Moment Inequalities for Equilibrium Measures in the Plane

A. Baernstein II, R. S. Laugesen and I. E. Pritsker

Dedicated to our friend Fred Gehring, on the occasion of his 80th birthday

Abstract: The equilibrium measure of a compact plane set gives the steady state distribution of charges on the conductor. We show that certain moments of this equilibrium measure, when taken about the electrostatic centroid and depending only on the real coordinate, are extremal for an interval centered at the origin. This has consequences for means of zeros of polynomials, and for means of critical points of Green's functions.

We also study moments depending on the distance from the centroid, such as the electrostatic moment of inertia.

Keywords: capacity, equilibrium potential, Green's function, extremal problem, polynomials, zeros, critical points.

1. **Introduction**

Let K be a compact non-polar subset of the complex plane $\mathbb{C}$, and μ_K be its equilibrium measure. For functions $\phi : K \to \mathbb{R}$, the integral

$$\int_K \phi(z) \, d\mu_K(z)$$

is called the ϕ *moment of K*, or of the probability measure μ_K. For example, when $\phi(z) = |z|^2$ the ϕ moment is the moment of inertia about the origin.

In this paper we take up some problems involving maximizing or minimizing ϕ moments when ϕ satisfies certain conditions. In all of our results the competing sets K will have the same logarithmic capacity, which, as a normalization, we take to be 1. That is:

$$\operatorname{cap}(K) = 1.$$

And we shall usually take the *conformal centroid* $\int_K z \, d\mu_K(z)$ of K to lie at the origin:

$$\int_K z \, d\mu_K(z) = 0.$$

In all of our results, the extremal ϕ moment will be achieved by a line segment of length 4. When sets in the class have conformal centroid at the origin, the interval L defined by

$$L = [-2, 2]$$

will be among the extremals.

In our two main results the function ϕ will in fact depend only on the real part of z. Here are those results.

Theorem 1. *Suppose $K \subset \mathbb{R}$ is compact with capacity $\operatorname{cap}(K) = 1$, and that its conformal centroid is at the origin. Then for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_K \phi(\operatorname{Re} z) \, d\mu_K(z) \geq \int_L \phi(\operatorname{Re} z) \, d\mu_L(z).$$

Moreover, if $K \setminus L$ has positive capacity and the restriction of ϕ to L is not a linear function, then strict inequality holds.

Theorem 2. *Suppose $K \subset \mathbb{C}$ is compact and connected with capacity $\operatorname{cap}(K) = 1$, and that its conformal centroid is at the origin. Then for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_K \phi(\operatorname{Re} z) \, d\mu_K(z) \leq \int_L \phi(\operatorname{Re} z) \, d\mu_L(z).$$

Moreover, if $K \neq L$ and the restriction of ϕ to L is not a linear function, then strict inequality holds.

Theorem 1 says that among all compact sets on a line with the same conformal centroid and the same capacity, the least spread out set, as measured by convex integral means, is a single interval. Contrary-wise, Theorem 2 says that among all plane continua with the same conformal centroid and the same capacity, the single interval is the most spread out.

For a lower estimate applicable in both Theorems 1 and 2, we observe

$$\int_K \phi(\operatorname{Re} z)\, d\mu_K(z) \geq \phi(0)$$

by Jensen's inequality, whenever K is compact with conformal centroid on the imaginary axis ($\operatorname{Re} \int_K z\, d\mu_K(z) = 0$) and ϕ is convex. Equality is attained whenever K is contained in the imaginary axis. Theorems 1 and 2 also require only that the conformal centroid be purely imaginary, but, for brevity, we shall continue to assume the conformal centroid is at the origin.

The proof of Theorem 2 is modeled on the proof of a theorem of Baernstein [2], p. 139, about maximizing integral means in certain classes of univalent functions in the unit disk. The novelty in the present Theorem 2 is that instead of working with symmetric decreasing rearrangements on circles, as in [2], one must devise a "*-function" appropriate to "symmetric increasing rearrangements" of functions defined and unbounded in all of $\mathbb{R}$. The proof of Theorem 1 follows the same general strategy as that of Theorem 2, but is simpler, in that no functions need to be rearranged.

Theorem 1 is motivated by considerations in number theory; it will be proved in §3. Some consequences will be presented in §4. Theorem 2, to be proved in §5, arose in an attempt to prove a conjecture stated in §6. The conjecture asserts, when both ϕ and ϕ' are convex, that

$$(1) \qquad \int_K \phi(\log|z|)\, d\mu_K(z) \leq \int_L \phi(\log|z|)\, d\mu_L(z)$$

provided K is compact and connected with $\operatorname{cap}(K) = 1$ and with its conformal centroid at the origin, and with the origin belonging to K. If true, the conjecture would prove a conjecture of Pommerenke [14] about integral means of univalent functions in the class Σ_0, and would also prove a conjecture of Pritsker and Ruscheweyh [17] about lower bounds for factors of polynomials.

We will present an example showing that (1) is false within the class of all convex functions. But if we add to K the assumption, stronger than having the

conformal centroid at the origin, that K is symmetric with respect to the origin, then (1) is true for all convex ϕ. This is Corollary 6.3. It follows from Theorem 6.2, which restates a result of Laugesen [11] (and which is itself a consequence of Baernstein's integral means result, Theorem 1 in [2]).

2. Potential theoretic preliminaries

For potential theoretic notions we shall mostly follow the approach in [18]. Let K be a compact subset of $\mathbb{C}$, and K^c be the complement of K on the Riemann sphere $\overline{\mathbb{C}}$. For a measure μ compactly supported in $\mathbb{C}$, the energy $I(\mu)$ is defined to be $I(\mu) = \int_{K \times K} \log |z - \zeta| \, d\mu(z) \, d\mu(\zeta)$. If $I(\mu) = -\infty$ for every μ supported on K then K is said to be *polar*. If K is non-polar, then there is a unique probability measure μ_K on K, called the equilibrium measure of K, which maximizes $I(\mu)$ over all probability measures μ on K. Clearly $I(\mu_K)$ is a finite real number, because K is bounded. The capacity $\operatorname{cap}(K)$ of K is defined to be $e^{I(\mu_K)}$. For polar K, define $\operatorname{cap}(K) = 0$. A general set E is said to be polar if $\operatorname{cap}(K) = 0$ for every compact $K \subset E$.

For non-polar K, denote by g the equilibrium potential of K. Then

$$g(z) = g_K(z) = \int_K \log |z - \zeta| \, d\mu_K(\zeta), \quad z \in \mathbb{C}.$$

Put $g(\infty) = +\infty$. Then g is harmonic in K^c except at ∞, where $g(z) = \log |z| + o(1)$. By Frostman's Theorem ([18], p. 59),

$$g \geq I(\mu_K) = \log \operatorname{cap}(K)$$

everywhere in $\mathbb{C}$, with equality on $K \setminus E$ for some polar set E. The potential g is related to the Green's function of K^c with pole at ∞ by

$$g(z) = \log \operatorname{cap}(K) + g(z, \infty, K^c), \quad z \in \mathbb{C}.$$

See [18], pp. 107, 132.

Set

$$\mathbb{B}(R) = \{z \in \mathbb{C} : |z| < R\}, \quad \overline{\mathbb{B}(R)} = \{z \in \mathbb{C} : |z| \leq R\},$$

and let

$$a_n = a_n(K) = \frac{1}{n} \int_K \zeta^n \, d\mu_K(\zeta), \quad n \geq 1.$$

In particular, a_1 is the conformal centroid of K.

Suppose that $K \subset \overline{\mathbb{B}(R)}$. In the definition of g, take $|z| > R$ and expand the log in powers of $|\zeta| \leq R$. We obtain

$$(2) \qquad g(z) = \log|z| - \operatorname{Re} \sum_{n=1}^{\infty} a_n z^{-n}, \quad |z| > R.$$

Next, suppose that K_1 and K_2 are two non-polar compact subsets of $\mathbb{C}$ with the same capacity and the same conformal centroid. Defining the potentials $g_j(z) = \int_{K_j} \log|z - \zeta| \, d\mu_{K_j}(\zeta)$, from (2) it follows that

$$(3) \qquad g_1(z) - g_2(z) = -\operatorname{Re} \sum_{n=2}^{\infty} b_n z^{-n},$$

where $b_n = a_n(K_1) - a_n(K_2)$ and the series converges for $|z| > R$ when $\overline{\mathbb{B}(R)}$ contains both K_1 and K_2. Thus $g_1(z) - g_2(z) = O(z^{-2})$ at ∞, and also, g_1 and g_2 are bounded on compact subsets of $\mathbb{C}$. It follows that the function

$$w(x) = \int_{-\infty}^{\infty} [g_1(x + iy) - g_2(x + iy)] \, dy, \quad x \in \mathbb{R},$$

is well defined as an absolutely convergent integral, and is continuous on $\mathbb{R}$. (To see the continuity, split the defining integral into two parts: inside and outside the disk $\mathbb{B}(2R)$. Outside the disk, $g_1 - g_2$ is represented by the absolutely and uniformly convergent series (3), and hence the integral is continuous in x. Inside the disk, one can first write down the definitions of g_1 and g_2 as potentials, and then use Fubini's theorem and integrate the logarithmic kernel with respect to Lebesgue measure on the vertical segment inside the disk. This eliminates the singularity, and thus this part of the integral is continuous in x too.)

Note that for each complete vertical line Γ not passing through 0 we have $\int_{\Gamma} z^{-n} \, dz = 0$, $n \geq 2$. With (3), this implies

$$(4) \qquad w(x) = 0, \quad |x| \geq R.$$

To prove Theorems 1 and 2, we shall make use of the following formula.

Formula. *Suppose K_1 and K_2 are compact non-polar subsets of $\mathbb{C}$ having the same capacity and the same conformal centroid, and contained in $\overline{\mathbb{B}(R)}$.*

Then for each $a \geq R$ and each function $\phi \in C^2(\mathbb{R})$, we have

$$\int_{K_1} \phi(\operatorname{Re} z) \, d\mu_{K_1}(z) - \int_{K_2} \phi(\operatorname{Re} z) \, d\mu_{K_2}(z) = \frac{1}{2\pi} \int_{-a}^{a} w(x) \phi''(x) \, dx.$$

Proof. For $a > R$, $b > R$ let $Q = [-a, a] \times [-b, b]$. In the sense of distributions, we have $\Delta g = 2\pi \, \mu_K$ in $\mathbb{C}$, where K denotes K_1 or K_2 and g denotes g_1 or g_2. See [18], Theorem 3.7.4. For $\psi \in C^2(\mathbb{C})$, Green's formula gives

$$2\pi \int_K \psi \, d\mu_K = \int_Q \psi \Delta g \, dx \, dy$$
$$= \int_Q g \, \Delta \psi \, dx \, dy + \int_{\partial Q} \{\psi \partial_n g - g \partial_n \psi\} |dz|$$

where ∂_n denotes outer normal derivative. Thus,

$$2\pi \Big\{ \int_{K_1} \psi \, d\mu_{K_1} - \int_{K_2} \psi \, d\mu_{K_2} \Big\}$$
$$= \int_Q (g_1 - g_2) \Delta \psi \, dx \, dy + \int_{\partial Q} \{\psi \partial_n (g_1 - g_2) - (g_1 - g_2) \partial_n \psi\} |dz|.$$

Write $z = x + iy$ and take $\psi(z) = \phi(x)$. Then

$$2\pi \Big\{ \int_{K_1} \phi(x) \, d\mu_{K_1}(z) - \int_{K_2} \phi(x) \, d\mu_{K_2}(z) \Big\}$$
$$= \int_Q (g_1 - g_2)(z) \phi''(x) \, dx \, dy + \int_{\partial Q} \{\phi \partial_n (g_1 - g_2) - (g_1 - g_2) \partial_n \phi\} |dz|.$$

Fix $a > R$ and let $b \to \infty$. Since $g_1(z) - g_2(z) = O(z^{-2})$ and $\nabla[g_1(z) - g_2(z)] = O(z^{-3})$ as $z \to \infty$, the integral over Q tends to $\int_{-a}^{a} w(x) \phi''(x) \, dx$ and the boundary integrals over the horizontal sides tend to 0.

Write $Q = Q(b)$ to show the dependence on b and denote the right hand vertical boundary side by $\partial Q^+(b)$. Then by (3),

$$\lim_{b \to \infty} \int_{\partial Q^+(b)} (g_1 - g_2) \partial_n \phi \, |dz| = -\phi'(a) \operatorname{Re} \int_{\mathbb{R}} \sum_{n=2}^{\infty} b_n (a + iy)^{-n} \, dy.$$

The last term equals $\phi'(a) w(a)$ which, by (4), is 0. Thus,

$$\lim_{b \to \infty} \int_{\partial Q^+(b)} (g_1 - g_2) \partial_n \phi \, |dz| = 0.$$

The three other vertical boundary integrals likewise have limit zero. The formula is proved when $a > R$. By continuity, the formula also holds for $a = R$. $\square$

Our proofs of Theorems 1 and 2 will make use of the following lemmas.

Lemma 2.1. *With K_1, K_2 as in the Formula, suppose that S is a vertical strip $-\infty < \gamma_1 < \operatorname{Re} z < \gamma_2 < \infty$. If $\mu_{K_1}(S) = 0$ then $w(x)$ is concave on (γ_1, γ_2). If $\mu_{K_2}(S) = 0$ then $w(x)$ is convex on (γ_1, γ_2).*

Proof. Assume $\mu_{K_1}(S) = 0$. Let ϕ be a nonnegative C^2 function on $\mathbb{R}$ with compact support in (γ_1, γ_2). Take $a \in \mathbb{R}$ so large that $a \geq R$ and $(\gamma_1, \gamma_2) \subset (-a, a)$. Then in the Formula, the integral over K_1 is zero. Since $\phi \geq 0$ and $\mu_{K_2} \geq 0$, the Formula implies that

$$0 \geq \int_{-a}^{a} w(x)\phi''(x)\,dx = \int_{\gamma_1}^{\gamma_2} w(x)\phi''(x)\,dx.$$

Since w is continuous on $\mathbb{R}$, the 1-dimensional version of Weyl's Lemma [9] or [4], p. 127, shows that w is concave on (γ_1, γ_2). The proof is similar when $\mu_{K_2}(S) = 0$. ∎

Lemma 2.2. *Suppose that a function u is subharmonic in the upper half plane $\mathcal{H}$, is continuous on $\mathcal{H} \cup \mathbb{R}$ and satisfies $u(x_0) > u(z)$ for all $z \in \mathcal{H}$ and some $x_0 \in \mathbb{R}$. Then*

$$\liminf_{y \to 0+} \frac{u(x_0) - u(x_0 + iy)}{y} > 0.$$

Proof. Let D be the open half disk $\mathcal{H} \cap \mathbb{B}(x_0, \epsilon)$. Then $u(x) \leq u(x_0)$ for all $x \in [x_0 - \epsilon, x_0 + \epsilon]$ and $u(z) < u(x_0)$ for all z in the circular part of ∂D. Let v solve the Dirichlet problem in D with boundary values u. Then v is nonconstant on ∂D, hence nonconstant in D. Also, $\sup_{\partial D} v = v(x_0)$, and so by the strong maximum principle, $v(x_0) > v(z)$ for all $z \in D$. By Hopf's Lemma, as stated at the top and bottom of [7], p. 34, the lim inf in Lemma 2.2 is positive for v. Since $u \leq v$ in D, the lim inf is also positive for u. ∎

3. **Proof of Theorem 1**

Let K be a compact subset of $\mathbb{R}$. We assume also that $\mathrm{cap}(K) = 1$ and that the conformal centroid of K is at the origin, $\int_K z\,d\mu_K(z) = 0$.

Recall that $L = [-2, 2]$. Then $\mathrm{cap}(L) = 1$ and the conformal centroid of L is at the origin. We shall apply the considerations of §2 with $K_1 = L$ and $K_2 = K$.

Write $G = g_1$ and $g = g_2$ for the respective Green's functions of L^c and K^c with poles at ∞. The function $w(x)$ introduced in §2 is defined on $\mathbb{R}$ by

$$w(x) = \int_{\mathbb{R}} [G(x + is) - g(x + is)] \, ds, \quad x \in \mathbb{R}.$$

Hence

$$w(x) = 2 \int_0^\infty [G(x + is) - g(x + is)] \, ds, \quad x \in \mathbb{R},$$

by symmetry of G and g in the real axis, recalling that $K, L \subset \mathbb{R}$.

As observed in §2, w is continuous on $\mathbb{R}$ and satisfies $w(x) = 0$ for $|x| \geq R$, where R is so large that K and L are contained in $\overline{\mathbb{B}(R)}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. The second distributional derivative of ϕ is a non-negative Borel measure on $\mathbb{R}$; call it ν. Via approximation, one sees that the Formula in §2 generalizes to

(5) $$\int_L \phi(\operatorname{Re} z) \, d\mu_L(z) - \int_K \phi(\operatorname{Re} z) \, d\mu_K(z) = \frac{1}{2\pi} \int_{\mathbb{R}} w \, d\nu.$$

Thus, to prove the inequality in Theorem 1, it suffices to prove that

$$w(x) \leq 0, \quad x \in \mathbb{R}.$$

To accomplish this, we solve the Dirichlet problem in $\mathcal{H}$ with boundary values w on $\mathbb{R}$ (and boundary value 0 at infinity), and call the resulting function $w(z) = w(x + iy)$. Then w is continuous on $\mathcal{H} \cup \mathbb{R}$, equals 0 on the real axis near infinity, and tends to 0 as $z \to \infty$ in $\mathcal{H}$. Moreover, we will show w has the representation

(6) $$w(z) = 2 \int_y^\infty [G(x + is) - g(x + is)] \, ds, \quad z = x + iy \in \mathcal{H} \cup \mathbb{R}.$$

To see that this representation is valid, call the function on the right $\widetilde{w}$. Then, see (3), $\widetilde{w}$ is bounded and continuous on $\mathcal{H} \cup \mathbb{R}$, and equals w on $\mathbb{R}$ and tends to 0 as $z \to \infty$ in $\mathcal{H}$. Further, g and G are harmonic in $\mathcal{H}$ (since $K, L \subset \mathbb{R}$), from which it follows that $\widetilde{w}_{xx} = 2(G_y - g_y)$. Also, differentiation of $\widetilde{w}$ twice with respect to y gives $\widetilde{w}_{yy} = 2(g_y - G_y)$. Thus $\widetilde{w}$ is harmonic in $\mathcal{H}$. By uniqueness of solutions to the Dirichlet problem in the halfplane, we have $\widetilde{w} = w$.

As just noted, (6) gives the identity

$$w_{xx}(z) = 2[G_y(z) - g_y(z)], \quad z \in \mathcal{H},$$

and also gives

(7)
$$w_y(z) = 2[g(z) - G(z)], \quad z \in \mathcal{H}.$$

Set

$$M = \sup_{\mathcal{H} \cup \mathbb{R}} w.$$

Then $M \geq 0$, since $w(x) = 0$ for $|x| \geq R$. Suppose that $M > 0$. Then by continuity of w and the strong maximum principle there exists $x_0 \in \mathbb{R}$ such that $w(x_0) = M$ and $w(z) < w(x_0)$ for each $z \in \mathcal{H}$. There are two possible locations for x_0.

Case 1. $x_0 \in (-\infty, -2] \cup [2, \infty)$. Since μ_L is supported on $[-2, 2]$, Lemma 2.1 in §2 implies that w is concave on each open bounded subinterval of $(-\infty, -2]$, hence is concave on $(-\infty, -2]$. Since $w(x) = 0$ for all $x \leq -R$, it follows that $w \leq 0$ on $(-\infty, -2]$. So if $x_0 \in (-\infty, -2]$, then $M = 0$. Similarly, if $2 \leq x_0 < \infty$ then $M = 0$. This contradicts our assumption that $M > 0$, and so Case 1 cannot occur.

Case 2. $x_0 \in (-2, 2)$. Since w is harmonic in $\mathcal{H}$, Lemma 2.2 implies that

(8)
$$\liminf_{y \to 0+} \frac{w(x_0) - w(x_0 + iy)}{y} > 0.$$

On the other hand, $w(x_0 + iy)$ is a continuous function of y on $[0, \infty)$ and is differentiable on $(0, \infty)$. By the mean value theorem, for each $y > 0$ there exists $y^* \in (0, y)$ such that

$$
\begin{aligned}
\frac{w(x_0) - w(x_0 + iy)}{y} &= -w_y(x_0 + iy^*) \\
&= 2[G(x_0 + iy^*) - g(x_0 + iy^*)] \qquad \text{by (7)} \\
&\leq 2G(x_0 + iy^*) \qquad\qquad\qquad \text{since } g \geq 0 \\
&\to 0
\end{aligned}
$$

as $y \to 0+$, because $x_0 \in L$ and $G = 0$ on L. This contradicts (8), and so Case 2 cannot occur. The inequality in Theorem 1 is proved.

To prove the strict inequality statement, assume $K \setminus L$ has positive capacity. Since g is harmonic in $\mathbb{C} \setminus \operatorname{supp} \mu_K$ and is nonnegative and nonconstant there, we have by the strong minimum principle that $g > 0$ on $\mathbb{C} \setminus \operatorname{supp} \mu_K$. Recalling that

$g = 0$ on $K \setminus E$ for some polar set E, we deduce $K \setminus E \subset \operatorname{supp} \mu_K$. Hence, if $\operatorname{supp} \mu_K \subset [-2, 2]$ then $K \setminus E \subset [-2, 2] = L$, which implies $K \setminus L \subset E$ is polar, meaning $K \setminus L$ has capacity zero in contradiction to our assumption. Therefore $\operatorname{supp} \mu_K \not\subset [-2, 2]$, so that some $t > 2$ exists with either $\mu_K([t, \infty)) > 0$ or $\mu_K((-\infty, -t]) > 0$. Say the former holds. Take $\phi(x) = (x - t)^+$ in (5). Then the distributional second derivative of ϕ is the unit point mass at t, so that

$$w(t) = -2\pi \int_K (x - t)^+ \, d\mu_K(x) < 0.$$

Thus, w is nonconstant in the closed upper half plane, and also $w \leq 0$ as we saw above.

If $w(x) = 0$ for some $x \in L$, then we can rerun the Hopf's lemma argument in the proof of Case 2 to get a contradiction. So, $w(x) < 0$ at every $x \in L$. If ϕ is convex on $\mathbb{R}$ and not linear on L, then the corresponding measure ν satisfies $\nu((-2, 2)) > 0$. Formula (5) implies $\int_L \phi(\operatorname{Re} z) \, d\mu_L(z) < \int_K \phi(\operatorname{Re} z) \, d\mu_K(z)$. □

4. Applications of Theorem 1

This section contains three direct applications of Theorem 1. They are related to the properties of Green's function and its derivatives, as well as to the asymptotic zero distribution of polynomials.

4.1. **Pointwise bounds for Green's function and its derivatives.** Suppose as before that $K \subset \mathbb{R}$ is a compact set, $\operatorname{cap}(K) = 1$ and $\int_K x \, d\mu_K(x) = 0$, where μ_K is the equilibrium measure of K. Recall that g denotes Green's function of $\overline{\mathbb{C}} \setminus K$ and G denotes Green's function of $\overline{\mathbb{C}} \setminus L$, with poles at ∞, where $L = [-2, 2]$. Then the equilibrium measure of L is given by $d\mu_L = dx/(\pi\sqrt{4 - x^2})$, and $G(z) = \log|z + \sqrt{z^2 - 4}| - \log 2$.

Corollary 4.1. *Let $x_0 \in \mathbb{R}$, $x_0 > 2$, be fixed. For any set K as above, with $\max K < x_0$, we have*

$$(9) \qquad \frac{\partial^m g}{\partial x^m}(x_0) \leq \frac{\partial^m G}{\partial x^m}(x_0) \qquad \text{when } m \geq 0 \text{ is even,}$$

$$(10) \qquad \frac{\partial^m g}{\partial x^m}(x_0) \geq \frac{\partial^m G}{\partial x^m}(x_0) \qquad \text{when } m \geq 1 \text{ is odd.}$$

Furthermore, if $z_0 = x_0 + iy_0$ and $\max K < x_0 - |y_0|$, then

$$(11) \qquad\qquad\qquad g(z_0) \leq G(z_0).$$

Equality holds in (9)–(11) *if and only if* $K \setminus L$ *has zero capacity.*

In words, inequality (9) with $m = 0$ says that the Green's function of K^c is smaller at x_0 than the Green's function of L^c, which is reasonable since K is more spread out than L and thus contains points closer to x_0.

Clearly, one can consider $x_0 < -2$ by symmetry, and make corresponding adjustments in the above corollary.

Proof. Recall that $g(z) = \int_K \log |z - s| \, d\mu_K(s)$. Since $\max K < x_0$, we have that $g \in C^\infty$ around $z = x_0$, and

$$\frac{\partial^m g}{\partial x^m}(x_0) = \int_K (-1)^{m+1}(m-1)!(x_0 - s)^{-m} \, d\mu_K(s), \quad m \in \mathbb{N}.$$

Note that the integrand can be extended to a strictly convex function of $s \in \mathbb{R}$, for odd $m \in \mathbb{N}$. Hence Theorem 1 gives (10). Similarly, the integrand is strictly concave for even $m \geq 0$, so that we obtain the reversed inequality (9).

For $z_0 = x_0 + iy_0$, we have

$$g(z_0) = \frac{1}{2} \int_K \log((x_0 - s)^2 + y_0^2) \, d\mu_K(s).$$

Thus the integrand is a strictly concave function of s for $s < x_0 - |y_0|$, and (11) is again a direct consequence of Theorem 1.

For the case of equality, suppose $\text{cap}(K \setminus L) = 0$. Then $\mu_K(K \setminus L) = 0$ (since otherwise the restriction of μ_K to $K \setminus L$ would give a finite energy), and so μ_K is supported in L. Hence g is harmonic in $\mathbb{C} \setminus L$, so that $g - G$ is harmonic in $\overline{\mathbb{C}} \setminus L$. Because $g - G$ is nonnegative on L and equals zero at infinity, the strong maximum principle implies $g - G \equiv 0$, so that equality holds in (9)–(11). On the other hand, if $\text{cap}(K \setminus L) > 0$ then strict inequalities hold in (9)–(11) by Theorem 1. $\square$

4.2. **Means of zeros of polynomials.** This part is inspired by the problem on the smallest limit point for the arithmetic means of zeros for polynomials with integer coefficients and positive zeros, considered by Schur [19] and Siegel [20]. They gave lower bounds for the arithmetic means of zeros, which improved the standard arithmetic-geometric means inequality.

We consider certain extremal polynomials on the real line here. The number theoretic aspects of the problem for integer polynomials will be treated in a separate paper.

Let $K \subset \mathbb{C}$ be an arbitrary compact set. It is well known that for any monic polynomial P_n of degree n, we have $\|P_n\|_K \geq (\operatorname{cap}(K))^n$, where the norm on K is the supremum norm (cf. [1]). Thus a sequence of monic polynomials P_n, $n \in \mathbb{N}$, is called *asymptotically extremal* for the set K if

$$\lim_{n \to \infty} \|P_n\|_K^{1/n} = \operatorname{cap}(K),$$

and all zeros of P_n, $n \in \mathbb{N}$, belong to K. This class includes many polynomials orthogonal with respect to various weights on K, and polynomials minimizing various L^p norms; see [1] and [21] for numerous examples. Among the classical families on the real line, we mention Legendre, Chebyshev and Jacobi polynomials (normalized to be monic). Asymptotically extremal polynomials have interesting asymptotic zero distributions. Let $\{\alpha_{k,n}\}_{k=1}^n$ be the zeros of P_n. Define the counting measure for the set $\{\alpha_{k,n}\}_{k=1}^n$ by

$$\tau_n = \frac{1}{n} \sum_{k=1}^n \delta_{\alpha_{k,n}},$$

where $\delta_{\alpha_{k,n}}$ is a unit point mass at $\alpha_{k,n}$. If $K \subset \mathbb{R}$, $\operatorname{cap}(K) \neq 0$, and the P_n are asymptotically extremal for K, then the τ_n form a sequence of positive unit Borel measures that converge in the weak* topology to the equilibrium measure of K; see Theorem 1.7 of [1, p. 55]. The definition of weak* convergence states that

$$\lim_{n \to \infty} \int_{\mathbb{C}} f \, d\tau_n = \int_{\mathbb{R}} f \, d\mu_K$$

for any continuous function f on $\mathbb{C}$. This enables us to obtain information on the limiting behavior of means of zeros of P_n. In particular, we have the following result stated for K normalized by $\operatorname{cap}(K) = 1$. (The case of arbitrary capacity can be reduced to this by a linear change of variable.)

Corollary 4.2. *Suppose that $\phi : \mathbb{C} \to \mathbb{R}$ is continuous, and ϕ is convex on $\mathbb{R}$. Assume that $K \subset \mathbb{R}$ is compact, $\operatorname{cap}(K) = 1$ and $\int_K x \, d\mu_K(x) = 0$. If P_n, $n \in \mathbb{N}$, is a sequence of asymptotically extremal polynomials for K, then we have for the*

ϕ-arithmetic means of their zeros that

$$(12) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(\alpha_{k,n}) = \int_K \phi(x)\, d\mu_K(x)$$

$$\geq \int_L \phi(x)\, d\mu_L(x) = \int_{-2}^{2} \frac{\phi(x)\, dx}{\pi\sqrt{4-x^2}} =: \ell(\phi).$$

If $K \setminus L$ has positive capacity and the restriction of ϕ to L is not a linear function, then strict inequality holds.

In particular, if $\phi(x) = |x|^m$, $m \in \mathbb{N}$, then

$$\ell(|x|^m) = 2^m \frac{\Gamma(m/2 + 1/2)}{\sqrt{\pi}\,\Gamma(m/2 + 1)},$$

because the change of variable $x = 2t^{1/2}$ reduces the integral for $\ell(|x|^m)$ to a beta integral. Hence $\ell(|x|) = 4/\pi$ and $\ell(x^2) = 2$.

Proof. Since ϕ is continuous on $\mathbb{R}$, the first equality in (12) follows from the weak* convergence of τ_n to μ_K. The inequality (and when it becomes equality) is immediate from Theorem 1. $\qquad\square$

We also state a version of this result for polynomials with positive zeros.

Corollary 4.3. *Assume $\phi : [0,\infty) \to \mathbb{R}$ and that $\phi(x^2)$ is convex on $\mathbb{R}$. Suppose $K \subset [0,\infty)$ is compact and $\mathrm{cap}(K) = 1$. If P_n, $n \in \mathbb{N}$, is a sequence of asymptotically extremal polynomials for K, and if each P_n has all its zeros positive, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(\alpha_{k,n}) = \int_K \phi(x)\, d\mu_K(x) \geq \int_0^4 \frac{\phi(x)\, dx}{\pi\sqrt{x(4-x)}} =: \ell_+(\phi).$$

If $K \setminus [0,4]$ has positive capacity and the restriction of $\phi(x^2)$ to $[0,4]$ is not a linear function, then strict inequality holds.

In particular, setting $\phi(x) = x^m$, $m \in \mathbb{N}$, gives

$$\ell_+(x^m) = \int_0^4 \frac{x^m\, dx}{\pi\sqrt{x(4-x)}} = 2^m \frac{1 \cdot 3 \cdot \ldots \cdot (2m-1)}{m!}.$$

The first few values of $\ell_+(x^m)$ are 2 for $m = 1$, 6 for $m = 2$, and 20 for $m = 3$.

Proof. The proof is essentially the same as for Corollary 4.2. For the inequality, one should apply the change of variable $x = t^2$ and define the compact set $\sqrt{K} = \{t \in \mathbb{R} : t^2 \in K\}$. Then $\sqrt{K}$ is symmetric about the origin, so that $\int_{\sqrt{K}} t \, d\mu_{\sqrt{K}}(t) = 0$. Furthermore, $d\mu_{\sqrt{K}}(t) = d\mu_K(t^2)$, $t \in \sqrt{K}$, and $\text{cap}(\sqrt{K}) = 1$; see [18, p. 134]. Now apply Theorem 1 to $\sqrt{K}$. $\square$

A consequence of Corollary 4.3 is that we also have information on the asymptotic behavior of the coefficients of P_n. For example, if $P_n(x) = x^n + a_{n-1,n}x^{n-1} + \ldots + a_{0,n} = \prod_{k=1}^{n}(x - \alpha_{k,n})$ then $a_{n-1,n} = -\sum_{k=1}^{n} \alpha_{k,n}$. Hence

$$\lim_{n \to \infty} \frac{a_{n-1,n}}{n} = -\int_K x \, d\mu_K(x) \le -2$$

under the assumptions of Corollary 4.3, with equality for $K = [0, 4]$.

4.3. Equilibrium measure and Green's function when K is the union of several intervals.

Let $K = \bigcup_{l=1}^{N}[a_l, b_l]$, where $a_1 < b_1 < a_2 < b_2 < \ldots < a_N < b_N$ are real numbers. Define the function $R(z) = \prod_{l=1}^{N}(z - a_l)(z - b_l)$. Consider the branch of $\sqrt{R(z)}$, satisfying $\lim_{z \to \infty} \sqrt{R(z)}/z^N = 1$, which is analytic in $\mathbb{C} \setminus \bigcup_{l=1}^{N}[a_l, b_l]$. For future reference, we describe the values of $\sqrt{R(z)}$ on the real line:

$$(13) \qquad \sqrt{R(x)} = \begin{cases} \sqrt{|R(x)|}, & x \ge b_N, \\ (-1)^{N+l} i \sqrt{|R(x)|}, & a_l \le x \le b_l, \ l = 1, \ldots, N, \\ (-1)^{N+l} \sqrt{|R(x)|}, & b_l \le x \le a_{l+1}, \ l = 1, \ldots, N-1, \\ (-1)^N \sqrt{|R(x)|}, & x \le a_1. \end{cases}$$

Here, the values of $\sqrt{R(x)}$ for $x \in \bigcup_{l=1}^{N}[a_l, b_l]$ are the limit values of $\sqrt{R(z)}$ when $\text{Im } z \to 0^+$.

When $K = L = [-2, 2]$, then $R(z) = z^2 - 4$ and for $-2 < x < 2$ we have

$$d\mu_L(x) = \frac{dx}{\pi\sqrt{4 - x^2}} = \frac{dx}{\pi i \sqrt{R(x)}}.$$

We give the following explicit representation for the equilibrium measure of the set K (see also [21] and [23]).

Proposition 4.4. *Let $K = \bigcup_{l=1}^{N} [a_l, b_l] \subset \mathbb{R}$. There exists a polynomial $T(x) = -x^{N-1} + \ldots \in \mathbb{R}_{N-1}[x]$, such that the equilibrium measure of K is given by*

$$(14) \qquad d\mu_K(x) = \frac{T(x)\, dx}{\pi i \sqrt{R(x)}}, \qquad x \in \bigcup_{l=1}^{N} [a_l, b_l].$$

Furthermore, when $N \geq 2$ we have $T(x) = -\prod_{j=1}^{N-1}(x - z_j)$ with $z_j \in (b_j, a_{j+1})$, $j = 1, \ldots, N-1$, and

$$(15) \qquad \int_K x\, d\mu_K(x) = \sum_{l=1}^{N} \frac{a_l + b_l}{2} - \sum_{l=1}^{N-1} z_l.$$

For the proof of Proposition 4.4, we need the following simple lemma.

Lemma 4.5. *Let $K = \bigcup_{l=1}^{N} [a_l, b_l]$. For any $T_{N-1} \in \mathbb{R}_{N-1}[x]$, we have*

$$(16) \qquad \frac{1}{\pi i} \int_K \frac{T_{N-1}(t)\, dt}{(t - z)\sqrt{R(t)}} = \begin{cases} 0, & z \in \bigcup_{l=1}^{N} (a_l, b_l), \\ T_{N-1}(z)/\sqrt{R(z)}, & z \in \mathbb{C} \setminus K, \end{cases}$$

where the integral is understood in the Cauchy principal value sense.

We remind the reader that when $t \in \mathbb{R}$, $\sqrt{R(t)}$ is defined to be $\lim_{s \to 0+} \sqrt{R(t + is)}$.

Proof of Lemma 4.5. For $z \in \mathbb{C} \setminus K$ define $f(z) = T_{N-1}(z)/\sqrt{R(z)}$. It is easy to see that the limit values of $\sqrt{R(z)}$ as z tends to a point of K from above and from below are negatives of each other, so the same is true for f. Thus, with obvious notation,

$$(17) \qquad f(z+) = f(z) = -f(z-), \quad z \in K.$$

Consider a contour Γ which consists of N simple closed curves, one around each of the intervals $[a_l, b_l]$, and located close to those intervals. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z}\, dt = f(z)$$

for z in the exterior of Γ, and for $z \in K$ the integral equals zero.

Taking $z \in \mathbb{C} \setminus K$, letting Γ shrink to K, and using (17), we obtain

$$f(z) = \frac{1}{\pi i} \int_K \frac{f(t)}{t - z}\, dt, \quad z \in K^c,$$

as asserted by the Lemma.

Next, take $z \in \cup_{l=1}^{N}(a_l, b_l)$. The existence of the Cauchy principal value at z for the function f follows from the results in Chapter 2 in [13], which also contains a discussion of Plemelj's formula. This formula asserts that the Cauchy principal value satisfies

$$\frac{1}{\pi i} \int_K \frac{f(t)}{t-z}\, dt = \frac{f(z+) + f(z-)}{2}, \quad z \in \cup_{l=1}^{N}(a_l, b_l).$$

By (17), the right hand side is zero. This completes the proof of the lemma. □

Proof of Proposition 4.4. We shall deduce (14) from Lemma 4.5. Select $T(t) = \sum_{j=0}^{N-1} c_j t^j \in \mathbb{R}_{N-1}[t]$ so that it satisfies the following equations:

$$(18) \qquad \int_{b_l}^{a_{l+1}} \frac{T(t)\, dt}{\sqrt{R(t)}} = \sum_{j=0}^{N-1} c_j \int_{b_l}^{a_{l+1}} \frac{t^j\, dt}{\sqrt{R(t)}} = 0, \quad l = 1, \dots, N-1,$$

and

$$(19) \qquad \frac{1}{\pi i} \int_K \frac{T(t)\, dt}{\sqrt{R(t)}} = \sum_{j=0}^{N-1} \frac{c_j}{\pi i} \int_K \frac{t^j\, dt}{\sqrt{R(t)}} = 1.$$

The polynomial $T(t)$ is defined by these equations uniquely, because the corresponding homogeneous system of linear equations (with zero on the right of (19)), in the coefficients c_j of $T(t)$, has only the trivial solution. Indeed, let $T_h(t)$ be a nontrivial solution of this homogeneous system. Since the sign of $\sqrt{R(t)}$ is constant on each (b_l, a_{l+1}) by (13), $T_h(t)$ must change sign on each $[b_l, a_{l+1}]$, $l = 1, \dots, N-1$, by (18). Hence $T_h(t)$ has a simple zero in each (b_l, a_{l+1}), $l = 1, \dots, N-1$, and its sign alternates on the intervals $[a_l, b_l]$, $l = 1, \dots, N$. (Note that the same is true for $T(t)$.) It follows from (13) that $T_h(t)/(\pi i \sqrt{R(t)})$ doesn't change sign on K, contradicting

$$\frac{1}{\pi i} \int_K \frac{T_h(t)\, dt}{\sqrt{R(t)}} = 0.$$

Thus $T(t)$ exists and is unique. In addition, the above argument and (19) show that $T(t)/(\pi i \sqrt{R(t)})$ keeps positive sign on K, that is, (14) actually defines a positive unit Borel measure on K.

As in the proof of Lemma 4.5, set $f = T/\sqrt{R}$. Let

$$h(z) = \frac{1}{\pi i} \int_K \frac{f(t)}{t-z}\, dt, \quad z \in \mathbb{C}.$$

Then h is the Cauchy transform of $f\mathbb{1}_K$ in $\mathbb{C}$ and is the Hilbert transform of $f\mathbb{1}_K$ on $\mathbb{R}$. It is easy to see that $f \in L^p(\mathbb{R})$ for each $1 < p < 2$. From M. Riesz's conjugate function theorem (see for example Stein-Weiss [22]), it follows that $h \in L^p(\mathbb{R})$.

From Lemma 4.5, we see that $h = 0$ on K except at endpoints, and $h = f$ on K^c. Define

$$u(z) = \frac{1}{\pi i} \int_K (\log |z - t|) f(t) \, dt.$$

Then u is continuous on $\mathbb{C}$ and $u_x = -\operatorname{Re} h$ in the open upper half plane. Since $h \in L^p(\mathbb{R})$, the function $h(\cdot + iy)$ converges to $h(\cdot)$ in $L^p(\mathbb{R})$ when $y \to 0+$, and hence converges to h in $L^1(a_1, b_N)$. Thus, for $x \in [a_1, b_N]$,

$$u(x + iy) - u(a_1 + iy) = -\operatorname{Re} \int_{a_1}^x h(t + iy) \, dt$$
$$\to -\operatorname{Re} \int_{a_1}^x h(t) \, dt = -\int_{a_1}^x h(t) \, dt.$$

The last equality holds because $h = 0$ on K and $h = f$ with f real in the gaps between the intervals of K. Combining this description of h with (18), we see that if $x \in K$ then the last integral is zero. Since $u(x + iy) \to u(x)$ as $y \to 0+$ for all x, we conclude that u is constant on K. Then Frostman's theorem and the uniqueness of the equilibrium measure imply that $f(x)/(\pi i) \, dx = T(x) \, dx/(\pi i \sqrt{R(x)})$ is the equilibrium measure for K.

We now show that the leading coefficient of T is -1. Observe that (16) gives for $z = 0$ and $T_{N-1}(x) = x^{j+1}$ that

$$\frac{1}{\pi i} \int_K \frac{t^j \, dt}{\sqrt{R(t)}} = 0, \qquad j = 0, \dots, N - 2.$$

Also, recall that near infinity

$$\left(1 - \frac{a}{z}\right)^{-1/2} = 1 + \frac{1}{2} \frac{a}{z} + \dots.$$

Therefore, we have the following Laurent expansion at infinity

(20) $$\frac{z^N}{\sqrt{R(z)}} = 1 + \frac{1}{2} \sum_{l=1}^N (a_l + b_l) \frac{1}{z} + \dots.$$

Applying the same argument as in the proof of Lemma 4.5 and evaluating the residue at infinity by (20), we obtain

$$\frac{1}{\pi i} \int_K \frac{t^{N-1} \, dt}{\sqrt{R(t)}} = -\frac{1}{2\pi i} \oint_{|z|=r} \frac{z^{N-1} \, dz}{\sqrt{R(z)}} = -1.$$

Hence (19) gives $c_{N-1} = -1$. Similarly, we have

$$\int_K x \, d\mu_K(x) = -\frac{1}{\pi i} \int_K \frac{x^N \, dx}{\sqrt{R(x)}} + c_{N-2} \frac{1}{\pi i} \int_K \frac{x^{N-1} \, dx}{\sqrt{R(x)}}$$

$$= \sum_{l=1}^{N} \frac{a_l + b_l}{2} - \sum_{l=1}^{N-1} z_l,$$

because $c_{N-2} = \sum_{l=1}^{N-1} z_l$. □

We remark that the zeros of the polynomial T are exactly the critical points of the Green's function $g(z, \infty, K^c)$ for the domain $K^c = \overline{\mathbb{C}} \setminus K$, with pole at infinity. Indeed, we have for $g(z, \infty, K^c) = \int_K \log |z - t| \, d\mu_K(t) - \log \text{cap}(K)$ that

$$g_x(x, \infty, K^c) = \frac{1}{\pi i} \int_K \frac{T(t) \, dt}{(x - t)\sqrt{R(t)}} = -\frac{T(x)}{\sqrt{R(x)}}, \quad x \in \mathbb{R} \setminus K,$$

by (16). Moreover, $g_y(z, \infty, K^c)$ is zero on $\mathbb{R} \setminus K$ and is never zero on $\mathbb{C} \setminus \mathbb{R}$.

Thus we can obtain interesting information about location of the critical points. For the background material on the critical points of Green's function see Chapter VII of Walsh [24]. If $K = [a_1, b_1] \cup [a_2, b_2]$ and $|b_1 - a_1| = |b_2 - a_2|$, then it follows by an elementary symmetry argument that $z_1 = (b_1 + a_2)/2$. Also, if $|b_1 - a_1| > |b_2 - a_2|$ then $z_1 > (b_1 + a_2)/2$. But the location of critical points becomes difficult to predict for three or more intervals.

The following inequality gives information on the average position of the critical points in terms of the midpoints of the gaps between the intervals of K.

Corollary 4.6. *Let* $K = \bigcup_{l=1}^{N} [a_l, b_l] \subset \mathbb{R}$ *satisfy* $\text{cap}(K) = 1$. *With the above notation, we have*

$$\sum_{l=1}^{N-1} \left(\frac{b_l + a_{l+1}}{2} - z_l \right) \geq 2 - \frac{b_N - a_1}{2},$$

where the sum is interpreted to be 0 for $N = 1$. *Equality holds above if and only if* K *is a segment of length 4.*

Proof. We consider the integral $\int_K x \, d\mu_K(x)$, and observe that translating the set K by a constant $c \in \mathbb{R}$ changes the integral by adding c. Hence we may assume that $a_1 = 0$, and must show that $\int_K x \, d\mu_K(x) \geq 2$, with equality only for $K = [0, 4]$. Define $\sqrt{K} = \{t \in \mathbb{R} : t^2 \in K\}$. Then, as in the proof of Corollary 4.3, $\sqrt{K}$ is symmetric about the origin, $\int_{\sqrt{K}} x \, d\mu_{\sqrt{K}}(x) = 0$, and $\mathrm{cap}(\sqrt{K}) = 1$. Moreover,

$$\int_{\sqrt{K}} t^2 \, d\mu_{\sqrt{K}}(t) = \int_K x \, d\mu_K(x) = \sum_{l=1}^{N} \frac{a_l + b_l}{2} - \sum_{l=1}^{N-1} z_l$$

by (15). Applying Theorem 1 with $\phi(t) = t^2$, we obtain that

$$\int_{\sqrt{K}} t^2 \, d\mu_{\sqrt{K}}(t) \geq \int_{-2}^{2} \frac{t^2 \, dt}{\pi \sqrt{4 - t^2}} = 2,$$

with equality possible only if $\sqrt{K} = [-2, 2]$ and $K = [0, 4]$. □

Using higher moments will give more complicated inequalities involving the endpoints of K and zeros (or coefficients) of T.

5. Proof of Theorem 2

Let K be a compact connected subset of $\mathbb{C}$ with $\mathrm{cap}(K) = 1$. The connectivity of K implies that each boundary point of the domain K^c is regular for the Dirichlet problem in K^c, which, in turn, implies that the Green function of K^c is continuous in $\mathbb{C}$.

We shall assume also that the conformal centroid of K is at the origin. That is:

$$\int_K z \, d\mu_K(z) = 0.$$

Then by Theorem 1.4 of [15, p. 19], we have

$$K \subset \overline{\mathbb{B}(2)}.$$

Recall that $L = [-2, 2]$. Then $\mathrm{cap}(L) = 1$ and the conformal centroid of L is at the origin. We shall apply the considerations of §2 with $K_1 = K$, $K_2 = L$ and $R = 2$.

Write $g = g_1$ and $G = g_2$ for the respective Green's functions of K^c and L^c with poles at ∞. The function $w(x)$ is defined on $\mathbb{R}$ by

$$(21) \qquad w(x) = \int_{\mathbb{R}} [g(x + is) - G(x + is)]\, ds, \quad x \in \mathbb{R}.$$

By the Formula in §2, to prove Theorem 2 it suffices to prove that

$$w(x) \leq 0, \quad x \in \mathbb{R}.$$

To accomplish this, we shall extend w to a certain function $w(z)$ which is subharmonic in the upper half plane $\mathcal{H}$.

For sets $E \subset \mathbb{R}$, let E^b denote the complement of E in $\mathbb{R}$:

$$E^b = \mathbb{R} \setminus E.$$

Also, let $|E|$ denote the one-dimensional Lebesgue measure of E, and for $y \geq 0$, let

$$I(y) = [-y, y].$$

For bounded $E \subset \mathbb{R}$ with $|E| = 2y$ and $x \in \mathbb{R}$, set

$$w(x, E) = \int_{\mathbb{R}} [\mathbb{1}_{E^b}(s) g(x + is) - \mathbb{1}_{I(y)^b}(s) G(x + is)]\, ds$$

$$= w(x) + \int_{I(y)} G(x + is)\, ds - \int_E g(x + is)\, ds,$$

where $\mathbb{1}$ denotes a characteristic or indicator function. The asymptotic behavior of g and G (discussed in §2) ensures that the first integral is absolutely convergent. The second equality follows from (21).

Now take $z = x + iy \in \mathcal{H}$, and define

$$(22) \qquad w(z) = \sup_E w(x, E),$$

where the sup is taken over all bounded measurable $E \subset \mathbb{R}$ with $|E| = 2y$.

For each x, we have $g(x + is) \geq 0$ and $\lim_{|s| \to \infty} g(x + is) = \infty$. The analysis on p.149 of [2] is applicable to $-g(x + is)$ as a function of s, and shows that for each $y \in [0, \infty)$ there exists a set $E \subset \mathbb{R}$ with $|E| = 2y$ for which the supremum of $-\int_E g(x + is)\, ds$ over all E with $|E| = 2y$ is attained. Note the minus sign in $-g$. Moreover, there exists a number $t \geq 0$ such that $\{s \in \mathbb{R} : g(x + is) < t\} \subset E \subset \{g(x + is) \leq t\}$, and E is bounded. We shall denote such a maximizing set by $E(z)$. Then

(23) $$w(z) = \int_{\mathbb{R}} [\mathbb{1}_{E(z)^b}(s)g(x+is) - \mathbb{1}_{I(y)^b}(s)G(x+is)]\, ds$$

(24) $$= w(x) + \int_{I(y)} G(x+is)\, ds - \int_{E(z)} g(x+is)\, ds$$

(25) $$= \int_{|s|>y} [g(x+is) - G(x+is)]\, ds$$

$$+ \int_{I(y)} g(x+is)\, ds - \int_{E(z)} g(x+is)\, ds$$

The following lemma provides information on the maximizing sets $E(z)$.

Lemma 5.1. *With the situation as above, there exist positive constants b and k depending only on K such that whenever $z = x + iy \in \mathcal{H}$:*

(a) if $y \geq b$ then $E(z) = I(y) + t = [-y + t, y + t]$, for some t with $|t| < k/y$;

(b) if $y \leq b$ then $E(z) \subset [-2b, 2b]$.

Proof of Lemma 5.1. By (2), we can write

(26) $$g(z) = \log|z| + h(z)$$

where h is harmonic outside $\overline{\mathbb{B}(2)}$. The conformal centroid of K is at the origin, and so the coefficient a_1 in (2) equals 0. Thus, h satisfies $|h(z)| \leq \frac{k}{8}|z|^{-2}$ and $|\nabla h(z)| \leq \frac{k}{8}|z|^{-3}$ for $|z| \geq 3$, for some positive constant k.

From $g_y = y|z|^{-2} + h_y$, it easily follows that there exists $b_0 \geq 3$ such that $g_y(x+iy) > 0$ whenever $y \geq b_0$ and $g_y(x+iy) < 0$ whenever $y \leq -b_0$.

Now we establish two estimates:

(27) $$|g(z) - g(\bar{z})| \leq \frac{k}{4}|z|^{-2},$$

(28) $$g(z+it) - g(z) \geq \frac{1}{2}ty|z|^{-2} - \frac{k}{4}|z|^{-2},$$

when $z = x + iy \in \mathcal{H}, |z| \geq 3$ and $t \in (0, |z|]$. The first estimate is obvious from (26). The second follows similarly, because $\mathrm{Re}(it/z) \in (0, 1]$ and so

$$\log|(z+it)/z| \geq \log(1 + \mathrm{Re}(it/z)) \geq \frac{1}{2}\mathrm{Re}(it/z) = \frac{1}{2}ty|z|^{-2}.$$

Moreover, there exists a number $b > b_0 \geq 3$ such that $g(x+is) > g(x+is_0)$ whenever $|s| \geq b$ and $|s_0| \leq b_0$, as we now show. For $|x| \leq 3$ one just takes b

large enough that $\max_{S_0} g < \min_{|x| \leq 3} g(x \pm ib)$ where $S_0 = [-3, 3] \times [-b_0, b_0]$, recalling here that g is continuous and finite in the plane. For $|x| > 3$, one estimates $|g(x + is) - \log|x + is|| \leq \frac{k}{8} x^{-2}$ and uses concavity of the function $t \mapsto \log \frac{1+b^2 t}{1+b_0^2 t}, t \in [0, \infty)$, together with monotonicity properties of g; note that for our purposes, $t = x^{-2} \in (0, 1/9)$. Details are left to the interested reader.

Now fix $x \in \mathbb{R}$ and visualize the graph of $p(s) = g(x + is)$. The function p is strictly increasing on $[b_0, \infty)$, strictly decreasing on $(-\infty, -b_0]$, and $p(s) > p(s_0)$ for every $|s_0| \leq b_0$ and $|s| \geq b$. For $\alpha > 0$, write $E_\alpha = \{s : p(s) < \alpha\}$. Then E_α is a maximal set of measure $|E_\alpha|$. Set $\alpha_0 = \min\{p(-b), p(b)\}$ and $y_0 = \frac{1}{2}|E_{\alpha_0}|$. Then E_{α_0} is a single interval which contains $[-b_0, b_0]$, and $y_0 \leq b$. Given $y \geq b$, there is a unique $\alpha \geq \alpha_0$ such that $|E_\alpha| = 2y$. Then $E(x + iy) = E_\alpha$, and this E_α also is a single interval containing $[-b_0, b_0]$. These facts imply that $E(x + iy)$ has the form $[-y + t, y + t]$, where $|t| \leq y - b_0$. Further, the maximality of $E(x + iy)$ and continuity of g imply that $p(-y + t) = p(y + t) = \alpha$.

Take $z = x + iy \in \mathcal{H}$ with $y \geq b$. Suppose the number t in the previous paragraph is nonnegative; the case $t \leq 0$ is handled analogously. Let $z_2 = z + it, z_1 = \bar{z} + it$. Then

$$\frac{1}{2} ty|z|^{-2} - \frac{k}{4}|z|^{-2} \leq g(z_2) - g(z) \qquad \text{by (28)}$$

$$\leq g(z_2) - g(\bar{z}) + \frac{k}{4}|z|^{-2} \qquad \text{by (27)}$$

$$< g(z_2) - g(z_1) + \frac{k}{4}|z|^{-2} \qquad \text{since } g(z_1) = \alpha < g(\bar{z})$$

$$= \frac{k}{4}|z|^{-2}$$

because $g(z_2) = g(z_1) = \alpha$ as above. Hence $ty < k$, proving part (a).

To prove (b), take $z = x + iy \in \mathcal{H}$ with $y \leq b$, and let $E(z)$ be a maximizing set for z, so that $|E(z)| = 2y$. Suppose $E(z)$ intersects the interval $(2b, \infty)$ in a set of measure $\epsilon > 0$. Then the set $[0, 2b] \setminus E(z)$ has measure at least ϵ. Since $g(x + is_0) < g(x + is)$ when $0 < s_0 < 2b < s$, we can strictly decrease $\int_{E(z)} g(x + is) \, ds$ if we move $E(z) \cap (2b, \infty)$ into some subset of $[0, 2b] \setminus E(z)$. This violates the definition of maximizing set, and shows that $E(z)$ cannot intersect the interval $(2b, \infty)$ in a set of positive measure. Similarly it cannot intersect $(-\infty, -2b)$. Thus $E(z) \subset [-2b, 2b]$, after possibly deleting a set of zero measure from E.

□

Here now is the main ingredient in the proof of the theorem.

Claim. *w is subharmonic in* $\mathcal{H}$.

Let us carry out the proof of Theorem 2 assuming the claim.

Firstly, the function w is continuous on $\mathcal{H} \cup \mathbb{R}$. It is continuous also at infinity, because $w(z) \to 0$ as $z \to \infty$ in $\mathcal{H}$, as we now show. From (25) it suffices to show that

$$\lim_{z \to \infty} \int_{|s| > y} [g(x + is) - G(x + is)] \, ds = 0$$

and

$$\lim_{z \to \infty} \left(\int_{I(y)} g(x + is) \, ds - \int_{E(z)} g(x + is) \, ds \right) = 0.$$

The first is a simple consequence of (3). The second follows from Lemma 5.1: when $y \geq b$ use part (a) of the lemma, and then decomposition (26), and when $y \leq b$ with $|x| \to \infty$, use part (b) of the lemma and then decomposition (26). A key fact for the latter case is that $|E(y)| = |I(y)|$. Details are left to the reader.

Continuing now with the proof of Theorem 2, set

$$M = \sup_{\mathcal{H} \cup \mathbb{R}} w,$$

where the supremum is finite because w is bounded at infinity. Note $M \geq 0$, since from §2 we know $w(x) = 0$ for $|x| \geq 2$.

If $M > 0$, then by continuity of w and the strong maximum principle, there exists $x_0 \in \mathbb{R}$ such that $w(x_0) = M$ and $w(z) < w(x_0)$ for each $z \in \mathcal{H}$. Since K is connected, its orthogonal projection onto the real axis is a single interval $[c_1, c_2]$, and since $K \subset \overline{\mathbb{B}(2)}$ we have $[c_1, c_2] \subset [-2, 2]$. By Lemma 2.1, $w(x)$ is concave on every bounded subinterval of $(-\infty, c_1)$, hence is concave on $(-\infty, c_1)$. Similarly, w is concave on (c_2, ∞). Since $w(x) = 0$ for $|x| \geq 2$, we must have $w \leq 0$ on $\mathbb{R} \setminus [c_1, c_2]$. Thus $x_0 \in [c_1, c_2]$. Since x_0 is a maximizing point for w, Lemma 2.2 implies we must have

(29) $$\liminf_{y \to 0+} \frac{w(x_0) - w(x_0 + iy)}{y} > 0.$$

On the other hand, from (24), we see that for $y > 0$,

$$(30) \qquad w(x_0) - w(x_0 + iy) = \int_{E(x_0+iy)} g(x_0 + is)\, ds - \int_{I(y)} G(x_0 + is)\, ds.$$

Since $E(x_0 + iy)$ maximizes integrals of $-g$, for each $y > 0$ and for each bounded $E \subset \mathbb{R}$ with $|E| = 2y$ we have

$$(31) \qquad\qquad 0 \le \int_{E(x_0+iy)} g(x + is)\, ds \le \int_{E} g(x + is)\, ds.$$

Further, because $x_0 \in [c_1, c_2]$ there exists $s_0 \in \mathbb{R}$ with $x_0 + is_0 \in K$, so that $g(x_0 + is_0) = 0$. Taking $E = [s_0 - y, s_0 + y]$ and using continuity of g, we see from (31) that

$$\lim_{y \to 0+} \frac{1}{y} \int_{E(x_0+iy)} g(x_0 + is)\, ds = 0.$$

Similarly, $G(x_0) = 0$, and hence $\lim_{y \to 0+} \frac{1}{y} \int_{I(y)} G(x_0 + is)\, ds = 0$. Thus, by (30),

$$\lim_{y \to 0+} \frac{w(x_0) - w(x_0 + iy)}{y} = 0,$$

which contradicts (29).

We conclude that $M > 0$ is impossible, and so $M = 0$, meaning $w \le 0$ in $\mathcal{H} \cup \mathbb{R}$. This completes the proof of the inequality in Theorem 2, modulo the Claim.

To prove the strict inequality statement in the theorem, let K be a compact set satisfying the hypotheses of Theorem 2 which does not coincide with L. Then K, which is contained in $\overline{\mathbb{B}(2)}$, cannot contain the points -2 or 2, because if it did then it would equal $[-2, 2] = L$ by the equality case of [15, Theorem 1.4]. Hence $-2 < c_1 \le c_2 < 2$. The argument that gave $w(t) < 0$ for some $t > 2$ in the proof of Theorem 1 works again here, except with K and L interchanged, producing that $w(t) < 0$ for every $t \in (-2, 2) \setminus [c_1, c_2]$.

Now we show $w(t) < 0$ for every $t \in [c_1, c_2]$. Suppose instead that $w(x_0) = 0$ for some $x_0 \in [c_1, c_2]$. Note $w \le 0$ is not identically zero in $\mathcal{H}$, by the preceding paragraph, and so $w < 0$ in $\mathcal{H}$ by the strong maximum principle. Now rerun the argument used above to rule out the case $M > 0$, to obtain a contradiction. Hence $w < 0$ on $[c_1, c_2]$.

We have shown $w < 0$ on $(-2, 2)$, and so formula (5) (with K and L interchanged) implies the strict inequality that

$$\int_K \phi(\operatorname{Re} z)\, d\mu_K(z) < \int_L \phi(\operatorname{Re} z)\, d\mu_L(z),$$

when ϕ is convex on $\mathbb{R}$ and is not a linear function on $[-2, 2]$.

Proof of the Claim. Fix $z = x + iy \in \mathcal{H}$. Let $E(z)$ be a corresponding maximal set of measure $2y$, as in (24). For brevity, we'll write

$$E(z) = E$$

and also

$$I(y) = I.$$

Then (23) says

(32) $$w(z) = \int_{\mathbb{R}} [\mathbb{1}_{E^b}(s)g(x + is) - \mathbb{1}_{I^b}(s)G(x + is)]\, ds.$$

Take $\rho \in (0, y)$. To prove subharmonicity of w it suffices to show that $w(z)$ is less than or equal to the mean value of w over the circle with center z and radius ρ.

The function g is subharmonic in $\mathbb{C}$ and G is harmonic in $\mathbb{C} \setminus [-2, 2]$. Thus

$$g(x + is) \le \frac{1}{2\pi} \int_0^\pi [g(x + is + \rho e^{i\phi}) + g(x + is + \rho e^{-i\phi})]\, d\phi, \quad s \in \mathbb{R}.$$

If $|s| > y$, then equality holds when g is replaced by G. Substitute the inequality and equality into (32), and switch the order of integration on the right. This gives the inequality

(33) $$2\pi w(z) \le \int_0^\pi [J(\phi) + J(-\phi)]\, d\phi,$$

where

$$J(\phi) = \int_{\mathbb{R}} [\mathbb{1}_{E^b}(s)g(x + is + \rho e^{i\phi}) - \mathbb{1}_{I^b}(s)G(x + is + \rho e^{i\phi})]\, ds.$$

Fix $\phi \in [0, \pi]$ and set $\epsilon = \rho \sin \phi$. In $J(\phi)$, substitute

$$x + is + \rho e^{i\phi} = x + \rho \cos \phi + i(s + \epsilon),$$

then make the change of variable $t = s + \epsilon$, and integrate over $\mathbb{R}$. We obtain

$$J(\phi) = \int_{\mathbb{R}} [\mathbb{1}_{E^b+\epsilon}(t)g(x + \rho\cos\phi + it) - \mathbb{1}_{I^b+\epsilon}(t)G(x + \rho\cos\phi + it)]\,dt.$$

The same equation holds when ϕ is changed to $-\phi$ and ϵ to $-\epsilon$. It follows that, for $\phi \in [0, \pi]$,

$$\begin{aligned}
J(\phi) + J(-\phi) = \int_{\mathbb{R}} \{&[\mathbb{1}_{E^b+\epsilon} + \mathbb{1}_{E^b-\epsilon}](t)g(x + \rho\cos\phi + it) \\
&- [\mathbb{1}_{I^b+\epsilon} + \mathbb{1}_{I^b-\epsilon}](t)G(x + \rho\cos\phi + it)\}\,dt.
\end{aligned}$$

The argument on the top half of p. 148 of [2] shows that for our set $E = E(x + iy)$ and for $0 < \epsilon < y$ there exist bounded measurable sets A and B in $\mathbb{R}$ such that $|A| = 2(y + \epsilon)$, $|B| = 2(y - \epsilon)$ and

$$\mathbb{1}_{E+\epsilon} + \mathbb{1}_{E-\epsilon} = \mathbb{1}_A + \mathbb{1}_B.$$

Using $\mathbb{1}_A = 1 - \mathbb{1}_{A^b}$, etc., one sees that this equation also holds when the four sets are replaced by their complements in $\mathbb{R}$. Furthermore, $(E \pm \epsilon)^b = E^b \pm \epsilon$, and, recalling that $I = I(y)$, one can check directly that $\mathbb{1}_{I^b+\epsilon} + \mathbb{1}_{I^b-\epsilon} = \mathbb{1}_{I(y+\epsilon)^b} + \mathbb{1}_{I(y-\epsilon)^b}$. Thus,

$$\begin{aligned}
J(\phi) &+ J(-\phi) \\
&= \int_{\mathbb{R}} \{[\mathbb{1}_{A^b} + \mathbb{1}_{B^b}](t)g(x + \rho\cos\phi + it) \\
&\qquad\qquad - [\mathbb{1}_{I(y+\epsilon)^b} + \mathbb{1}_{I(y-\epsilon)^b}](t)G(x + \rho\cos\phi + it)\}\,dt \\
&= \int_{\mathbb{R}} [\mathbb{1}_{A^b}(t)g(x + \rho\cos\phi + it) - \mathbb{1}_{I(y+\epsilon)^b}(t)G(x + \rho\cos\phi + it)]\,dt \\
&\quad + \int_{\mathbb{R}} [\mathbb{1}_{B^b}(t)g(x + \rho\cos\phi + it) - \mathbb{1}_{I(y-\epsilon)^b}(t)G(x + \rho\cos\phi + it)]\,dt \\
&\leq w(x + \rho\cos\phi + i(y + \epsilon)) + w(x + \rho\cos\phi + i(y - \epsilon)),
\end{aligned}$$

by the definition of w as a supremum, in (22). Substitution in (33) gives

$$\begin{aligned}
2\pi w(z) &\leq \int_0^\pi [w(x + \rho\cos\phi + i(y + \epsilon)) + w(x + \rho\cos\phi + i(y - \epsilon))]\,d\phi \\
&= \int_0^\pi [w(z + \rho e^{i\phi}) + w(z + \rho e^{-i\phi})]\,d\phi,
\end{aligned}$$

recalling $\epsilon = \rho\sin\phi$. Thus, w satisfies the sub-mean value property at z, and the Claim is proved.   □

6. **Moments involving** $|z|$

In Theorem 2 we obtained sharp upper bounds for moments of the form $\int_K \phi(\operatorname{Re} z)\, d\mu_K(z)$, where K is a continuum satisfying certain hypotheses. In this section, we again take K to be a continuum, and seek sharp upper bounds for moments of the form $\int_K \phi(|z|)\, d\mu_K(z)$. It turns out to be convenient to state the results in terms of $\phi(\log|z|)$ instead of $\phi(|z|)$.

Let K be a compact, connected subset of $\mathbb{C}$ that contains the origin, and satisfies $\operatorname{cap}(K) = 1$. As before, let $K^c = \overline{\mathbb{C}} \setminus K$. Also, set $\Omega = \{z \in \mathbb{C} : 1/z \in K^c\}$. Then the plane domain Ω is the image of the unit disk $\mathbb{B}(1)$ under a function f belonging to the class S of univalent analytic functions (conformal mappings) in the disk with $f(0) = 0$ and $f'(0) = 1$. See, for example, [18, p. 133]. The Koebe one-quarter theorem asserts that $\Omega = f(\mathbb{B}(1))$ contains the disk $\mathbb{B}(1/4)$, so that

$$K \subset \overline{\mathbb{B}(4)}.$$

The interval

$$\widetilde{L} = [0, 4]$$

satisfies all our assumptions on K, and shows that the "4" on the previous line is the smallest possible constant.

As in §2, let $g(z) = \int_K \log|z - \zeta|\, d\mu_K(\zeta)$ be the Green's function of K^c with pole at ∞. From (2), we have

$$g(z) = \log|z| - \operatorname{Re} \sum_{n=1}^{\infty} a_n z^{-n}, \quad |z| > 4,$$

where $a_n = n^{-1} \int_K \zeta^n\, d\mu_K$. From the expansion, it follows that

$$(34) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta})\, d\theta = \log r, \qquad\qquad r \geq 4,$$

$$(35) \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(re^{i\theta})\, d\theta = r^{-1}, \qquad\qquad r \geq 4.$$

Next, we give a representation of logarithmic moments of μ_K in terms of integrals involving g.

Proposition 6.1. *Assume that $K \subset \mathbb{C}$ is compact and connected, with $\mathrm{cap}(K) = 1$. Suppose $\phi \in C^2(\mathbb{R})$ is constant near $-\infty$. Then for each $R \geq 4$ we have*

$$\int_K \phi(\log|z|)\, d\mu_K(z) = \frac{1}{2\pi} \int_{\mathbb{B}(R)} g(z)\phi''(\log|z|)|z|^{-2}\, dx\, dy$$
$$+ \phi(\log R) - \phi'(\log R) \log R.$$

Proof. As in the proof of the Formula in §2, we start with $2\pi\mu_K = \Delta g$, then apply Green's formula to the integral on the left, this time in the disk $\mathbb{B}(R)$. Set $\psi(z) = \phi(\log|z|)$. Then ψ is constant on circles, and from (34) and (35) the boundary terms have the form stated. Also

$$\Delta\psi(z) = \phi''(\log|z|)|z|^{-2},$$

so the integral over $\mathbb{B}(R)$ has the form stated.   □

Our Theorem 2 takes K to be conformally centered at the origin. The next theorem assumes instead just that K contains the origin, and proves that the logarithmic moments are maximal when K equals the segment $\widetilde{L} = [0, 4]$ with one endpoint at the origin (rather than $L = [-2, 2]$, which is centered at the origin).

Let $\widetilde{G}$ denote the Green's function of $\widetilde{L}^c$ with pole at ∞.

Theorem 6.2. *Suppose $K \subset \mathbb{C}$ is compact, connected, contains the origin, and has $\mathrm{cap}(K) = 1$. Then for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_K \phi(\log|z|)\, d\mu_K(z) \leq \int_{\widetilde{L}} \phi(\log|z|)\, d\mu_{\widetilde{L}}(z).$$

This result is due to Laugesen [11, Corollary 6]. We give below a brief version of that proof, relying on Baernstein's result on integral means.

Proof. When ϕ is linear, the theorem holds with equality because

$$\int_K \log|z|\, d\mu_K(z) = g(0) = 0$$

(and similarly for L), using that $0 \in K$ by hypothesis and that every point of K is regular for the Dirichlet problem in K^c. For general convex ϕ, we can reduce by approximation to the case where ϕ is linear near $-\infty$, and hence to the case where $\phi \equiv 0$ near $-\infty$. Then by mollification we may further assume ϕ is smooth.

Then by Proposition 6.1, to prove Theorem 6.2 it suffices to show that for every $r \in (0, \infty)$,

$$\int_{-\pi}^{\pi} g(re^{i\theta}) \, d\theta \leq \int_{-\pi}^{\pi} \widetilde{G}(re^{i\theta}) \, d\theta.$$

As noted in the second paragraph of this section, there is a function f in the class S which maps $\mathbb{B}(1)$ onto the domain $\Omega = \{z \in \mathbb{C} : 1/z \in K^c\}$. Denoting the Green's function of Ω with pole at 0 by $g(z, 0, \Omega)$, the conformal invariance of Green's functions shows that

$$g(z, 0, \Omega) = g(1/z), \quad z \in \mathbb{C}.$$

Let $\widetilde{\Omega}$, $\widetilde{f}$ and $\widetilde{G}(z, 0, \widetilde{\Omega})$ be the corresponding objects for $\widetilde{L}$. Then $\widetilde{\Omega} = \mathbb{C} \setminus [1/4, \infty)$ and $\widetilde{f}(z) = \frac{z}{(1+z)^2}$, the Koebe function with omitted set on the positive real axis, and

$$g(z, 0, \widetilde{\Omega}) = \widetilde{G}(1/z), \quad z \in \mathbb{C}.$$

Thus, the conclusion of Theorem 6.2 will hold if for every $r \in (0, \infty)$,

(36)
$$\int_{-\pi}^{\pi} g(re^{i\theta}, 0, \Omega) \, d\theta \leq \int_{-\pi}^{\pi} g(re^{i\theta}, 0, \widetilde{\Omega}) \, d\theta.$$

But this inequality is true, since it is the special case $\varphi = \pi$ in inequality (35) of [2]. (The functions called there $u^*(re^{i\pi})$ and $v^*(re^{i\pi})$ equal the left and right sides of (36), respectively.) Theorem 6.2 is proved. $\qquad\square$

Corollary 6.3. *Suppose that $K \subset \mathbb{C}$ is compact, connected, contains the origin and satisfies $\mathrm{cap}(K) = 1$, and in addition that K is symmetric with respect to the origin. Then for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_K \phi(\log |z|) \, d\mu_K(z) \leq \int_L \phi(\log |z|) \, d\mu_L(z).$$

Here, as before, $L = [-2, 2]$.

Proof. To prove the Corollary, use the same construction as in the proofs of Corollaries 4.3 and 4.6. That is, let $\widetilde{K} = \{z^2 : z \in K\}$. Then $\widetilde{K}$ satisfies the hypotheses of Theorem 6.2, and $\mu_{\widetilde{K}}$ is the push forward of μ_K by the map $z \mapsto z^2$.

Thus by Theorem 6.2,

$$\int_K \phi(\log|z|)\, d\mu_K(z) = \int_{\widetilde{K}} \phi\left(\frac{1}{2}\log|z|\right) d\mu_{\widetilde{K}}(z)$$

$$\leq \int_{\widetilde{L}} \phi\left(\frac{1}{2}\log|z|\right) d\mu_{\widetilde{L}}(z) = \int_L \phi(\log|z|)\, d\mu_L(z).$$

The inequality in the middle is justified since $\phi(\frac{1}{2}\,\cdot)$ is convex. Corollary 6.3 is proved. $\qquad\square$

There are strict inequality statements for Theorem 6.2 and Corollary 6.3, for which we refer to [11].

To get a closer parallel to Theorem 2, it would be nice if in Corollary 6.3 we could drop the symmetry assumption on K and replace it by the much weaker assumption that the conformal centroid of K is at the origin. But the example below shows that no such result can exist.

Example. Hayman [8, p. 262] built on work of Jenkins [10] and showed existence of a map $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ in the class S for which $A_2 = 0$ and $M(r,f) \sim c(1-r)^{-2}$ as $r \to 1$, where $M(r,f) = \max_\theta |f(re^{i\theta})|$ and c is some positive constant. Let $K = \{1/z : z \notin f(\mathbb{B}(1))\}$. Then K is compact and connected, contains the origin, and $\mathrm{cap}(K) = 1$. The Green's function $g(z) = g(z, \infty, K^c)$ is related to f by $g(z) = \log 1/|f^{-1}(1/z)|$, where f^{-1} is the inverse function of f, from which one calculates that

$$a_1 = -A_2 = 0.$$

We saw in §2 that a_1 is the conformal centroid of K, and thus the conformal centroid of K is 0.

The behavior of $M(r,f)$ as $r \to 1$ implies that $M(r,g) \sim c_1 r^{1/2}$ as $r \to 0$. Since g is subharmonic in $\mathbb{C}$, it follows that g is majorized in any disk by its Poisson integral over the boundary. Thus,

$$c_2 r^{1/2} \leq M(r,g) \leq \frac{3}{2\pi} \int_{-\pi}^{\pi} g(2re^{i\theta})\, d\theta.$$

On the other hand, the Green's function G of L^c satisfies $M(r,G) \leq c_3 r$ for all $r \in [0, \infty)$. We conclude that

$$\int_{-\pi}^{\pi} g(re^{i\theta})\, d\theta > \int_{-\pi}^{\pi} G(re^{i\theta})\, d\theta, \quad r \in (0, r_0),$$

for some $r_0 \in (0, 1)$.

Take a smooth, convex ϕ which is constant on $(-\infty, 2 \log r_0)$, strictly convex on $(2 \log r_0, \log r_0)$, and linear on $(\log r_0, \infty)$. Then Proposition 6.1 gives

$$\int_K \phi(\log |z|) \, d\mu_K(z) > \int_L \phi(\log |z|) \, d\mu_L(z),$$

which is the reverse of the moment inequality we might have hoped would be true.

This example shows the full analogue of Corollary 6.3 does not hold if the symmetry constraint is relaxed to the centroid constraint. We now propose a substitute, "averaged" result. Assume that K is compact, connected, contains the origin, has $\mathrm{cap}(K) = 1$, and also satisfies the centroid constraint

$$\int_K z \, d\mu_K(z) = 0.$$

Then, as noted in §5, $K \subset \overline{\mathbb{B}(2)}$, and thus the formula in Proposition 6.1 is valid for all $R \geq 2$. Fix $R \geq 2$, and define

$$I(r) = I(r, K) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta}) \, d\theta, \qquad r \in [0, \infty),$$

$$J(r) = J(r, K) = \int_r^R I(t, K) \frac{dt}{t}, \qquad r \in [0, R].$$

Conjecture 1. *Suppose $K \subset \mathbb{C}$ is compact and connected with $\mathrm{cap}(K) = 1$, and that $0 \in K$ and the conformal centroid of K lies at the origin. Then for all $R \geq 2$ we have*

$$J(r, K) \leq J(r, L), \quad r \in [0, R].$$

An equivalent inequality is

$$\int_{r < |z| < R} g(z) \, |z|^{-2} \, dx \, dy \leq \int_{r < |z| < R} G(z) \, dx \, dy, \quad r \in [0, R].$$

Since $I(r, K) = \log r$ for $r \geq 2$, by (34), it follows that $I(r, K) = I(r, L)$ for $r \geq 2$ and hence another equivalent inequality is

$$\int_{r < |z| < \infty} [g(z) - G(z)] |z|^{-2} \, dx \, dy \leq 0, \quad r \in [0, \infty).$$

There is still another equivalent version of Conjecture 1 involving functions ϕ, which we'll call Conjecture 2.

Conjecture 2. *Suppose $K \subset \mathbb{C}$ is compact and connected with $\mathrm{cap}(K) = 1$, and that $0 \in K$ and the conformal centroid of K lies at the origin. Then for all functions $\phi \in C^1(\mathbb{R})$ such that both ϕ and ϕ' are convex, we have*

$$\int_K \phi(\log|z|)\, d\mu_K \leq \int_L \phi(\log|z|)\, d\mu_L.$$

To see the equivalence, first reduce to the case of smooth ϕ with $\phi \equiv 0$ near $-\infty$, by arguing as in the proof of Theorem 6.2. Then go to Proposition 6.1 and express the integral over $\mathbb{B}(R)$ in polar coordinates, and integrate it by parts with respect to r. The resulting formula is

$$\int_K \phi(\log|z|)\, d\mu_K(z)$$
$$= \int_0^R \phi'''(\log t)\, \frac{J(t)}{t}\, dt + \phi(\log R) - \phi'(\log R) \log R,$$

where $R \geq 2$. Now it is immediate that Conjecture 1 implies Conjecture 2. As for the converse, one need only take $\phi(t) = [(t - \log r)^+]^2$, noting $\phi \equiv 0$ near $-\infty$ and ϕ''' is a positive point mass at $\log r$.

To conclude, we describe two special cases of Conjecture 2 which have appeared in the literature as separate conjectures.

The first concerns the class Σ_0 of all univalent meromorphic functions F in the exterior $\mathbb{B}^c$ of the unit disk $\mathbb{B}$, with $F(z) = z + O(z^{-1})$ as $z \to \infty$. The function $F_0(z) = z + z^{-1}$ belongs to Σ_0, and maps the exterior of the unit disk onto the domain $L^c = \overline{\mathbb{C}} \setminus [-2, 2]$.

Conjecture 3 (Pommerenke [14]). *If $F \in \Sigma_0$ and $0 \in K = F(\mathbb{B}^c)^c$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|\, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_0(e^{i\theta})|\, d\theta = \frac{4}{\pi}.$$

The best known estimate [14] is $\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|\, d\theta \leq 4.02/\pi$. One would like to replace 4.02 by 4.

Note in Pommerenke's Conjecture that $\mathrm{cap}(K) = 1$ and K satisfies the other hypotheses of Conjecture 2. Moreover, $d\mu_K$ is the harmonic measure of $K^c = F(\mathbb{B}^c)$ at ∞ and $\frac{d\theta}{2\pi}$ is the harmonic measure of $\mathbb{B}^c$ at ∞ (see [18, p. 105]). By

conformal invariance of harmonic measure, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})| \, d\theta = \int_{K} |z| \, d\mu_K,$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F_0(e^{i\theta})| \, d\theta = \int_{L} |z| \, d\mu_L.$$

Thus if Conjecture 2 is true with $\phi(x) = e^x$, then so is Pommerenke's Conjecture.

Incidentally, the case $\phi(x) = e^{2x}$ of Conjecture 2 says

$$\int_{K} |z|^2 \, d\mu_K \leq \int_{L} |z|^2 \, d\mu_L,$$

which is equivalent as above to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_0(e^{i\theta})|^2 \, d\theta = 2.$$

This case of Conjecture 2 can be proved as follows: write $F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$ and observe

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|^2 \, d\theta = 1 + \sum_{n=1}^{\infty} |b_n|^2$$

$$\leq 1 + \sum_{n=1}^{\infty} n |b_n|^2$$

$$\leq 2$$

by the area theorem [15, Theorem 1.3] . Clearly equality holds if and only if $|b_1| = 1$ and $b_n = 0$ for all $n \geq 2$, which means K is a rotate of L.

The second special case of Conjecture 2 concerns norms of polynomials. Let M_K be the smallest number M such that

$$\prod_{j=1}^{m} \|p_j\|_K \leq M^n \|p\|_K$$

for all polynomials p of degree $n \geq 1$ and all polynomials $p_1, \ldots, p_m$ such that $\prod_{j=1}^{m} p_j = p$. Here $\| \cdot \|_K$ denotes the sup norm on K. The constant M_K was evaluated in [16] as

$$M_K = \frac{\exp\left(\int_{K} \log d_K(z) \, d\mu_K(z) \right)}{\text{cap}(K)}$$

where $d_K(z) = \max_{t \in K} |z - t|$ is the farthest point distance function for K. (Further properties of d_K have been studied by Laugesen and Pritsker [12], and Gardiner and Netuka [5, 6].)

The following natural extremal conjecture for M_K was stated in Pritsker and Ruscheweyh's paper [17]:

Conjecture 4. *For all compact connected $K \subset \mathbb{C}$ with more than one point, we have $M_K \leq M_L$.*

The constant M_K is invariant under similarity transformations, and so when studying the conjecture it suffices to assume $\mathrm{cap}(K) = 1$ and that the conformal centroid of K lies at the origin.

Assuming in addition that K contains its conformal centroid, the authors of [17] showed that $M_K < (1.022)M_L$. In the direction of the conjectured sharp bound (with constant 1), they observed

$$\log M_K \leq \int_K \log(2 + |z|) \, d\mu_K(z),$$

with equality when $K = L$.

Now, the function $\phi(t) = \log(2 + e^t)$ is convex on $\mathbb{R}$, and ϕ' is convex on $(-\infty, \log 2]$. Replacing ϕ on $(\log 2, \infty)$ by an appropriate quadratic, we obtain a function $\widetilde{\phi}$ which, along with its derivative, is convex on all of $\mathbb{R}$. Suppose Conjecture 2 is true. Then the inequality in it holds with $\widetilde{\phi}$ in place of ϕ. Moreover, $K \subset \overline{\mathbb{B}(2)}$, and so the integrals are the same for ϕ and $\widetilde{\phi}$, which would establish Conjecture 4.

References

[1] V. V. Andrievskii and H.-P. Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer–Verlag, New York, 2002.

[2] A. Baernstein, *Integral means, univalent functions and circular symmetrization*, Acta Math. 133 (1974), 139–169.

[3] A. Baernstein, *The *-Function in Complex Analysis*, in Handbook of Complex Analysis: Geometric Function Theory, Volume 1, edited by R. Kühnau, 231–271, North–Holland, Amsterdam, 2002.

[4] W. F. Donoghue Jr., Distributions and Fourier Transforms, Academic Press, New York, 1969.

[5] S. J. Gardiner and I. Netuka, *The farthest-point distance function*, in Complex and Harmonic Analysis, Proceedings of the International Conference 2006 at Aristotle University of Thessaloniki, Destech Publications, Lancaster, PA, 2007.

[6] S. J. Gardiner and I. Netuka, *Potential theory of the farthest point distance function*, J. Anal. Math. 101 (2006), 163–177.

[7] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of the Second Order, Second Edition, Springer–Verlag, New York, 1983.

[8] W. K. Hayman, *The asymptotic behavior of p-valent functions*, Proc. London Math. Soc. 5 (1955), 257–284.

[9] L. Hörmander, Notions of Convexity, Birkhäuser, Boston, 1995.

[10] J. A. Jenkins, *On a problem of Gronwall*, Annals of Math. 59 (1954), 490–504.

[11] R. Laugesen, *Extremal problems involving logarithmic and Green capacity*, Duke Math. J. 70 (1993), 445–480.

[12] R. S. Laugesen and I. E. Pritsker, *Potential theory of the farthest-point distance function*, Canad. Math. Bull. 46 (2003), 373–387.

[13] N. I. Muskhelishvili, Singular Integral Equations, Dover, New York, 1992.

[14] Ch. Pommerenke, *Über einige Klassen meromorpher schlichter Funktionen*, Math. Zeit. 78 (1962), 263–284.

[15] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

[16] I. E. Pritsker, *Products of polynomials in uniform norms*, Trans. Amer. Math. Soc. 353 (2001), 3971–3993.

[17] I. E. Pritsker and S. Ruscheweyh, *Inequalities for products of polynomials I*, Math. Scand. 104 (2009), 147–160.

[18] T. Ransford, Potential Theory in the Plane, Cambridge Univ. Press, Cambridge, 1995.

[19] I. Schur, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Zeit. 1 (1918), 377–402.

[20] C. L. Siegel, *The trace of totally positive and real algebraic integers*, Annals of Math. 46 (1945), 302–312.

[21] H. Stahl and V. Totik, General Orthogonal Polynomials, Cambridge Univ. Press, Cambridge, 1992.

[22] E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, N.J., 1971.

[23] V. Totik, *Polynomial inverse images and polynomial inequalities*, Acta Math. 187 (2001), 139–160.

[24] J. L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, AMS, New York, 1950.

A. Baernstein II

Department of Mathematics,

Washington University,

St. Louis, MO 63130, U.S.A.

E-mail: Al@math.wustl.edu

R. S. Laugesen
Department of Mathematics,
University of Illinois,
Urbana, IL 61801, U.S.A.
E-mail: Laugesen@illinois.edu

I. E. Pritsker
Department of Mathematics,
Oklahoma State University,
Stillwater, OK 74078, U.S.A.
E-mail: Igor@math.okstate.edu