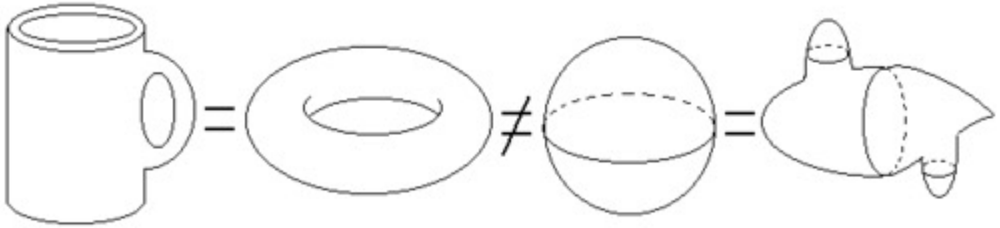
Configuration Space: Topology

- If $\text{dof} = n$, then is the C-space is $\mathbb{R}^n$?
  - No! Spaces can have different shapes or **topologies**
- A circle is written as a S or S^1
- A line can be written as $\mathbb{E}$ or $\mathbb{E}^1$ or $\mathbb{R}$
- A choice of n coordinates to represent an n-dimensional space is called an **explicit parametrization**.



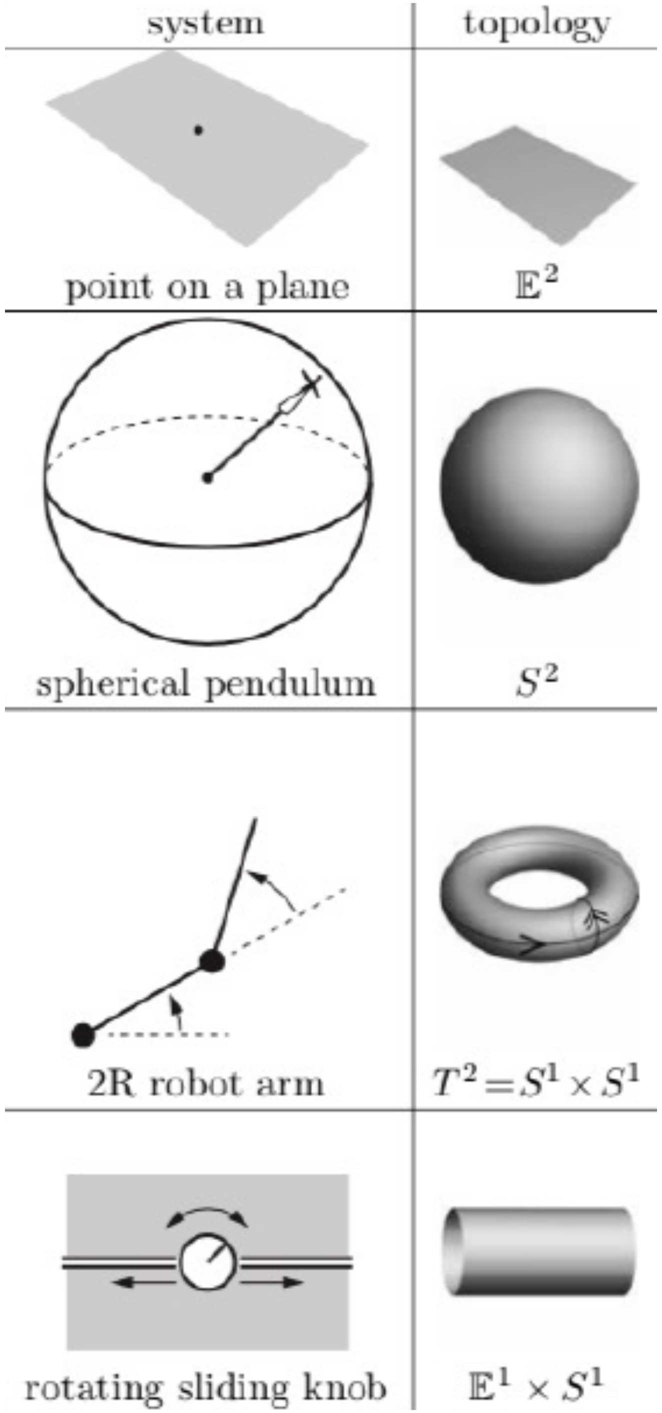


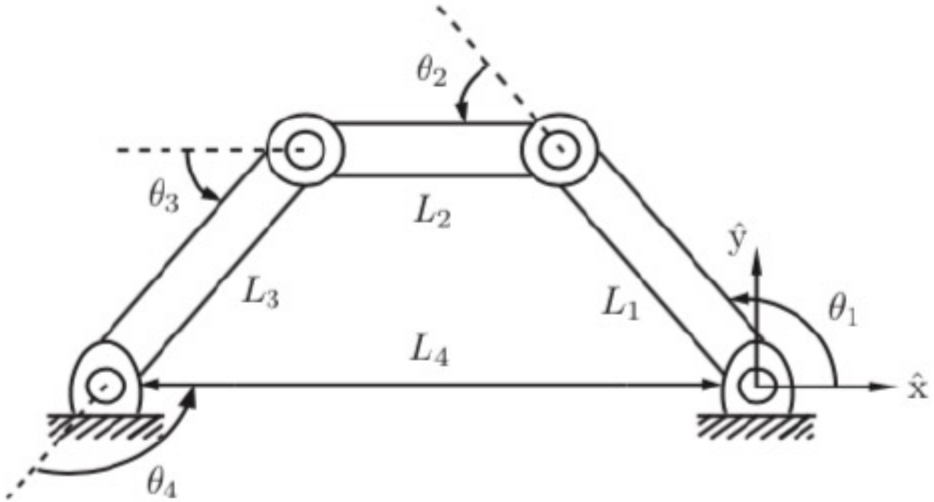
Image Credit: Wikipedia
and Renaissance Universal

Configuration Space Representation

- An **implicit representation** is given by constrained coordinates
- It is often easier to obtain than an **explicit representation**
  - For instance, instead of a spherical representation, we can use a higher dimensional space like (x, y, z) and subject it to constraints to reduce the DoF $(x^2 + y^2 + z^2 = 1)$

Configurations and Constraints

How many DoF?



$$L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \cdots + L_4 \cos(\theta_1 + \cdots + \theta_4) = 0$$
$$L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \cdots + L_4 \sin(\theta_1 + \cdots + \theta_4) = 0$$
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi = 0$$

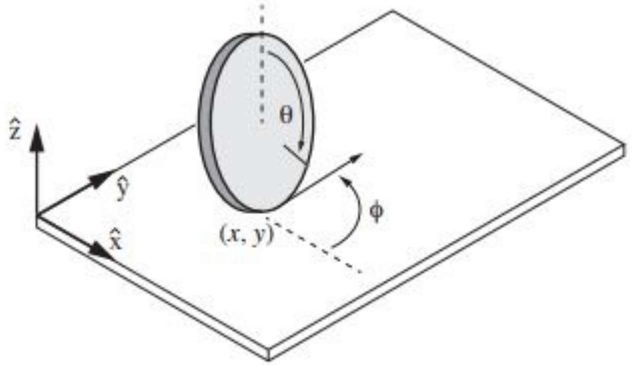
Types of Constraints

Holonomic constraints decrease the dimension of the C-space, while **non-holonomic** constraints do not.

Types of Constraints

- Suppose we are given Pfaffian constraints $A(\theta)\dot{\theta} = 0$
- If we can find a function g such that $\frac{\partial g}{\partial \theta} = A$, the (integrable) constraints g are **holonomic**
  - Why? If such g exists, the constraints $A(\theta)\dot{\theta} = 0$ are the same as the constraints on the position variables $g(\theta)$
- If no such g exists, the constraints are called **non-holonomic**

Rolling Penny Example



Summary

- Introduced fundamental robotics definitions like **configuration space, task space,** and **workspace**
- Learned methods for determining the number of **degrees of freedom** for a rigid body and robot mechanism
- Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot

ECEB 2036 3-5 PM
HW1 is due at 11.59 PM on Fri.


1.
2.

Credit: Wikipedia

Who was Benjamin Olinde Rodrigues?

$\boxed{\text{Chasles}}$

- A mathematician who turned to finance for employment
  - PhD in math and became a banker
- His dissertation introduced the Rodrigues formula
  - Formerly known as the Ivory–Jacobi formula, as it was independently introduced by Olinde Rodrigues (1816), Sir James Ivory (1824), and **Carl Gustav Jacobi (1827)**

Exponential coordinates for rotation

Types of Constraints

Constraints : remove DOFs.

general rule : 1 constraint removes 1 DOF.

- It can be broken in several ways.
  - Constraints can be dependent $\rightsquigarrow$ does not remove a DOF.

θ_1, θ_2

$$\begin{cases} g_1(\theta_1, \theta_2) = \theta_1 + \theta_2 - 1 = 0 \\ g_2(\theta_1, \theta_2) = (\theta_1 + \theta_2 - 1)^2 = 0 \end{cases}$$

Jacobian test for constraints independence.

$\frac{\partial g}{\partial \theta}$ is full row rank if constraints are lin. indep.

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ (\theta_1+\theta_2-1) & (\theta_1+\theta_2-1) \end{bmatrix}$$

↳ rank 1 ⇒ constraints are dependent ↳ row 2 = row 1 $\cdot$ $(\theta_1+\theta_2-1)$

- $\theta_1^2 + \theta_2^2 = 0$

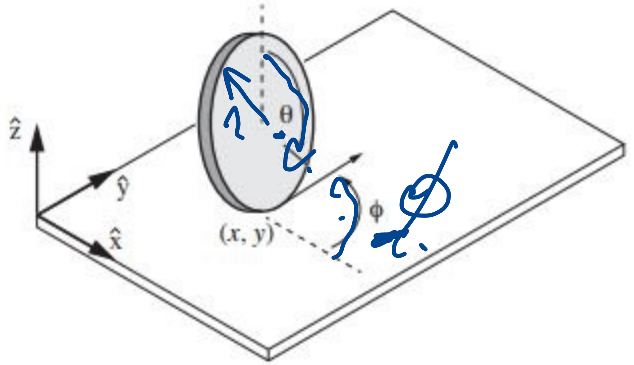
1 constraint removed 2 DOF!

↳ singularity [won't look at this].

Rolling Penny Example

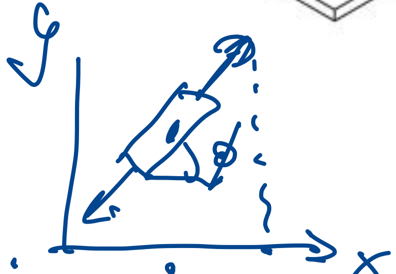
$$\dot{x} = \frac{d}{dt} x$$

$$\dot{\theta} = \frac{d}{dt} \theta.$$



4 DOF (x, y, ϕ, θ)

$$\left|\begin{aligned} \dot{x} &= r\dot{\theta}\cos\phi \Leftrightarrow \dot{x} - r\cos\phi\,\dot{\theta} = 0 \\ \dot{y} &= r\dot{\theta}\sin\phi. \Leftrightarrow \dot{y} - r s_\phi \dot{\theta} = 0 \end{aligned}\right.$$



$\dot{x} = r\dot{\theta}\cos\phi$

$\dot{y} = \text{"} \sin\phi$

Pfaffian form: $A(\theta)\dot{\theta} = 0.$

$\Theta = [x, y, \phi, \theta]$:

if g ∃, then: $\frac{\partial g_1}{\partial \theta}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \boxed{-r c_\phi} \\ 0 & 1 & 0 & -r s_\phi \end{bmatrix}}_{A(\theta)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = 0$$

$\cos\phi$

cohomology

$\frac{\partial g_1}{\partial \theta_1} = 1 \quad \frac{\partial g_2}{\partial \theta_1}$

Q: ∃? g s.t.

$$\frac{\partial g}{\partial \theta} = A \ ?$$

$$\begin{bmatrix} 1 & 0 & 0 & -r c_\phi \\ 0 & 1 & 0 & -r s_\phi \end{bmatrix}$$

$\rightsquigarrow$ If $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ $\exists$, then

$$\frac{\partial g_1}{\partial x} = 1 \qquad \frac{\partial g_1}{\partial y} = 0 \qquad \frac{\partial g_1}{\partial \phi} = 0 \qquad \frac{\partial g_1}{\partial \theta} = -r c_\phi$$

this implies that:

$$\frac{\partial^2 g_1}{\partial \theta \partial \phi} = r s_\phi \neq \frac{\partial^2 g_1}{\partial \phi \partial \theta} = 0.$$

$\rightsquigarrow$ contradiction : g does not exist.

$\Rightarrow$ constraint is non-holonomic.

Example: $\dot\theta_1 \theta_1 + \dot\theta_2 \theta_2 = 0$

$$\underbrace{\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}}_{A(\theta)} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

$$\frac{\partial g_1}{\partial \theta_1} \qquad \frac{\partial g_1}{\partial \theta_2}$$

$$\frac{\partial^2 g_1}{\partial\theta_1\partial\theta_2} = 0 \Rightarrow \frac{\partial^2 g_1}{\partial\theta_2\partial\theta_1} \quad \checkmark$$

Cannot Conclude

$$g = \frac{\theta_1^2}{2} + \frac{\theta_2^2}{2} \rightsquigarrow \text{holonomic.}$$

ex: $\Rightarrow \dfrac{\dot\theta_1\theta_2 - \dot\theta_2\theta_1}{\theta_1^2 + \theta_2^2} = 0.$

Summary

- Introduced fundamental robotics definitions like **configuration space, task space,** and **workspace**
- Learned methods for determining the number of **degrees of freedom** for a rigid body and robot mechanism
- Discussed **holonomic and non-holonomic constraints** which may affect the configuration space and mobility of a robot

Rigid Body Motions

Frames of Reference

$\{s\}$- frame

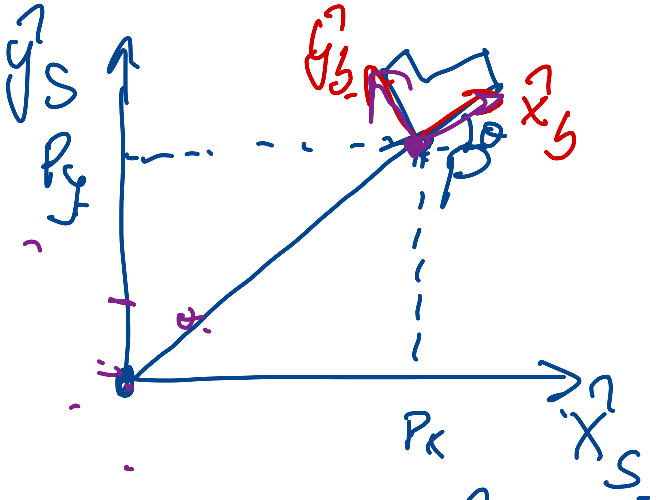
$\hat{x}$ ← means it is normalized.

$\hat{x}_b$ ~ body.

$\{p_s, \hat{x}_b, \hat{y}_b\}$ is called a frame!

$$p = p_x \hat{x}_s + p_y \hat{y}_s.$$

$$\hat{x}_b = \begin{pmatrix} c_\theta \\ s_\theta \end{pmatrix} \qquad \hat{y}_b = \begin{pmatrix} -s_\theta \\ c_\theta \end{pmatrix}$$



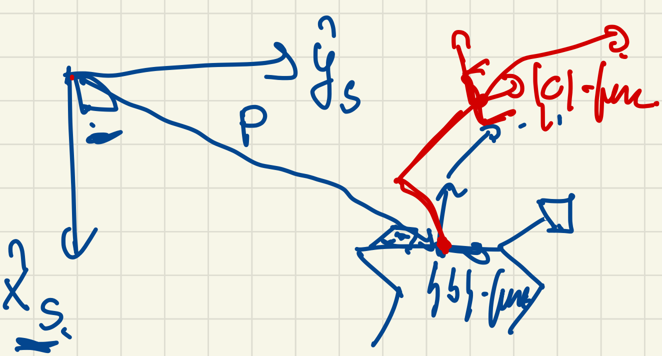
$$p_s = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

coord of $\hat{x}_b$ in $\{s\}$-frame

coord of $\hat{y}_b$ in $\{s\}$-frame

$$T_{sb} = \begin{bmatrix} c_\theta & -s_\theta & p_x \\ s_\theta & c_\theta & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

R_{sb} = rotation matrix.

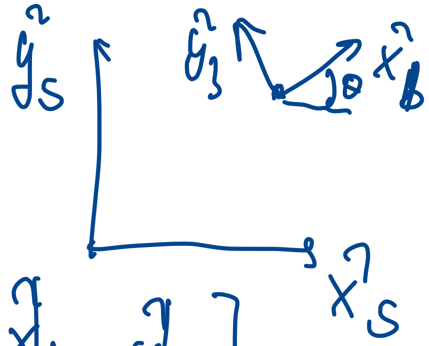


$\underline{T_{bc}}$ = forward kinematics will allow us to calculate it.

$$T_{sb} = \begin{pmatrix} R_{sb} & P \\ 0 & 1 \end{pmatrix}$$

$$\boxed{T_{s\cancel{b}}\, T_{\cancel{b}c} = T_{sc}}$$

Rotation Matrices (1)



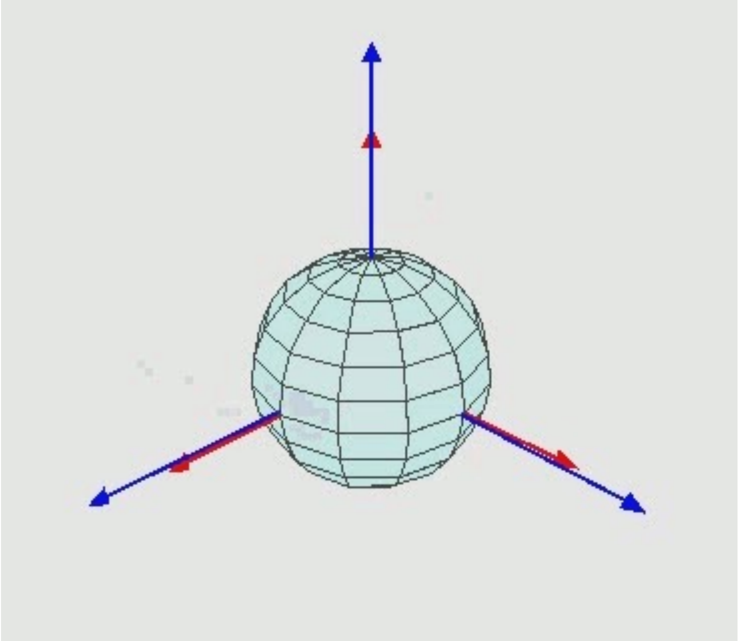
$$R_{sb} := \begin{bmatrix} \hat{x}_b & \hat{y}_b \end{bmatrix}$$

- 2D rotations:

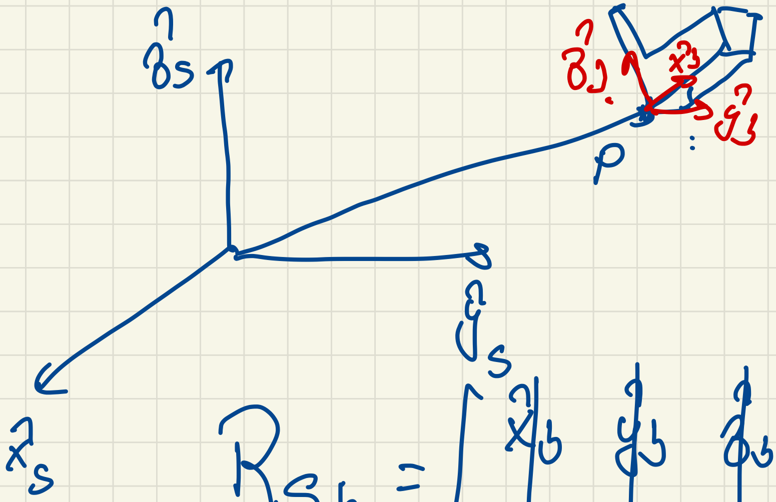
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- 3D rotations about a main axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Credit: Wikipedia

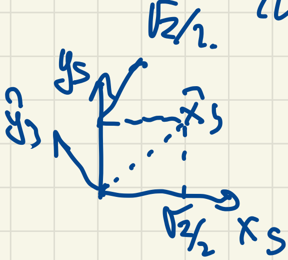


$$R_{sb} = \begin{bmatrix} | & | & | \\ \hat{x}_b & \hat{y}_b & \hat{z}_b \\ | & | & | \end{bmatrix}$$

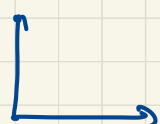
$$T_{sb} = \begin{bmatrix} \underline{R_{sb}} & p_s \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

b frame in 3D.

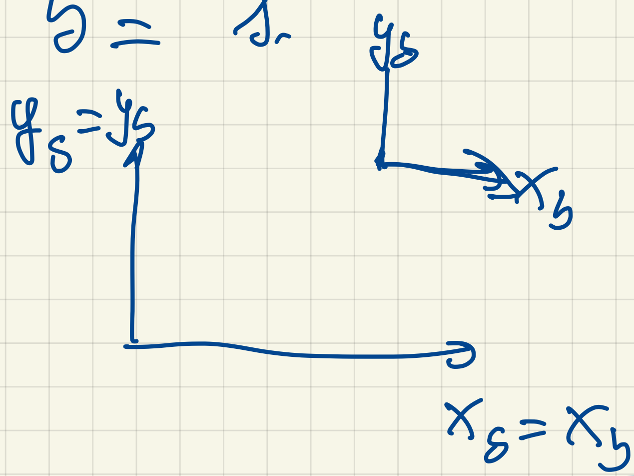
R_{sb} : put in columns the coord of vectors of $\{b\}$-frame in $\{s\}$-frame.



$$R_{sb} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

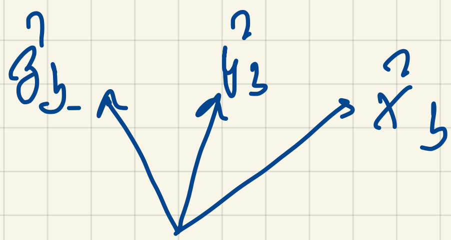


if $b = s$.



$$R_{sb} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3D case:



$$\|\hat{x}_b\|^2 = 1 = \hat{x}_b^T \hat{x}_b$$

$$\hat{x}_b^T \hat{y}_b = 0.$$

$$R_{sb} = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix}$$

$$R_{sb}^T R_{sb} = \begin{bmatrix} \hat{x}_b^T \\ \hat{y}_b^T \\ \hat{z}_b^T \end{bmatrix} \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

in 2D:
$$R_{sb} = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$$

$$R_{sb}^T R_{sb} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R^T R = \underline{I} \iff R$ is an orthogonal matrix.

$\det R = ?$: $\det(R^T R) = \det(R^T) \det(R) = 1$

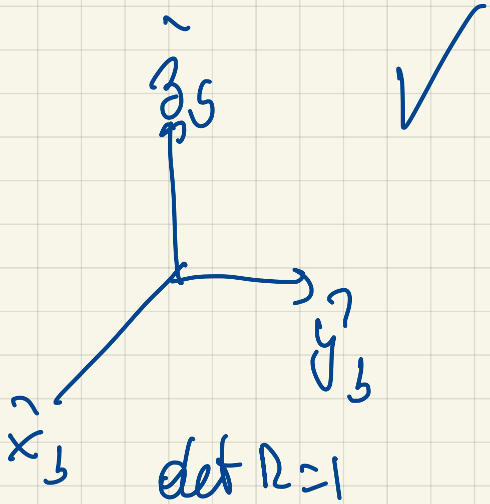
$\leadsto [\det R]^2 = 1 \leadsto \boxed{\det R = \pm 1}$

if $\det R = 1$ & $R^T R = I$

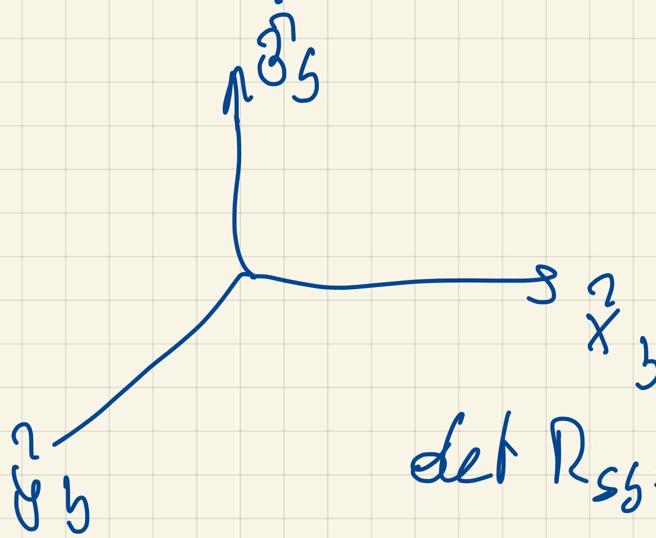
R is said to belong to the

$\boxed{\text{special orthogonal group}}$

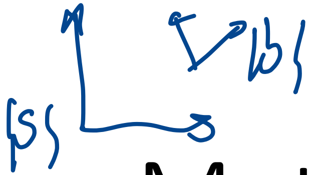
$SO(n)$ $\quad n = 2, 3$



$\det R = 1$



$\det R_{sb} = -1$



$R_{aa} = I$

if $b = s$

$R_{sb} = \begin{bmatrix} x_b & y_b & z_b \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Rotation Matrices (2)

- We use the convention R_{ab} for the matrix whose **columns are coordinates of the frame *b* expressed in the frame *a***
- We have the following laws:

$$R_{ab}R_{bc} = R_{ac} \text{ and } R_{ab}^{-1} = R_{ba}$$

- If the vector p_a is the vector p expressed in frame a, then

$$R_{ba}p_a = p_b$$

- If we rotate a vector v around the $\widehat{\omega}$ axis by angle θ, we say $\mathrm{Rot}(\widehat{\omega}, \theta)v$
  - For a rotation about the x-axis: $\mathrm{Rot}(\hat{x}, \theta)v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} v$
  - Rotations about the y and z axis follow

Special Orthogonal Groups

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\boldsymbol{SO(3)}$, the group that represents all spatial rotation matrices, is the set of all 3×3 real matrices R that satisfy

1. $R^\top R = I$
2. $\det R = 1$

- Note that (2) implies that both (1) holds and that we obey the righthand rule

Similarly, $\boldsymbol{SO(2)}$, the group describes all planar rotation matrices, is the set of all 2×2 real matrices R, satisfying the same conditions.

- EXAM DATES have been fixed!
  - * EXAM 0 [ ]
- HW2 will be posted later this week.

- Constraints.
  - $$\begin{cases} g_1(\theta_1, \dots, \theta_n) = 0 \\ \vdots \\ g_k(\theta_1 \dots \theta_n) = 0 \end{cases}$$
- They are dependent if

$$\frac{\partial g}{\partial \theta} := \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \cdots & \frac{\partial g_1}{\partial \theta_n} \\ \vdots & & \\ \frac{\partial g_k}{\partial \theta_1} & \cdots & \frac{\partial g_k}{\partial \theta_n} \end{bmatrix}$$ is of

rank $k' < k$

$\rightsquigarrow$ they remove k' DOF.

- Non-holonomic constraints
  - Deals with velocity constraints!
- If a constraint is NON-HOLONOMIC, it does not decrease the # of DOF's.

$$\dot\theta_1 \theta_1 + \dot\theta_2 \theta_2 = 0.$$

H or NH ? ONLY way for argue.

H : why ? H is the provide g.

$\exists g$ s.t. $\frac{d}{dt} g(\theta(t))$

= constraint = 0.

$$g = \frac{\theta_1^2}{2} + \frac{\theta_2^2}{2} + r$$

indeed $\frac{d}{dt} g = \frac{2}{2}\theta_1\dot\theta_1 + \frac{2}{2}\dot\theta_2\theta_2 = 0$ ✓

$$\dot\theta_1 \sin\theta_2 + \dot\theta_2 \sin\theta_1 = 0$$

H or NH ?

try: $g = \cos\theta_2 + \cos\theta_1$

$\frac{d}{dt} g = -\sin\theta_2\,\dot\theta_2 - \sin\theta_1\,\dot\theta_1$ ✗

$$A = \begin{bmatrix} S_2 & S_1 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = 0$$

$\boxed{S_i = \sin\theta_i}$

If H, then $\exists g$ st $\frac{\partial g}{\partial \theta} = A = \left[\frac{\partial g}{\partial \theta_1} \quad \frac{\partial g}{\partial \theta_2} \right]$

$\hookrightarrow \frac{\partial g}{\partial \theta_1} = S_2 \qquad \frac{\partial g}{\partial \theta_2} = S_1$

$$\frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} = \frac{\partial}{\partial \theta_1} \frac{\partial g}{\partial \theta_2} = \frac{\partial}{\partial \theta_1} S_1 \simeq \underset{\cos\theta_1}{C_1}$$

$$\neq$$

$$\frac{\partial^2 g}{\partial \theta_2 \partial \theta_1} = \frac{\partial}{\partial \theta_2} \frac{\partial g}{\partial \theta_1} = \frac{\partial}{\partial \theta_2} S_2 = C_2$$

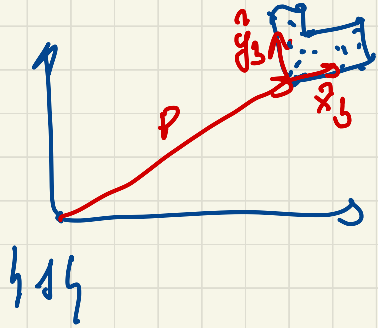
$\rightsquigarrow$ NH.

. $\theta_1^2 + 3\theta_1 \cos\theta_2 = 0$ g!

$\boxed{H}$ or $\underline{NH}$? no $\dot\theta_1$ if is H.

. $g(\theta_1, \theta_2) = 0$; $\dot\theta_1, \theta_2$

$4\frac{\theta_1^2}{2} + C = 0 \rightarrow g$



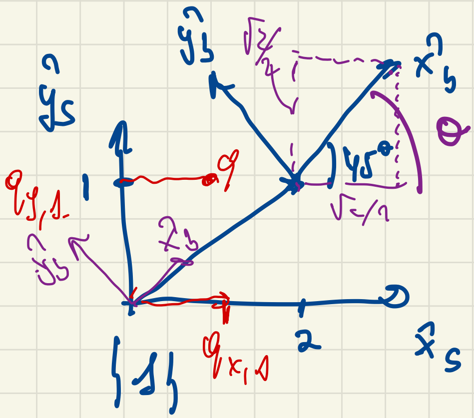
$$\underline{p \,,\ \hat{x}_b \ \hat{y}_b}$$

$$R_{sb} = \begin{pmatrix} \hat{x}_b & \hat{y}_b \end{pmatrix}$$

Coord of $\underline{\hat{x}_b \text{ in}}$ $\{s\}$ frame

Coord of $\hat{y}_b$ in $\{s\}$ frame

2D



$$p = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad R_{sb} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$R_{sb} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\underline{R_{sb}}$ = rotation matrix.

- $q = q_{x,s}\hat{x}_s + q_{y,s}\hat{y}_s$

$$q = \begin{bmatrix} \hat{x}_s & \hat{y}_s \end{bmatrix} \begin{bmatrix} q_{x,s} \\ \underline{q_{y,s}} \end{bmatrix}$$

$$\boxed{q_b = R_{bs}\, q_s}$$

$$\underline{R_{bs}} = \begin{bmatrix} \hat{x}_s & \hat{y}_s \end{bmatrix}$$

↳ coord of basis $\{s\}$ in $\{b\}$

- R_{ab} : write in colums the coord of basis $\{b\}$ in $\{a\}$

- relations: $\begin{cases} q_b = R_{ba}\, q_a \\ R_{ac} = R_{ab}\, R_{bc} \end{cases}$

$*\ R_{ab} = R_{ba}^{-1} = R_{ba}^T.$

- $R_{ab} = \begin{bmatrix} x_b & y_b & z_b \end{bmatrix}$ $\quad R_{ab}^T = \begin{bmatrix} x_b^T \\ y_b^T \\ \hat{z}_b \end{bmatrix}$

$$R_{ab}^T R_{ab} = I. \Rightarrow R_{ab}^T = R_{ab}^{-1}$$

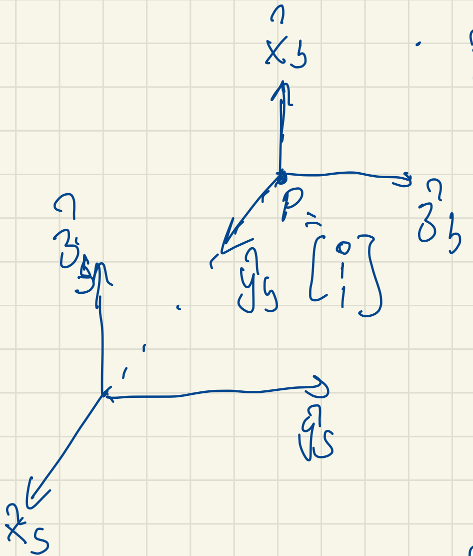
- <u>Frame</u>: data of rotation & point.

$$T_{ab} = \begin{bmatrix} R_{ab} & P_a \\ 0\ 0\ 0 & 1 \end{bmatrix} : 4 \text{ by } 4 \text{ matrix in 3D}$$

$$= \begin{bmatrix} R_{ab} & P_a \\ 0\ 0 & 1 \end{bmatrix} : 3\times 3 \text{ matrix in 2D.}$$

$T_{ab} =$ · R_{ab} : same as before
(write {b}-frame in {a})

· P_a = coord of point p in {a} frame.



$$T_{sb} = \begin{bmatrix} R_{sb} & p_s \\ 0\,0\,0 & 1 \end{bmatrix}$$

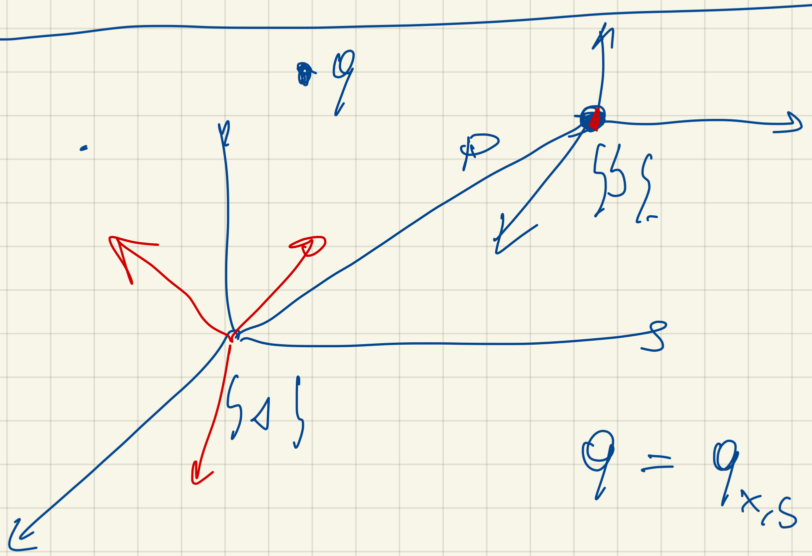
$$R_{sb} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad p_s = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$T_{sb}^{-1} = ?$ $\qquad T_{sb} T_{sb}^{-1} = I.$

$$T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T = R_{bs} & -R_{bs} p_s \\ 0\;0\;0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_{sb} & p_s \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb}^{-1} & -R_{sb}^{-1} p_s \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} I & -p_s + p_s \\ 0\,0\,0 & 1 \end{bmatrix} = I.$$

$$T_{sb}^{-1} \neq T_{sb}^{T}$$



$$q = q_{x,s}\,\hat{x}_s + q_{y_s}\,\hat{y}_s + q_{z_s}\,\hat{z}_s$$

coord of q in {b} frame?

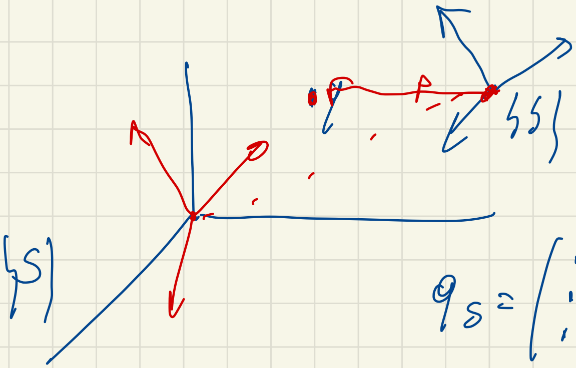
$$q_s = \begin{pmatrix} q_{x,s} \\ q_{y_s} \\ q_{z_s} \\ 1 \end{pmatrix} \qquad q_b = T_{bs}\, q_s \qquad \begin{pmatrix} q_x \\ q_y \\ q_z \\ 1 \end{pmatrix}$$

$$T_{ab} = \begin{bmatrix} R_{ab} & p_a \\ 0\;0\;0 & 1 \end{bmatrix}$$

R_{ab} : col. of {b} in {a} frame

q_a = coord of q in {a}-frame

$p = 0.$

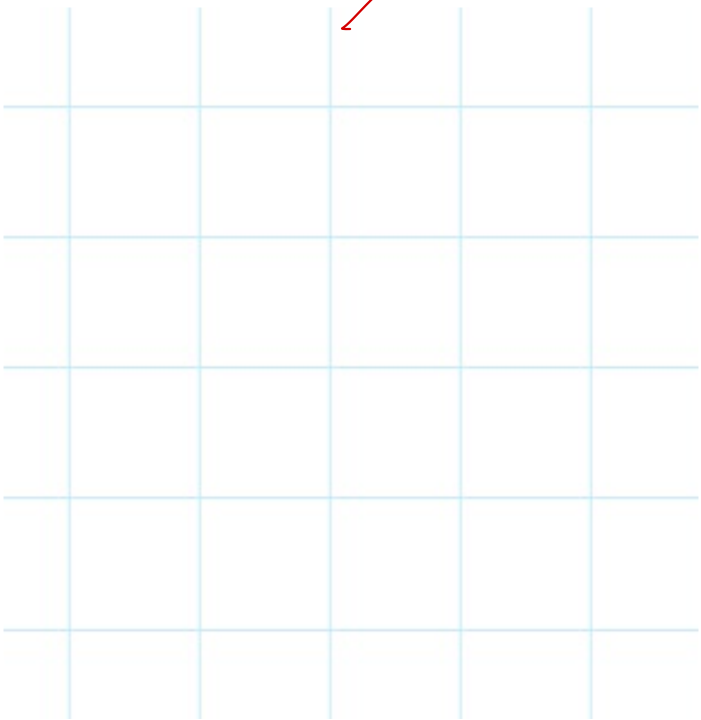


$q_s = \begin{pmatrix} \vdots \end{pmatrix}$

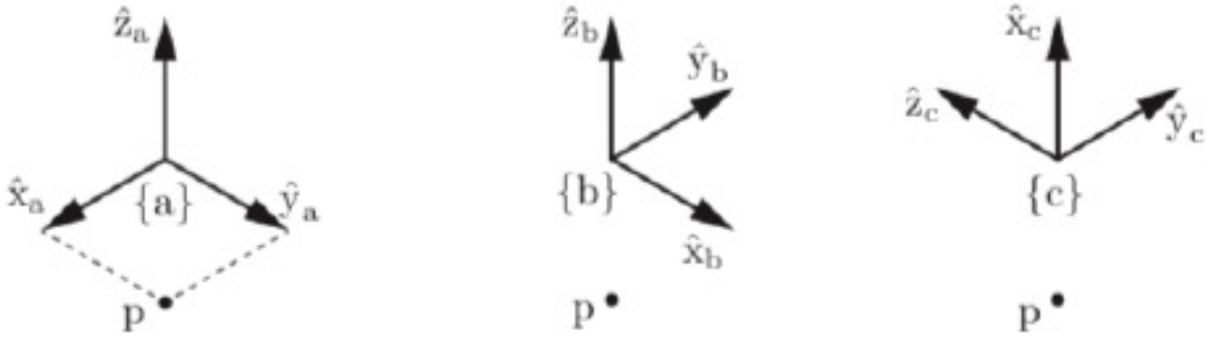
$$\boxed{T = \begin{bmatrix} R_{sb} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\,0\,0 & 1 \end{bmatrix}}$$

q

Changing frames



Frames in 3D

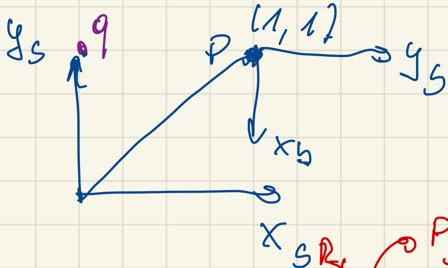


$$P_a = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$P_b = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$R_{ab} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{ab} P_b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = P_a$$



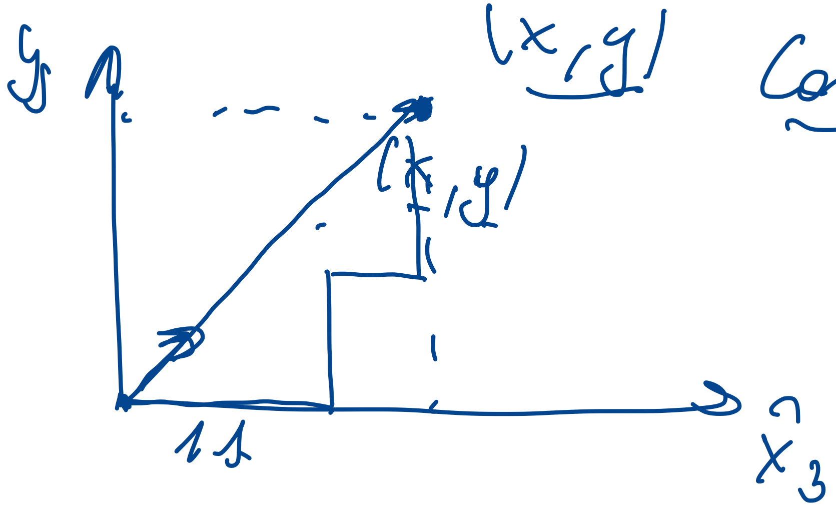
$$T_{sb} = \left(\begin{array}{cc|c} \multicolumn{2}{c}{\overbrace{\boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}}}^{R_{sb}}} & \overbrace{\begin{matrix} 1 \\ 1 \end{matrix}}^{p_s} \\ 0 & 0 & 1 \end{array}\right)$$

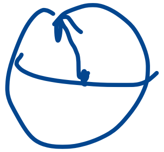
$$q_b = \begin{pmatrix} 0 \\ -1 \\ \hline 1 \end{pmatrix}$$ projective coord.

$$T_{sb}\, q_b = \begin{bmatrix} 0 \\ 1 \\ \hline 1 \end{bmatrix} = q_s$$

[Exponential Coordinates.]

(x, y) Cartesian





Rigid Body Motions

Recall the cross product

- The cross-product is defined as
$$a \times b = \|a\| \|b\| \sin\theta \, \hat{n}$$

- In coordinates:
$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ -\det \begin{bmatrix} a_1 & b_1 \\ a_3 & b_3 \end{bmatrix} \\ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix}$$

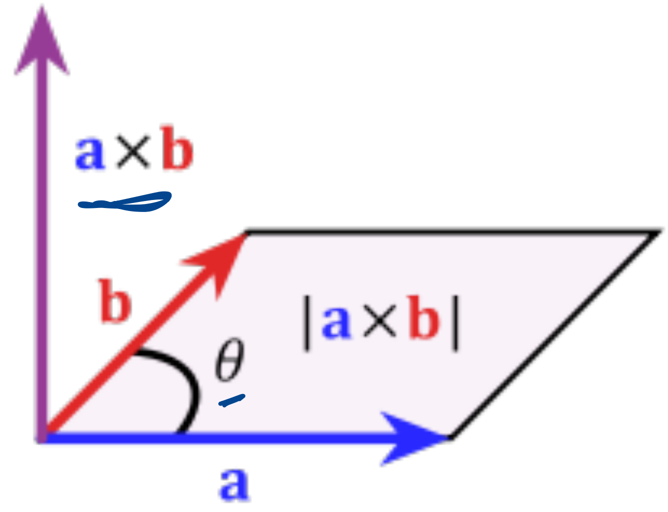


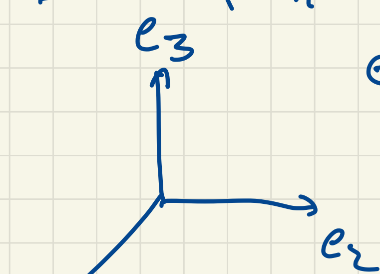
Image Credit: Wikipedia

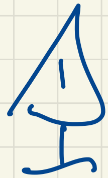
$$a \times b = -b \times a$$

- $$(a \times b) \times c \neq a \times (b \times c)$$

<u>ex:</u> $(e_1 \times e_1) \times e_2 = 0$

$$e_1 \times (\underbrace{e_1 \times e_2}_{e_3}) = -e_2$$

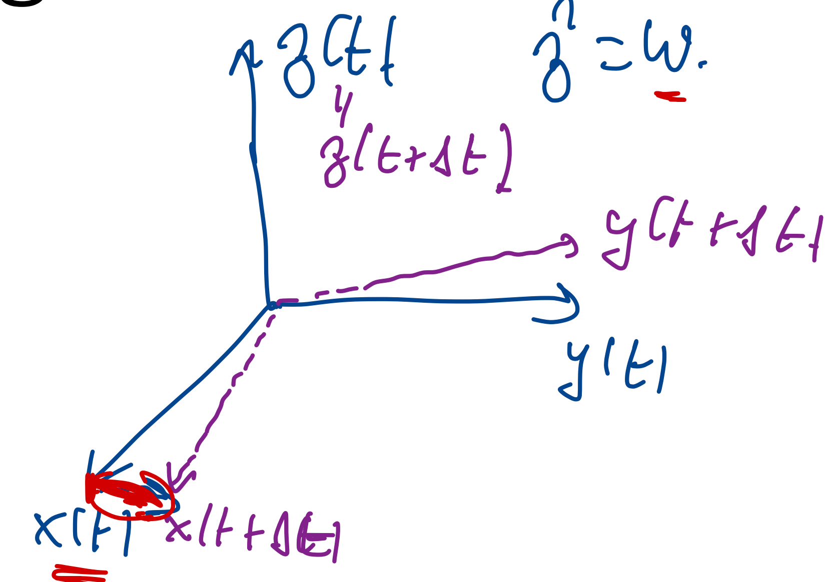




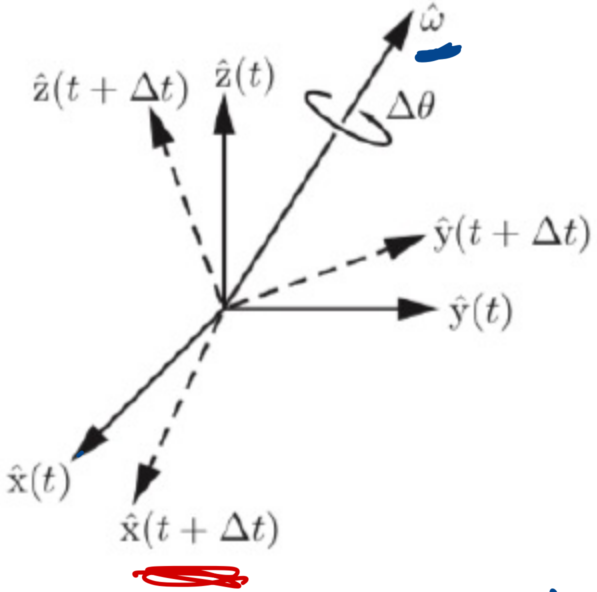
Cross product is

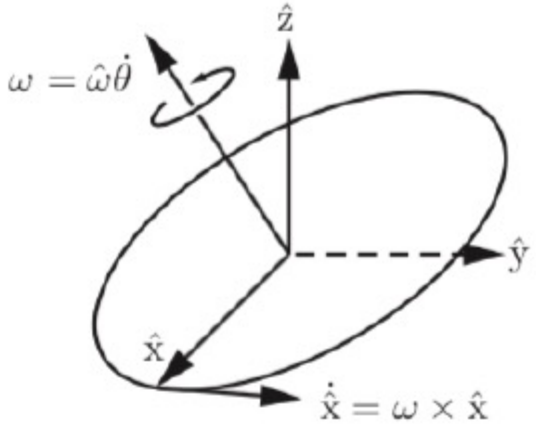
NOT associative.

Angular Velocities



$\hat{z} = \omega.$



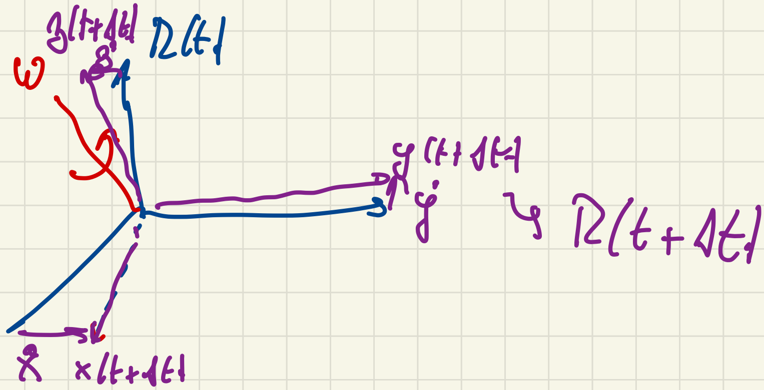


$$\boxed{x(t+\Delta t) = x(t) + (\omega \times x(t))\,\Delta t.}$$

$$\frac{d}{dt}\begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = \omega \times \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}$$

$$\rightsquigarrow \frac{x(t+\Delta t) - x(t)}{\Delta t} = \omega \times x(t) \rightsquigarrow \dot{x}(t) = \omega \times x(t)$$

$$\frac{d}{dt} R(t) = \omega \times R(t)$$



· We don't like $\times$.

$$\omega \times x = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \times \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}\omega_2 x_3 - \omega_3 x_2\\ \omega_3 x_1 - \omega_1 x_3\\ \omega_1 x_2 - \omega_2 x_1\end{bmatrix}$$

3×3

Define a matrix $[\omega]$ s.t.

$$[\omega]\, x = \omega \times x.$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \rightsquigarrow [\omega] := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\rightarrow \boxed{\frac{d}{dt} R = [\omega] R}$$

Rotation.

$$\boxed{\frac{d}{dt} x(t) = v}$$

Cartesian.