Project IV — Fourier Series*

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Goal of the project

To develop understanding of how many terms of a Fourier series are required in order to well-approximate the original function, and of the differences between the Fourier series of functions with jumps, functions with no jumps but with corners, and functions with no jumps and no corners.

Instructions

Answer Questions 1 and 2 in the spaces provided, in Section 2.

1 Preliminaries

Launch Iode, and get into the Fourier series module.

1. When you start the module, it plots two graphs. The upper one shows an odd 2π-periodic square wave $f(x)$. (Notice you are shown *two* periods of this function, over a length 4π.) It also shows, in red, a partial sum

$$\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

of the Fourier series. The final terms in this partial sum are $\cos(Nx)$ and $\sin(Nx)$, and so Iode calls N the "top harmonic". The current value of the top harmonic is displayed in the middle of the plotting window, and you can increase or decrease it by clicking on the arrow buttons; doing so repeatedly creates an "animation" effect. Or, you can just enter a new top harmonic number directly into the box. When you increase the value of the top harmonic, the partial sum should better approximate the function.

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2. Now use the `Function` menu to enter a new function, perhaps $f(x) = |x|$ (the Matlab code for this absolute value function is `abs(x)`). Try increasing and decreasing top harmonic, to see the effect on the partial sums.

3. The lower graph in the window shows the "error" between f and the partial sum of its Fourier series, defined just to be the difference

$$\text{error}\,(x) = f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)\right].$$

   When we make the top harmonic N bigger, we expect the error to get smaller. Try it and see. (*Note.* The vertical scale on the error plot changes, when the error gets smaller, in order to keep the error visible.)

4. Now go back to the `Function` menu and again select `Enter function`. Change the function to `sin(x)`, and enter new numbers for the left and right ends of the period interval, perhaps `1` and `4`. These numbers define one period of the function, and then Iode extends the function periodically.

   Look at the upper graph, showing the graph of the function. Is it different from the usual sine curve? Why does the graph now have a jump?

5. Click on `Plot coefficients A_n and B_n` in the middle of the window. This plots the A_n and B_n Fourier coefficients versus n, for $n > 0$. (The $n = 0$ coefficient A_0 is a special case, and is not graphed.) Do you observe some patterns in the coefficients? Patterns are usually clear when top harmonic is 25, but you may want to go up to 50.

   By the way, Iode evaluates the Fourier coefficients approximately, by doing the integrals numerically with the trapezoidal rule (which you might have learned in calculus).

6. Finally, remember that you can always change the name of the variable `x`, using `Relabel variables` in the `Function` menu.

Tip. For more information on the Fourier series module, browse the Iode manual online.

Self assessment. If you've read down to here, but haven't actually carried out any of the above tasks, then it's time to re-think your approach to the project...

2 Questions

1 Fill in the table that follows.

- Column 1. All functions are assumed to be 2π periodic, with the formula in the first column defining the function for $-\pi < x < \pi$. So when you enter these functions into Iode, take the left end of the period interval to be `-pi` and the right end to be `pi`.
- Column 2. If f has a jump (f is not continuous), then state the x-value(s) at which it jumps. Similarly in Column 3 for points where f has a corner (*i.e.* f is continuous but the slope f' jumps). This can usually be determined graphically. (**Aside.** Jumps and corners frequently occur at $x = \pi$ in this exercise. Why?)
- Column 4. To determine "naked eye convergence", look at the plots of the partial sums and errors, in Iode. In this Project, we say that "naked eye convergence" occurs if two things happen:
  - the plot of the function and the plot of the partial sum of the Fourier series look exactly the same to your naked eye (without any magnification, 2 feet away from the screen), and
  - the error (as shown in the error plot) gets small everywhere, in comparison to the size of the function.

  If naked eye convergence seems to happen for some value of top harmonic, then write this value in Column 4. If it doesn't seem to happen, then write a dash "—" in Column 4.

  There are no exactly right answers, but you should try to apply a consistent standard in your work.
- Column 5. Relative error is defined as

$$\frac{\text{size of error}}{\text{size of function}}.$$

  Here "size of function" means the difference between the maximum and mimimum values of the function, and similarly, "size of error" is defined to be the difference between the maximum and minimum values of the error (as seen from the error plot). Do not try to compute the relative error very precisely; just estimate it to within about a factor of 2. Note that the minimum value of the error will be negative.

A technical aside, not part of the Project. The above definition of relative error is specialized to Fourier series, and relies on the fact that the first term of the Fourier series equals the average value of the function:

$$\frac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = (\text{average value of } f).$$

It follows that every partial sum of the Fourier series has the same average value as f, and so the absolute size of the function plays no role when trying to approximate it with the higher partial sums, and it is only the *variations* of f around its average that matter. Similarly, the error has average value zero, and so always takes on both positive and negative values.

[1, continued.] Fill in the table below, following the **directions on the previous pages**. For your own records, print a plot of the partial sum of each Fourier series, showing the naked eye convergence if it occurs. But you need not hand in any graphs.

$f = 2\pi$-periodic extension of:	jumps of f, for $-\pi < x \leq \pi$	corners of f, for $-\pi < x \leq \pi$	Top harmonic N value for naked eye convergence ?	Relative error when top harmonic = 25?
$(x^2 - \pi^2)^2$	—	—	$\approx 6 - 10$	$\approx 10^{-4}$
$\lvert x + 1 \rvert$				
$e^{-x}\text{sign}(2 - x)$				
x^2				
$\sin((2/\pi)x^2 + \pi/2)$				
$x^3 - \pi^3 \sin(x/2)$				
$(x - \pi)\sinh(x + \pi)$				
$(\log(x + 6))^2$				
$\cos(\frac{1}{2}(x - 1)^2)$				

Reminders. In Matlab, the absolute value function is `abs()`, the exponential is `exp()`, and log (the natural logarithm) is just `log()`. To multiply you need a "`*`", and π is `pi`. The `sign` function picks off the sign of x, in other words

$$\text{sign}\,(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0\,. \\ -1, & \text{if } x < 0 \end{cases}$$

Obviously this is **not** the same as the sine function $\sin(x)$!

2 Answer the following questions, based on what you have observed in Question 1 above. *Advice.* Think hard about your observations and write a rough draft before composing the final version. Carefully written answers will receive more credit.

a Compare and contrast the naked eye convergence you observed in

i functions with jumps (discontinuities)

ii functions that are continuous but have corners (that is discontinuities in the first derivative.)

iii functions that have no jumps and no corners (continuous, and with the first derivative continuous as well).

b Compare and contrast the relative error you observed in

i functions with jumps

ii continuous functions with corners

iii functions with no jumps and no corners.

c For functions with jumps, discuss more carefully the kind of convergence you observed: can the Fourier series be said to converge at all? What happens to the error away from the discontinuities?

d Suppose that f is a function with both a jump and a corner. What would you predict will be the relative error for f when the top harmonic $= 25$? Explain your answer. (If you think it is impossible to predict, explain why.)