rangle \\ &= |\det b| \sum_{\ell \in \mathbb{Z}^d} \int_{\mathcal{C}_0 b^{-1}} z_\ell e^{-2\pi i \xi b \ell} \widehat{f}(\xi a_j) \overline{\widehat{\phi}(\xi) e^{-2\pi i \xi b k}}\, d\xi,\end{aligned}$$

by Plancherel and the compact support of $\widehat{\phi}$. Substituting in the definition of ζ from (5) and then using the definition of μ_j, we find

$$\begin{aligned}&(z * T_j f)_k \\ &= |\det b| \int_{\mathbb{R}^d} \frac{-i\xi}{|\xi|} \nu(\xi b) \widehat{f}(\xi a_j) \overline{\widehat{\phi}(\xi) e^{-2\pi i \xi b k}}\, d\xi \\ &= |\det b| \int_{\mathbb{R}^d} \operatorname{sign}(\alpha_j) (\mu_j * Rf)\widehat{\ }(\xi a_j) \overline{\widehat{\phi}(\xi) e^{-2\pi i \xi b k}}\, d\xi \qquad \text{since } a_j = \alpha_j I \\ &= |\det b| \operatorname{sign}(\alpha_j) \langle R(\mu_j * f), \phi_{j,k} \rangle\end{aligned}$$

by Plancherel, and this is the k-th term of $\operatorname{sign}(\alpha_j) T_j R(\mu_j * f)$, as desired. □

Proposition 21 (Analysis into h^1)**.** *Take ϕ and ν as in Lemma 20. Then for each j,*

$$T_j : H^1 \to h^1 \qquad \textit{with norm } \|T_j\| \le \| P|\phi| \|_\infty \|\check{\nu}\|_1.$$

Proof of Proposition 21. If $f \in H^1$ then

$$\|T_j f\|_{\ell^1} \le \| P|\phi| \|_\infty \|f\|_1$$

by Proposition 16 with $p = 1$, while

$$\begin{aligned}\|z * T_j f\|_{\ell^1} = \|RT_j f\|_{\ell^1} &\le \| P|\phi| \|_\infty \|R(\mu_j * f)\|_1 \\ &\le \| P|\phi| \|_\infty \|\mu_j\|_1 \|Rf\|_1\end{aligned}$$

by combining Lemma 20 and Proposition 16. Add these two estimates and observe $\|\mu_j\|_1 = \|\check{\nu}\|_1 \geq |\nu(0)| = 1$, by definition of $\widehat{\mu_j}$ in Lemma 20. $\square$

Aside. Our compact support assumption on the Fourier transform of the analyzer ϕ, in Lemma 20 and Proposition 21, seems rather strong. Perhaps it can be weakened. But notice it ensures that $\widehat{\phi}(\ell b^{-1}) = 0$ for all row vectors $\ell \in \mathbb{Z}^d \setminus \{0\}$, which implies $P\phi \equiv$ const. This constant periodization condition is necessary, as follows. If T_j maps H^1 into h^1, then for all $f \in H^1$ we have

$$\int_{\mathbb{R}^d} f(x)\overline{P\phi(a_j x)}\,dx = \sum_{k\in\mathbb{Z}^d} (T_j f)_k = 0$$

by the zero-mean property of $T_j f \in h^1$ (see Appendix B). Taking $f \in H^1$ to approach a difference of delta functions implies $P\phi$ is constant.

9.3. **Scale-averaged approximation in H^1.**

In this section we prove scale-averaged convergence in H^1, which we need for the proof of Theorem 8.

Theorem 22. *Assume $\psi \in L^1$ with $\int_{\mathbb{R}^d} \psi\,dx = 1$ and*

$$\|\psi - \psi(\cdot - y)\|_{H^1} \to 0 \qquad \text{as } y \to 0, \tag{17}$$

which is hypothesis (8)*. Assume ϕ is a Schwartz function with $\widehat{\phi}$ supported in $\mathcal{C}_0 b^{-1}$ and $\int_{\mathbb{R}^d} \phi\,dx = 1$. Let $f \in H^1$.*
(a) [Constant periodization] *If $P\psi = 1$ a.e. then*

$$S_j T_j f \to f \qquad \text{in } H^1 \text{ as } j \to \infty.$$

(b) [Scale-averaged approximation] *If the dilations a_j expand exponentially, then*

$$\frac{1}{J}\sum_{j=1}^{J} S_j T_j f \to f \qquad \text{in } H^1 \text{ as } J \to \infty.$$

The theorem was proved in our earlier paper [7, Theorem 1] for analyzers ϕ with compact support and $P\phi \equiv$ const., by comparing $S_j T_j f$ with an approximate identity formula. The new proof below is more conceptually satisfying, as it is based on commuting the Riesz transform through the analysis and synthesis operators.

Proof of Theorem 22. Convergence in L^1, for parts (a) and (b), follows immediately from our L^p result Theorem 17, with $p = 1$.

To prove convergence in H^1, we want to show

$$RS_j T_j f \to Rf \quad \text{in } L^1, \text{ for part (a), and} \tag{18}$$

$$\frac{1}{J}\sum_{j=1}^{J} RS_j T_j f \to Rf \quad \text{in } L^1, \text{ for part (b).} \tag{19}$$

To begin, suppose $\widehat{\mu}(\xi) = \nu(\xi b)$ and λ are as in Lemma 18, and decompose

$$\begin{aligned} S_j T_j f = S_{j,\psi} T_j f &= S_{j,\psi - \psi*\lambda*\mu} T_j f + S_{j,\psi*\lambda*\mu} T_j f \\ &= A_j + B_j, \quad \text{say,} \end{aligned}$$

where we write $S_{j,\psi}$ and so on to emphasize the synthesizer being used, in each part of the formula. We have

$$\begin{aligned} RB_j &= \operatorname{sign}(\alpha_j) S_{j,\psi*\lambda} R T_j f \qquad \text{by Lemma 18} \\ &= S_{j,\psi*\lambda} T_j R(\mu_j * f) \qquad \text{by Lemma 20} \\ &= S_{j,\psi*\lambda} T_j (\mu_j * (Rf) - Rf) + S_{j,\psi*\lambda} T_j R f \\ &= C_j + D_j, \quad \text{say.} \end{aligned}$$

We estimate C_j by

$$\begin{aligned} \|C_j\|_1 &\le (\text{const.}) \|\mu_j * (Rf) - Rf\|_1 \qquad \text{by Theorem 1 and Proposition 16} \\ &\to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

since $\mu_j(x) = |\det b^{-1} a_j| \breve{\nu}(b^{-1} a_j x)$ is an approximation to the identity (recalling $a_j = \alpha_j I$ with $|\alpha_j| \to \infty$).

Theorem 17 implies $D_j \to Rf$ in L^1 in part (a), and implies in part (b) that $\frac{1}{J} \sum_{j=1}^{J} D_j \to Rf$ in L^1.

Thus to prove (18)–(19), it suffices to show $RA_j \to 0$ in part (a), and that $\frac{1}{J} \sum_{j=1}^{J} RA_j \to 0$ in part (b). To accomplish this, first compute

$$\begin{aligned} RA_j &= R S_{j,\psi - \psi*\lambda*\mu} T_j f \\ &= \operatorname{sign}(\alpha_j) S_{j,R(\psi - \psi*\lambda*\mu)} T_j f, \end{aligned} \tag{20}$$

where it is permissible here to pass the Riesz transform through the synthesis operator because the series for $S_{j,\psi-\psi*\lambda*\mu} T_j f$ converges absolutely in H^1: the coefficient sequence $T_j f$ belongs to ℓ^1 by Proposition 16, and the synthesizer $\psi - \psi*\lambda*\mu$ belongs to H^1 by Lemma 19 (noting (17) implies (6) by [7, §3.4]).

Next notice $\int_{\mathbb{R}^d} R(\psi - \psi * \lambda * \mu)\, dx = 0$, since all Hardy space functions and their Riesz transforms integrate to zero; cf. (4). Thus in part (b) of the theorem we deduce that $\frac{1}{J} \sum_{j=1}^{J} RA_j \to 0$, from (20) and Theorem 17(b) (and also splitting the sum $\sum_{j=1}^{J}$ into two pieces, where $\alpha_j > 0$ and $\alpha_j < 0$ respectively).

In part (a) of the theorem we deduce that $RA_j \to 0$, by using (20) and Theorem 17(a), and the following observation. If $P\psi = 1$ a.e., then by computing the Fourier coefficients of $P\psi$ we find $\widehat{\psi}(\ell b^{-1}) = 0$ for all $\ell \in \mathbb{Z}^d \setminus \{0\}$, and thus

$$(R(\psi - \psi * \lambda * \mu))\widehat{\ }(\ell b^{-1}) = 0. \tag{21}$$

Of course (21) holds for $\ell = 0$ too, as observed in the preceding paragraph. Hence $PR(\psi - \psi * \lambda * \mu) = 0$ a.e., by computing the Fourier coefficients of this periodic function.

This finishes the proof. □

9.4. **Proof of Theorem 8 — synthesis onto H^1.**

We can assume the dilations expand exponentially, like we did in the proof of Theorem 2.

Let $f \in H^1$ and $J \in \mathbb{N}$, and define the sequence c_J by (12) where now ϕ is assumed to be a Schwartz function with $\widehat{\phi}$ supported in $\mathcal{C}_0 b^{-1}$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$. Then $c_J \in \ell^1(h^1)$ by applying Proposition 21 for each $j = 1, \dots, J$, giving

$$\|c_J\|_{\ell^1(h^1)} \le \frac{1}{J} \sum_{j=1}^{J} \|P|\phi|\|_\infty \|\widetilde{\nu}\|_1 \|f\|_{H^1} = \|P|\phi|\|_\infty \|\widetilde{\nu}\|_1 \|f\|_{H^1}. \tag{22}$$

Observe $Sc_J = \frac{1}{J} \sum_{j=1}^{J} S_j T_j f \to f$ in H^1 as $J \to \infty$ by Theorem 22. (This is where the hypotheses on ψ are needed, and that the dilations expand exponentially.) Combining this with (22), we see the open mapping theorem in Appendix A implies S is open, and that for each $\varepsilon > 0$ there exists $c \in \ell^1(h^1)$ with $Sc = f$ and $\|c\|_{\ell^1(h^1)} \le \|P|\phi|\|_\infty \|\widetilde{\nu}\|_1 \|f\|_{H^1} + \varepsilon$. Choosing $C = \|P|\phi|\|_\infty \|\widetilde{\nu}\|_1$ proves the theorem.

9.5. **Proof of Corollary 10 — analysis norm for H^1.**

Take ν as in the definition of h^1 in Section 4. Then by boundedness of the analysis operator in Proposition 21,

$$\sup_j \|T_j f\|_{h^1} \le \|P|\phi|\|_\infty \|\widetilde{\nu}\|_1 \|f\|_{H^1}.$$

To prove the other direction of the equivalence, choose ψ to be a Schwartz function with $P\psi \equiv 1$. Then $S_j T_j f \to f$ in H^1 as $j \to \infty$ by Theorem 22(a), noting that hypothesis (17) is known to hold for the Schwartz function ψ (cf. [7, §3.3]). Therefore it follows from Theorem 7 (bounded synthesis) that

$$\|f\|_{H^1} \le \sup_j \|S_j T_j f\|_{H^1} \le C \sup_j \|T_j f\|_{h^1},$$

where C is independent of f.

10. Sobolev space proofs

Throughout this section we assume the dilations are isotropic and expanding, meaning $a_j = \alpha_j I$ for some nonzero real numbers α_j with $|\alpha_j| \to \infty$.

We introduce the notation g^{*r} for the convolution of a function g with itself r times (for example, $g^{*2} = g * g$), and we write

$$\Delta_y g = g - g(\cdot - y), \qquad y \in \mathbb{R}^d,$$

for the backwards difference of g by y. Also we define

$$\beta_1(x) = \delta(x_1) \mathbb{1}_{[0,1)}(x_2) \cdots \mathbb{1}_{[0,1)}(x_d)$$

and similarly define functions $\beta_2, \dots, \beta_d$, so that the partial derivatives of the unit cube indicator function are

$$D_t \mathbb{1}_{\mathcal{C}} = \Delta_{e_t} \beta_t, \qquad t = 1, \dots, d.$$

10.1. **Proof of Theorem 11 — synthesis $\ell^1(w^{m,p},\alpha)\to W^{m,p}$.**

First assume $b=I$, so that $\beta=\mathbb{1}_{\mathcal{C}}$. Fix $c\in\ell^1(w^{m,p},\alpha)$.

The initial task is to show that $Sc\in W^{m,p}$. By differentiating the definition (9) of ψ we convert derivatives to differences:

$$\begin{aligned} D^\rho\psi &= (D_1\mathbb{1}_{\mathcal{C}})^{*\rho_1}*\cdots*(D_d\mathbb{1}_{\mathcal{C}})^{*\rho_d}*\beta^{*\,m-|\rho|}*\eta \\ &= \Delta_{e_1}^{\rho_1}\cdots\Delta_{e_d}^{\rho_d}(\beta_1^{*\rho_1}*\cdots*\beta_d^{*\rho_d}*\beta^{*\,m-|\rho|}*\eta)\in L^p. \end{aligned} \tag{23}$$

Obviously (23) implies that $\psi\in W^{m,p}$. Further, since $P|\eta|\in L^p_{loc}$ by hypothesis, we have

$$\begin{aligned} P|\beta_1^{*\rho_1}*\cdots*\beta_d^{*\rho_d}*\beta^{*\,m-|\rho|}*\eta| &\le \beta_1^{*\rho_1}*\cdots*\beta_d^{*\rho_d}*\beta^{*\,m-|\rho|}*P|\eta| \\ &\in L^p_{loc}, \end{aligned} \tag{24}$$

and therefore $P|D^\rho\psi|\in L^p_{loc}$ by (23). This allows us to use $D^\rho\psi$ as a synthesizer when applying Theorem 1, below.

It is straightforward to show that Sc (which belongs to L^p by Theorem 1) has weak derivatives given by "differentiating through the sum", namely

$$D^\rho(S_\psi c)=S_{D^\rho\psi}(\alpha^{|\rho|}c). \tag{25}$$

Note that the righthand side belongs to L^p by Theorem 1, since $c\in\ell^1(w^{m,p},\alpha)$ ensures $\alpha^{|\rho|}c\in\ell^1(\ell^p)$. Thus the function Sc belongs to $W^{m,p}$, completing the first task in the proof.

The next task is to prove that

derivatives commute with synthesis,

in the sense that

$$D^\rho S_\psi c = S_{\eta_\rho}(\alpha^{|\rho|}\Delta^\rho c), \tag{26}$$

where we have introduced the function

$$\eta_\rho=\beta_1^{*\rho_1}*\cdots*\beta_d^{*\rho_d}*\beta^{*\,m-|\rho|}*\eta$$

(so that for example, $\eta_0=\psi$). Indeed

$$\begin{aligned} (D^\rho S_\psi c)(x) &= \sum_{j>0}\sum_{k\in\mathbb{Z}^d}\alpha_j^{|\rho|}c_{j,k}(D^\rho\psi)_{j,k}(x) \quad \text{by (25)} \\ &= \sum_{j>0}\alpha_j^{|\rho|}\sum_{k\in\mathbb{Z}^d}c_{j,k}|\det a_j|^{1/p}(\Delta_{e_1}^{\rho_1}\cdots\Delta_{e_d}^{\rho_d}\eta_\rho)(a_jx-k) \quad \text{by (23)} \\ &= \sum_{j>0}\alpha_j^{|\rho|}\sum_{k\in\mathbb{Z}^d}(\Delta^\rho c)_{j,k}|\det a_j|^{1/p}\eta_\rho(a_jx-k) \qquad (27) \\ &\qquad\qquad \text{by summation by parts, the key step in the proof,} \\ &= S_{\eta_\rho}(\alpha^{|\rho|}\Delta^\rho c)(x), \end{aligned}$$

which proves (26).

We will deduce an estimate on the synthesis operator of the form

$$\|Sc\|_{W^{m,p}}\le\|P|\eta|\|_{L^p(\mathcal{C})}\|c\|_{\ell^1(w^{m,p},\alpha)}, \tag{28}$$

which completes the proof of the theorem when $b = I$. Start by observing

$$\begin{aligned}\|Sc\|_{W^{m,p}} &= \sum_{|\rho|\le m} \|D^\rho Sc\|_p \\ &= \sum_{|\rho|\le m} \|S_{\eta_\rho}(\alpha^{|\rho|}\Delta^\rho c)\|_p \qquad \text{by (26)} \\ &\le \sum_{|\rho|\le m} \||P|\eta_\rho|\|_{L^p(\mathcal{C})} \|\alpha^{|\rho|}\Delta^\rho c\|_{\ell^1(\ell^p)}\end{aligned}$$

by Theorem 1 (boundedness of synthesis into L^p). To complete the proof of (28), notice

$$\||P|\eta_\rho|\|_{L^p(\mathcal{C})} \le \||P|\eta|\|_{L^p(\mathcal{C})}$$

by (24).

To prove the theorem when $b \ne I$, first rescale the definition (9) of ψ to obtain that

$$M_b\psi = \overbrace{\mathbb{1}_{\mathcal{C}} * \cdots * \mathbb{1}_{\mathcal{C}}}^{m \text{ factors}} * M_b\eta.$$

That is, (9) holds with $M_b\psi, M_b\eta$ and I instead of ψ, η and b, so that $M_b\psi$ and $M_b\eta$ satisfy the hypotheses of the theorem for "$b = I$".

By the "$b = I$" case of the theorem already proved, then, we have $M_b\psi \in W^{m,p}$ (so that $\psi \in W^{m,p}$), and for each sequence $c \in \ell^1(w^{m,p}, \alpha)$ we have $S_{M_b\psi,I}c \in W^{m,p}$ with norm estimate

$$\|S_{M_b\psi,I}c\|_{W^{m,p}} \le \||P_I|M_b\eta|\|_{L^p(\mathcal{C})} \|c\|_{\ell^1(w^{m,p},\alpha)}.$$

Further,

$$\begin{aligned}(S_{M_b\psi,I}c)(x) &= \sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}|\det a_j|^{1/p}(M_b\psi)(a_jx - k) \\ &= \sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}|\det a_j|^{1/p}|\det b|\psi(b(a_jx - k)) \\ &= |\det b|\sum_{j>0}\sum_{k\in\mathbb{Z}^d} c_{j,k}|\det a_j|^{1/p}\psi(a_jbx - bk) \\ &\qquad\qquad \text{(noting } b \text{ commutes with } a_j = \alpha_j I) \\ &= (M_bS_{\psi,b}c)(x). \qquad (29)\end{aligned}$$

Hence

$$\begin{aligned}\|M_bS_{\psi,b}c\|_{W^{m,p}} &= \|S_{M_b\psi,I}c\|_{W^{m,p}} \\ &\le \||P_I|M_b\eta|\|_{L^p(\mathcal{C})}\|c\|_{\ell^1(w^{m,p},\alpha)} \\ &= |\det b|^{-1/p}\||P|\eta|\|_{L^p(b\mathcal{C})}\|c\|_{\ell^1(w^{m,p},\alpha)},\end{aligned}$$

which finishes the proof.

10.2. **Analysis $W^{m,p} \to w^{m,p}$.**

The proof of Theorem 12 relies on boundedness of analyzers acting on Sobolev space, as developed in the next proposition. For simplicity we assume $b = I$, so that the analysis operator at scale j (defined in Section 2) is just $(T_j f)_k = \langle f, \phi_{j,k} \rangle$.

Proposition 23 (Analysis into $w^{m,p}$)**.** *Assume $1 \le p \le \infty$ and $m \in \mathbb{N}$. Take $\phi \in L^q$ with $P|\phi| \in L^\infty$. Fix $j > 0$, and assume $b = I$.*

Then $T_j : W^{m,p} \to w^{m,p}$, with norm controlled by the estimate

$$\sum_{|\rho| \le m} |\alpha_j|^{|\rho|} \|\Delta^\rho T_j f\|_{\ell^p} \le \| P|\phi| \|_\infty \|f\|_{W^{m,p}}, \qquad f \in W^{m,p}.$$

Proof of Proposition 23. Let $f \in W^{m,p}$ and observe

$$(T_j f)_k = \langle f, \phi_{j,k} \rangle = \int_{\mathbb{R}^d} f(y + \alpha_j^{-1} k) |\det a_j|^{1/q} \overline{\phi(a_j y)}\, dy.$$

Hence for each multiindex ρ of order $\le m$ we have

$$\begin{aligned} \alpha_j^{|\rho|} \Delta^\rho (T_j f)_k &= \int_{\mathbb{R}^d} \alpha_j^{|\rho|} \Delta^\rho f(y + \alpha_j^{-1} k) |\det a_j|^{1/q} \overline{\phi(a_j y)}\, dy \\ &= \int_{\mathbb{R}^d} (\Delta^\rho_{\alpha_j} f)(y + \alpha_j^{-1} k) |\det a_j|^{1/q} \overline{\phi(a_j y)}\, dy \end{aligned}$$

where the function $\Delta^\rho_{\alpha_j} f$ in this last line denotes the ρ-th backwards difference quotient of f with step size α_j^{-1}. Changing variable with $y \mapsto y - \alpha_j^{-1} k$ gives

$$\alpha_j^{|\rho|} \Delta^\rho (T_j f)_k = \langle \Delta^\rho_{\alpha_j} f, \phi_{j,k} \rangle = (T_j \Delta^\rho_{\alpha_j} f)_k, \tag{30}$$

which says that

differences commute with the analysis operator.

Thus for each fixed j, taking the ℓ^p-norm with respect to k in (30) implies

$$\begin{aligned} \|\alpha_j^{|\rho|} \Delta^\rho T_j f\|_{\ell^p} &\le \| P|\phi| \|_\infty \|\Delta^\rho_{\alpha_j} f\|_p \qquad \text{by Proposition 16} \\ &\le \| P|\phi| \|_\infty \|D^\rho f\|_p, \end{aligned}$$

by using the fundamental theorem of calculus. Summing over $|\rho| \le m$ now proves the proposition. □

10.3. **Scale-averaged approximation in $W^{m,p}$.**

Here we prove scale-averaged convergence in $W^{m,p}$, which we need for the proof of Theorem 12. Just like for L^p and the Hardy space, the idea is to analyze with ϕ, then synthesize with ψ, and then average over all dilation scales.

Theorem 24. *Assume $1 \le p < \infty$ and $\eta \in L^p$ with $P|\eta| \in L^p_{loc}$ and $\int_{\mathbb{R}^d} \eta\, dx = 1$. Let $m \in \mathbb{N}$ and define ψ by* (9)*. Take $\phi \in L^q$ with $P|\phi| \in L^\infty$ and $\int_{\mathbb{R}^d} \phi\, dx = 1$. Assume $b = I$. Let $f \in W^{m,p}$.*

(a) [Constant periodization] *If $P\eta = 1$ a.e. then*

$$S_j T_j f \to f \qquad \text{in } W^{m,p} \text{ as } j \to \infty.$$

(b) [Scale-averaged approximation] *If the dilations a_j expand exponentially, then*

$$\frac{1}{J}\sum_{j=1}^{J} S_j T_j f \to f \qquad \text{in } W^{m,p} \text{ as } J \to \infty.$$

The theorem was proved already in our paper [8, Theorem 1], by comparing $S_j T_j f$ with an approximate identity formula. The proof we give below is considerably easier, and is based on passing derivatives and differences through the analysis and synthesis operators. On the other hand, our assumptions on the synthesizer ψ are noticeably stronger than the Strang–Fix type assumptions in [8], because here we assume ψ has the special convolution form (9). This is explained in more detail in [8, *Notes on Theorem 1*].

Theorem 24(a) is due to Di Guglielmo [29, Théorème $2'$], when η has compact support. Further in this direction, Strang–Fix theory establishes approximation rates of the firm $O(|\alpha_j|^{k-m})$ in $W^{k,p}$, for $k < m$, which improves on the rate $o(1)$ in Theorem 24(a). See Jia [31, Theorem 3.1], and our discussion of the literature in [8, §3.5] (where some results special to $p = 2$ are cited also).

Theorem 24(b) is needed below when proving Theorem 12.

Proof of Theorem 24. Let ρ be a multiindex of order $\le m$. Then

$$\begin{aligned} D^\rho S_j T_j f &= S_{j,\eta_\rho}(\alpha^{|\rho|}\Delta^\rho T_j f) && \text{by (26)} \\ &= S_{j,\eta_\rho} T_j \Delta^\rho_{\alpha_j} f && \text{by (30)} \\ &= S_{j,\eta_\rho} T_j (\Delta^\rho_{\alpha_j} f - D^\rho f) + S_{j,\eta_\rho} T_j D^\rho f \\ &= A_j + B_j, \qquad \text{say.} \end{aligned}$$

For A_j we have

$$\begin{aligned} \|A_j\|_p &\le \|P|\eta_\rho|\|_{L^p(\mathcal{C})} \|T_j(\Delta^\rho_{\alpha_j} f - D^\rho f)\|_{\ell^p} && \text{by Theorem 1} \\ &\le \|P|\eta_\rho|\|_{L^p(\mathcal{C})} \|P|\phi|\|_\infty \|\Delta^\rho_{\alpha_j} f - D^\rho f\|_p && \text{by Proposition 16} \\ &\to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

For B_j, we first note that if $P\eta = 1$ a.e. then $P\eta_\rho = 1$ a.e. Thus Theorem 17 implies in part (a) that $B_j \to D^\rho f$ in L^p. In part (b), Theorem 17 implies that if the dilations expand exponentially, then $\frac{1}{J}\sum_{j=1}^{J} B_j \to D^\rho f$ in L^p, since $\int_{\mathbb{R}^d} \eta_\rho \, dx = \int_{\mathbb{R}^d} \eta \, dx = 1$. This proves the theorem. □

10.4. **Proof of Theorem 12 — synthesis onto $W^{m,p}$.**

In proving S is surjective, we can assume the dilations expand exponentially, like we did in the proof of Theorem 2.

First we prove surjectivity assuming $b = I$. Let $f \in W^{m,p}$ and $J \in \mathbb{N}$. Like in the proof of Theorem 2 (but with $b = I$), we take $\phi = \mathbb{1}_{\mathcal{C}}$ and define c_J by (12). Then

$$\|c_J\|_{\ell^1(w^{m,p},\alpha)} \le \|f\|_{W^{m,p}} \tag{31}$$

by using Proposition 23 for each $j = 1, \dots, J$. And $S_{\psi,I} c_J \to f$ in $W^{m,p}$ as $J \to \infty$, by Theorem 24(b).

Hence the hypotheses of the open mapping theorem in Appendix A are satisfied with $A = 1$, by (31). Therefore $S_{\psi,I} : \ell^1(w^{m,p}, \alpha) \to W^{m,p}$ is open and surjective, and for each $f \in W^{m,p}$ and $\varepsilon > 0$ there exists $c \in \ell^1(w^{m,p}, \alpha)$ with $S_{\psi,I}c = f$ and $\|c\|_{\ell^1(w^{m,p},\alpha)} \le \|f\|_{W^{m,p}} + \varepsilon$. This proves the theorem when $b = I$.

For the general case where $b \neq I$, we rescale like in the proof of Theorem 11: since $M_b\psi$ and $M_b\eta$ satisfy formula (9) with "$b = I$", the case of the theorem already proved tells us that for each $f \in W^{m,p}$ and $\varepsilon > 0$, a sequence $c \in \ell^1(w^{m,p}, \alpha)$ exists such that $\|c\|_{\ell^1(w^{m,p},\alpha)} \le \|f\|_{W^{m,p}} + \varepsilon$ and $f = S_{M_b\psi,I}c$. The calculation (29) now implies $M_b^{-1}f = S_{\psi,b}c$, as desired.

10.5. **Proof of Corollary 15 — analysis norm for $W^{m,p}$.**

By boundedness of the analysis operator in Proposition 23,

$$\sup_j \sum_{|\rho|\le m} |\alpha_j|^{|\rho|} \|\Delta^\rho T_j f\|_{\ell^p} \le \|P|\phi|\|_\infty \|f\|_{W^{m,p}}.$$

On the other hand, choosing η as in Theorem 24(a), we see that $S_jT_jf \to f$ in $W^{m,p}$ as $j \to \infty$. It then follows from Theorem 11 (bounded synthesis) that

$$\begin{aligned}\|f\|_{W^{m,p}} &\le \sup_j \|S_jT_jf\|_{W^{m,p}} \\ &\le \|P|\eta|\|_{L^p(\mathcal{C})} \sup_j \sum_{|\rho|\le m} |\alpha_j|^{|\rho|} \|\Delta^\rho T_j f\|_{\ell^p}.\end{aligned}$$

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APPENDIX A. The open mapping theorem

The open mapping theorem in the following form is used to prove surjectivity of the synthesis operator, at various points in the paper.

Theorem 25. *Let X and Y be Banach spaces, and suppose $S : X \to Y$ is bounded and linear. Assume*

$$\overline{S(B_X(A))} \supset B_Y(1)$$

for some $A > 0$. That is, assume for each $y \in Y$ that a sequence $\{x_J\} \subset X$ exists with $Sx_J \to y$ as $J \to \infty$ and $\|x_J\|_X \le A\|y\|_Y$ for all J.

Then S is an open mapping, and $S(X) = Y$. Indeed, given $y \in Y$ and $\varepsilon > 0$ there exists $x \in X$ with $Sx = y$ and $\|x\|_X \le A\|y\|_Y + \varepsilon$.

For a proof, see [35, Theorem 4.13] with $A = 1/\delta$.

The hypothesis in Theorem 25 can clearly be weakened, to assume only for some *dense* subset of y-values that a sequence $\{x_J\} \subset X$ exists with $Sx_J \to y$ as $J \to \infty$ and $\|x_J\|_X \le A\|y\|_Y$ for all J.

APPENDIX B. **Discrete Hardy spaces**

In this appendix we study properties of the discrete Hardy space h^1, which was defined in Section 4. We will show h^1 is independent of the cut-off function used in its definition, and that it coincides (when $b = I$) with the discrete Hardy space $H^1(\mathbb{Z}^d)$ studied by Q. Y. Sun [39] and C. Eoff [16] in dimension 1, and later by S. Boza and M. J. Carro [3, 4] in all dimensions.

We will need the following result on Riesz transforms of Schwartz functions, which is a special case of [4, Corollary 2.4]. Recall that $Z(x) = C_d x/|x|^{d+1}$ for $x \neq 0$, and $Z(0) = 0$.

Lemma 26. *If θ is a Schwartz function then*

$$R\theta(x) = \widehat{\theta}(0)Z(x) + O\left(\frac{1}{|x|^{d+1}}\right) \qquad \text{for all } x \in \mathbb{R}^d \setminus \{0\}.$$

The next theorem replaces the kernel z defining h^1 with a discretization of the Riesz kernel, namely the sequence $z^b = \{Z(bk)\}$ having kth term $z^b_k = Z(bk)$.

Proposition 27. *In the definition of the space h^1, if we replace the sequence $z = \{z_k\}$ by $z^b = \{Z(bk)\}$ then we obtain the same space, with an equivalent norm.*

Proof of Proposition 27. Let ν be a cut-off function as in Section 4. Let μ be the Schwartz function with $\widehat{\mu} = \nu$. Note $\widehat{\mu}(0) = \nu(0) = 1$. Then

$$\zeta(\xi) = -i\frac{\xi b^{-1}}{|\xi b^{-1}|}\nu(\xi) = \widehat{Z}(\xi b^{-1})\widehat{\mu}(\xi) = \widehat{K\mu}(\xi), \qquad \xi \in \mathcal{C}_0,$$

where K is the singular integral operator with kernel

$$K(x) = |\det b|Z(bx) = |\det b|C_d\frac{bx}{|bx|^{d+1}}.$$

Thus by definition of the sequence z in Section 4, we have $z_k = \widehat{\zeta}(-k) = K\mu(k)$.

Next, observe that for each $x \in \mathbb{R}^d$,

$$\begin{aligned} K\mu(x) &= |\det b|C_d \,\text{p.v.}\int_{\mathbb{R}^d} \frac{by}{|by|^{d+1}}\mu(x-y)\,dy \\ &= C_d \,\text{p.v.}\int_{\mathbb{R}^d} \frac{y}{|y|^{d+1}}\mu(b^{-1}(bx-y))\,dy \\ &= R\theta(bx), \end{aligned}$$

where $\theta(y) = \mu(b^{-1}y)$ is a Schwartz function with $\widehat{\theta}(0) \neq 0$. In particular

$$\begin{aligned} z_k = K\mu(k) = R\theta(bk) &= \widehat{\theta}(0)Z(bk) + O\left(\frac{1}{|bk|^{d+1}}\right) \qquad \text{by Lemma 26} \\ &= \widehat{\theta}(0)z^b_k + O\left(\frac{1}{|k|^{d+1}}\right) \end{aligned}$$

for all $k \neq 0$. Hence $z - \widehat{\theta}(0)z^b \in \ell^1$, so that z and z^b define identical h^1 spaces with equivalent norms. □

The proposition and its proof yield a large class of kernel sequences that generate h^1, for they show that if μ is any Schwartz function with $\widehat{\mu}(0) \neq 0$, then we can replace the sequence $\{z_k\}$ by $\{K\mu(k)\}$ in the definition of h^1.

Now we can show independence of h^1 from the cut-off function.

Corollary 28. *The space h^1 does not depend on the cut-off function ν used to define it, and different cut-off functions produce equivalent norms.*

Proof of Corollary 28. This follows from Proposition 27, because z^b does not depend on ν.

Alternatively, consider two different cut-off functions ν_1 and ν_2, giving rise to periodic functions ζ_1 and ζ_2 as in Section 4. Then $\zeta_1 - \zeta_2$ is smooth and compactly supported in $\mathcal{C}_0$ and hence has Fourier coefficients in ℓ^1. Therefore the kernel sequences associated with ν_1 and ν_2 differ by only an ℓ^1 sequence, and so they define the same h^1 space, with comparable h^1 norms. □

Corollary 29. *If $b = I$ then $h^1 = H^1(\mathbb{Z}^d)$, with equivalent norms.*

This corollary simply restates Proposition 27 with $b = I$, because the discrete Hardy space $H^1(\mathbb{Z}^d)$ is defined (following [4]) by the kernel sequence $z^I = \{Z(k)\}$; in other words,

$$H^1(\mathbb{Z}^d) = \{s \in \ell^1 : z^I * s \in \ell^1\}$$

with a norm $\|s\|_{H^1(\mathbb{Z}^d)} = \|s\|_1 + \|z^I * s\|_1$. Note that in dimension 1, the sequence $z^I = \{\mathbb{1}_{\{k\neq 0\}}/\pi k\}$ is called the *Hilbert sequence* and was considered by R. E. Edwards and G. I. Gaudry [15], who proved boundedness of $s \mapsto z^I * s$ on $\ell^p(\mathbb{Z})$, for $1 < p < \infty$.

S. Boza and M. J. Carro [4] proved the space $H^1(\mathbb{Z}^d)$ admits a characterization by maximal functions in the sense of Fefferman–Stein [17], and an atomic decomposition in the sense of Coifman–Weiss [12]. The atomic decomposition in one dimension was also stated in [12]. It is an interesting problem to investigate these characterizations for our space h^1 when b is not the identity matrix.

Remark on vanishing means in h^1. If $s \in h^1$ then $\sum_{k\in\mathbb{Z}^d} s_k = 0$. *Proof:* Writing $\sigma(\xi) = \sum_{k\in\mathbb{Z}^d} s_k e^{-2\pi i\xi k}$ we see that $\sigma(\xi)\zeta(\xi) = \sum_{k\in\mathbb{Z}^d}(s * z)_k e^{-2\pi i\xi k}$ in $L^2(\mathcal{C}_0)$. This last series is continuous because $s * z \in \ell^1$, and $\sigma(\xi)$ is continuous too. But $\zeta(\xi)$ is not continuous at the origin and so $\sigma(0)$ must equal zero, as claimed.

Conversely if $b = I$ and $s \in \ell^1$ is finitely supported with $\sum_{k\in\mathbb{Z}^d} s_k = 0$, then $s \in h^1$; cf. [4, Theorem 3.3]. In other words, atoms belong to $H^1(\mathbb{Z}^d)$.

We end this appendix with a question: is h^1 independent of the choice of "translation" matrix b? We suspect not. Of course there is a trivial result: one can always replace b by a multiple of b without affecting the resulting space h^1.

APPENDIX C. **Banach frames**

This appendix explains how Banach frames arise from the analysis norms earlier in the paper.

Let Y be a Banach space, and let Z be a Banach space whose elements are complex sequences indexed by a countable set I. Let $\{g_i\}_{i\in I}$ be a subset of Y^*, the dual space of Y, and let $S_* : Z \to Y$ be a bounded linear operator. We say that $(\{g_i\}, S_*)$ is a *Banach frame* for Y with respect to Z if the following three conditions are satisfied:

(i) $\{\langle f, g_i\rangle\} \in Z$, for all $f \in Y$,
(ii) $\|f\|_Y \approx \|\{\langle f, g_i\rangle\}\|_Z$, for all $f \in Y$,
(iii) $S_*(\{\langle f, g_i\rangle\}) = f$ for all $f \in Y$.

In other words, "analyzing" with the $\{g_i\}$ maps Y to Z with comparable norms, and then "synthesizing" with S_* recovers the identity map on Y. The above definition is due to K. Gröchenig [27]; see the treatment in [10, §17.3].

The next result reformulates our L^p-analysis norm in Corollary 6 as a Banach frame result.

Corollary 30 (Banach frame for L^p)**.** *Assume $1 \le p < \infty$ and let ϕ and ψ satisfy the assumptions of Theorem 17 with $\gamma = 1$ (scale-averaged approximation in L^p). Assume the dilations expand exponentially.*

Then $(\{|\det b|\phi_{j,k}\}, S_*)$ is a Banach frame for L^p with respect to

$$Z = \{c \in \ell^\infty(\ell^p) : S_*c = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k} \text{ exists in } L^p\}.$$

One can similarly reformulate the Hardy and Sobolev space results (Corollaries 10 and 15), for isotropic dilations $a_j = \alpha_j I$.

Proof of Corollary 30. Write $c_j = \{c_{j,k}\}_{k\in\mathbb{Z}^d}$, so that $S_j c_j = \sum_{k\in\mathbb{Z}^d} c_{j,k}\psi_{j,k}$. Define $Bc = \{S_j c_j\}_{j>0}$. Then B is a bounded linear operator from $\ell^\infty(\ell^p)$ into $\ell^\infty(L^p)$ (by the proof of Theorem 1). Therefore Lemma 31 below tells us Z is a closed subspace of $\ell^\infty(\ell^p)$, and hence is a Banach space itself. Moreover, S_* is bounded from Z into L^p. Corollary 6 and Theorem 17 now show that $(\{|\det b|\phi_{j,k}\}, S_*)$ is a Banach frame for L^p with respect to Z, noting in particular by Theorem 17 that

$$S_*(Tf) = \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} S_j T_j f = f \qquad \text{in } L^p, \text{ for all } f \in L^p.$$

□

The final lemma states that the preimage of the space of Cesàro-convergent sequences in a Banach space forms a closed subspace. Let Y be a Banach space, and write $\ell^\infty(Y)$ for the Banach space of all sequences $y = \{y_j\}_{j>0}, y_j \in Y$, such that $\|y\|_{\ell^\infty(Y)} = \sup_{j>0} \|y_j\|_Y < \infty$.

Lemma 31. *Let X and Y be Banach spaces and $B : X \to \ell^\infty(Y)$ be a bounded linear operator. Then the subspace*

$$\{x \in X : \text{the limit } \lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} (Bx)_j \text{ exists in } Y\}$$

is closed in X.

We omit the proof.

REFERENCES

[1] W. Abu-Shammala, J.-L. Shiu and A. Torchinsky. *Characterizations of the Hardy space H^1 and BMO.* Preprint, 2006.

[2] A. Aldroubi, Q. Sun and W.-S. Tang. *p-frames and shift invariant subspaces of L^p.* J. Fourier Anal. Appl. 7:1–21, 2001.

[3] S. Boza and M. J. Carro. *Discrete Hardy spaces.* Studia Math. 129:31–50, 1998.

[4] S. Boza and M. J. Carro. *Hardy spaces on $\mathbb{Z}^N$.* Proc. Roy. Soc. Edinburg, Sect. A 132:25–43, 2002.

[5] J. Bruna. *On translation and affine systems spanning $L^1(\mathbb{R})$.* J. Fourier Anal. Appl., 12:71–82, 2006.

[6] H.-Q. Bui and R. S. Laugesen. *Affine systems that span Lebesgue spaces.* J. Fourier Anal. Appl. 11:533–556, 2005.

[7] H.-Q. Bui and R. S. Laugesen. *Approximation and spanning in the Hardy space, by affine systems.* Constructive Approximation, to appear. `www.math.uiuc.edu/~laugesen/publications.html`.

[8] H.-Q. Bui and R. S. Laugesen. *Sobolev spaces and approximation by affine spanning systems.* Math. Ann., accepted `www.math.uiuc.edu/~laugesen/publications.html`, 2005.

[9] H.-Q. Bui and M. Paluszyński. *On the phi and psi transforms of Frazier and Jawerth*. Research Report UDCDMS2004/11, University of Canterbury, 2004.

[10] O. Christensen. An introduction to frames and Riesz bases. Birkhäuser, Boston, 2003.

[11] C. K. Chui and Q. Sun. *Affine frame decompositions and shift-invariant spaces.* Appl. Comput. Harmon. Anal. 20:74–107, 2006.

[12] R. R. Coifman and G. Weiss. *Extensions of Hardy spaces and their use in analysis.* Bull. Amer. Math. Soc. 83:569–645, 1977.

[13] W. Czaja, G. Kutyniok and D. Speegle. *The geometry of sets of parameters of wave packet frames.* Appl. Comput. Harmon. Anal. 20:108–125, 2006.

[14] I. Daubechies. Ten lectures on wavelets. CBMS-NSF Regional Conf. Ser. in Appl. Math. 61, SIAM, 1992.

[15] R. E. Edwards and G. I. Gaudry. Littlewood-Paley and multiplier theory. Springer-Verlag, Berlin, Heldelberg, New York, 1977.

[16] C. Eoff. *The discrete nature of the Paley-Wiener spaces.* Proc. Amer. Math. Soc. 123:505–512, 1995.

[17] C. Fefferman and E. M. Stein. *H^p spaces of several variables.* Acta Math. 129:137–193, 1972.

[18] H. G. Feichtinger and M. Fornasier. *Flexible Gabor–wavelet atomic decompositions for L^2-Sobolev spaces.* Ann. Mat. Pura Appl. (4) 185:105–131, 2006.

[19] H. G. Feichtinger and K. Gröchenig. *Banach spaces related to integrable group representations and their atomic decompositions I.* J. Funct. Anal. 86:307-340, 1989.

[20] H. G. Feichtinger and T. Strohmer (eds.). Gabor Analysis and Algorithms, Theory and Applications. Birkhäuser, Boston, 1998.

[21] V. I. Filippov and P. Oswald. *Representation in L_p by series of translates and dilates of one function.* J. Approx. Theory 82:15–29, 1995.

[22] V. I. Filippov. *On the completeness and other properties of some function systems in $L_p, 0 < p < \infty$.* J. Approx. Theory 94: 42–53, 1998.

[23] M. Frazier and B. Jawerth.A discrete transform and decomposition of distribution spaces. J. Funct. Analysis 93:34–170, 1990.

[24] M. Frazier, B. Jawerth and G. Weiss. Littlewood-Paley theory and the study of function spaces. CBMS Reg. Conf. Ser. in Math., No. 79, Amer. Math. Soc., Providence, RI, 1991.

[25] J. E. Gilbert, Y. S. Han, J. A. Hogan, J. D. Lakey, D. Weiland and G. Weiss. *Smooth molecular decompositions of functions and singular integral operators.* Memoirs AMS 156, no. 742, 74 pp., 2002.

[26] J. E. Gilbert, J. A. Hogan and J. D. Lakey. *BMO, boundedness of affine operators, and frames*. Appl. Comp. Harmon. Anal. 18:3–24, 2005.

[27] K. Gröchenig. *Describing functions: frames versus atomic decompositions.* Monatsh. Math. 112:1–41, 1991.

[28] K. Gröchenig. Foundations of time-frequency analysis. Birkhäuser, Boston, 2001.

[29] F. di Guglielmo. *Construction d'approximations des espaces de Sobolev sur des réseaux en simplexes.* Calcolo 6:279–331, 1969.

[30] E. Hernández and G. Weiss. A first course on wavelets. CRC Press, Boca Raton, Florida, 1996.

[31] R.-Q. Jia. *Approximation with scaled shift-invariant spaces by means of quasi-projection operators.* J. Approx. Theory 131:30–46, 2004.

[32] D. Labate, G. Weiss and E. Wilson. *An approach to the study of wave packet systems.* Wavelets, frames and operator theory, 215–235, Contemp. Math., 345. Amer. Math. Soc., Providence, RI, 2004.

[33] R. S. Laugesen. *Affine synthesis onto L^p when $0 < p \leq 1$.* Preprint `www.math.uiuc.edu/~laugesen/publications.html`, 2007.

[34] Y. Meyer. Wavelets and operators. Cambridge University Press, Cambridge, 1992.

[35] W. Rudin. Functional analysis. Second edition. McGraw–Hill, New York, 1991.

[36] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, New Jersey, 1970.

[37] E. M. Stein. Harmonic analysis, real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, Princeton, New Jersey, 1993.

[38] G. Strang and G. Fix. *A Fourier analysis of the finite element variational method.* In: Constructive aspects of functional analysis (ed. G. Geymonat), pp. 793–840. C.I.M.E., 1973.

[39] Q. Y. Sun. *Sequence spaces and stability of integer translates.* Z. Anal. Anwendungen 12:567–584, 1993.

[40] P. A. Terekhin. *Inequalities for the components of summable functions and their representations by elements of a system of contractions and shifts.* (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. no. 8:74–81, 1999; translation in Russian Math. (Iz. VUZ) 43 no. 8:70–77, 1999.

[41] P. A. Terekhin. *Translates and dilates of function with nonzero integral.* Mathematics, Mechanics (published by Saratov University) 1:67–68, 1999. (Russian.)

[42] H. Triebel. *Towards a Gausslet analysis: Gaussian representations of functions*. Function spaces, interpolation theory, and related topics (Lund, 2000), 425–449. De Gruyter, Berlin, 2002.

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