Project V — Fourier Coefficients*

Richard S. Laugesen

Goal of the project

To further develop understanding of how many terms of a Fourier series are required in order to well-approximate the original function. We do this by studying the decay rates of Fourier coefficients of: functions with jumps, functions with no jumps but with corners, and functions with no jumps and no corners.

Instructions

Answer Question 1 in the table provided, and Questions 2–7 on a different sheet of paper.

*Copyright © 2002, The Triode (iode@math.uiuc.edu). Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is available at `http://www.fsf.org/copyleft/fdl.html`.

1 Preliminaries

Launch Matlab and Iode, and get into the Fourier series module of Iode. You are already familiar with this module from Project IV.

Tip. For more information, you can browse the manual at the Iode website.

2 Questions

1 Fill in the table on the next page.

Remarks.

- Column 1. All functions are assumed to be 2π periodic, with the formula in the first column defining the function for $-\pi < x < \pi$. So when you enter the functions in Iode, set the left endpoint to `-pi` and the right endpoint to `pi`.
- Columns 2 and 3. Answer Yes or No. Remember to check whether f has jumps or corners at $x = \pi$.
- Column 4. The nth cosine coefficient is A_n, the nth sine coefficient is B_n, and the combined amplitude of the two is

$$C_n = \sqrt{A_n^2 + B_n^2}.$$

The term "amplitude" makes sense here because we know that

$$A_n \cos nx + B_n \sin nx = C_n \cos(nx - \alpha_n)$$

for some α_n. Thus C_n measures the total amplitude of the part of the Fourier series that has frequency n.

You can plot the C_n versus n by clicking in the middle of the window on the button `Plot coefficients C_n`. You will observe C_n generally gets smaller as n gets bigger, although perhaps with "blips" upward. Column 4 asks you to gauge how fast the amplitudes C_n decay to zero as n increases. By examining several of the different functions in the Table, you'll start to see that C_n falls off much more quickly for some functions than for others.

Your answer in Column 4 should say either "slow", or "rapid", or "very rapid". A good benchmark is: what is the last value of n for which you can tell graphically that C_n is positive?

Note. Look at several of the different functions before filling in any answers, because our notions of "slow" and "rapid" are relative!

[1, continued.] Fill in the following table.

$f = 2\pi$-periodic extension of:	f has jumps? (Y/N)	f has corners? (Y/N)	Decay of Fourier amplitude C_n?
$(x^2 - \pi^2)^2$	No	No	Very rapid
$\|x + 1\|$			
$e^{-x}\text{sign}(2 - x)$			
x^2			
$\sin((2/\pi)x^2 + \pi/2)$			
$x^3 - \pi^3 \sin(x/2)$			
$(x - \pi)\sinh(x + \pi)$			
$(\log(x + 6))^2$			
$\cos(\frac{1}{2}(x - 1)^2)$			

Reminders. In Matlab, the absolute value function is `abs()`, the exponential is `exp()`, and log (the natural logarithm) is just `log()`. To multiply you need a "`*`", and π is `pi`. The `sign` function picks off the sign of x, in other words

$$\text{sign}\,(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0\,. \\ -1, & \text{if } x < 0 \end{cases}$$

Obviously this is **not** the same as the sine function $\sin(x)$!

2 Based on what you observed in Question 1 above, compare and contrast the rate of decay of the Fourier coefficients of:

i. functions with discontinuities (f has jumps);

ii. functions that are continuous but have discontinuities in the first derivative (f has no jumps, but does have corners);

iii. functions that are continuous and have continuous first derivative as well (no jumps and no corners).

3 In Project 4 you compared and contrasted the number of terms typically needed to obtain naked eye convergence of the Fourier series for:

i. functions with discontinuities (f has jumps);

ii. functions that are continuous but have discontinuities in the first derivative (f has no jumps, but does have corners);

iii. functions that are continuous and have continuous first derivative as well (no jumps and no corners).

Now justify that answer again, qualitatively, with reference to your answer to Question 2.

4 Consider the three functions

i. f =odd extension of x, $0 < x < \pi$, (the sawtooth, as on p. 613–614 of Edwards & Penney); `f(x)=x` in Iode.

ii. f =even extension of x, $0 < x < \pi$ (as on p. 613–614 of Edwards & Penney); `f(x)=abs(x)` in Iode.

iii. f =odd extension of $\cos(2x) - 1$, $0 < x < \pi$;
`f(x)=(cos(2*x)-1)*(x>0)-(cos(2*x)-1)*(x<0)` in Iode.

For each function i–iii, sketch the graph and evaluate A_n, B_n, C_n by hand. You can use formulas from the text, wherever appropriate, and for part iii you should find $B_n = 16/\pi n(n^2 - 4)$ when n is odd.

5 Further, for each function i, ii, iii, find the rate of decay of the Fourier amplitudes C_n, that is, find the value of p for which $C_n = O(1/n^p)$.

Notation. Here the notation $C_n = O(1/n^p)$ means that C_n behaves like a constant multiple of $\frac{1}{n^p}$, when n is large. (You pronounce "$C_n = O(1/n^p)$" as "C_n has order $1/n^p$", or just "C_n is big Oh of $1/n^p$".) More precisely, the definition is that

$$n^p C_n \text{ is bounded as } n \to \infty.$$

For example, justify to yourself that

$$\frac{1}{5n^2+3} = O(\frac{1}{n^2}),$$

by showing that n^2 times the lefthand side is bounded as $n \to \infty$.

6 Check your answers to Problem 5 parts i,ii,iii as follows. For each part, enter the function in Iode, then plot the C_n coefficients, then use the `Function` menu to `Enter comparison function`. In particular, take the value of p found in Problem 5 and enter `1/n^p` as a comparison function. If the comparison plot (which will show up in the lower graph) is neither growing nor decaying, overall for large n-values, then you have found the correct decay rate. *[Nothing to hand in for this problem.]*

Then try entering the *wrong* decay rate, just to see what happens...

7 Do the functions in Problem 4 agree with the pattern you observed in Problem 2? That is, are the decay rates found in Problem 5 consistent with your conclusions in Problem 2? Explain.

Hint. Suppose $C_n = O(1/n^p)$. Then the bigger p is, the more rapidly the coefficients C_n decay. For example, $\frac{1}{n} > \frac{1}{n^2} > \frac{1}{n^3} > 0$ for all $n \geq 2$, and so $\frac{1}{n^3}$ decays more rapidly than $\frac{1}{n^2}$.