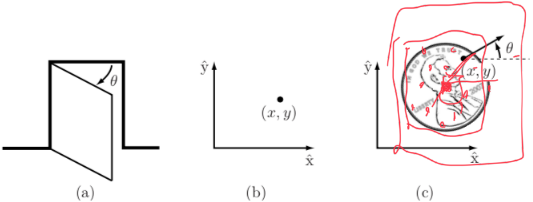
Introduction to Robotics
Lecture 1: Degrees of Freedom and Grübler formula

Configuration and DoFs

- A robot is a device constructed by connecting **rigid** bodies (called **links**) using **joints**. A robot moves thanks to **actuators** providing forces and torques.
- Given that we know the shape of all links, joints and actuators, how can we describe every point in space that is part of the robot? This description is called the **configuration** of the robot.
- E.g.: configuration of point in the plane is given by 2 coordinates: (x, y).

Examples

(x, y, θ)



Soft Robots

Configuration and DoFs

Definition

The configuration of a robot is a complete specification of the position of every point of the robot. The minimum number n of real-valued coordinates needed to represent the configuration is the number of degrees of freedom (dof) of the robot. The n-dimensional space containing all possible configurations of the robot is called the configuration space (C-space). The configuration of a robot is represented by a point in its C-space.

- A robot may have an end-effector, such as a hand or a gripper. The C-space of the end-effector is called the **task space**.

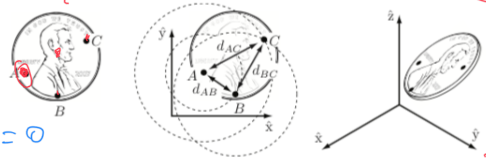
Degrees of freedom of a rigid body

- Given a rigid body in 2-d: knowing the position of 3 *generic* points is sufficient to know the position of all points. More points will do as well.
- DoF= (number of variables to describe the position of the points) - (number of *independent* constraints)
- 2d-rigid body: 6 variables to describe position of 3 points, say A, B and C.
- The distances $d(A, B), d(B, C)$ and $d(A, C)$ are constant: these are 3 constraints $\rightarrow$ 3 degrees of freedom. We can use (x_A, y_A, θ).

$3 \times 2 = 6 - 3 = 3$

$d(A,B) + d(B,C)$

2



(a) (b) (c)

$$\begin{cases} d(A,B) = \ell_1 \\ d(B,C) = \ell_2 \\ d(A,B) + d(B,C) = \ell_1 + \ell_2 \end{cases}$$

$g_1(x_A, x_B, x_C) = 0$

$g_2(x_A, x_B, x_C) = 0$

Independent constraints

$g_1(x,y,z)$
$g_2(x,y,z)$ $\rightsquigarrow \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & & \end{bmatrix}$

- In the example above, $d(A, B) + d(B, C)$ is also constant, why not add it to the constraints?
- Because it is not *independent* from the other constraints.
- Given constraints $g_i(x_A, y_A, x_B, y_B, x_C, y_C) = 0$ where g_i are differentiable functions, they are **independent** if the matrix of partial derivatives

$$\boxed{\frac{\partial g}{\partial x} = \left(\frac{\partial g_i}{\partial x_j}\right)}$$

  is of *full row rank*.

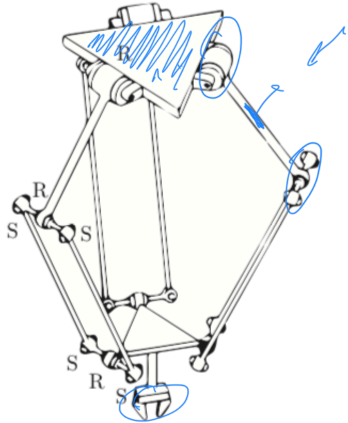
$g_1(x,y) = x^2+y^2+1$
$g_2(x,y) = x+2y$

$\det \begin{bmatrix} 2x & 2y \\ 1 & 2 \end{bmatrix} = 4x-2y$

- **Recall:** A matrix $A \in \mathbb{R}^{n\times m}$ with $m \geq n$ is of full row rank if its rows are linearly independent or, equivalently, if $\det(AA^\top) \neq 0$.
- A **rigid body in 3d** has **6 degrees of freedom**: 3 for "translation" and 3 for "rotation". You can find this using the method above; we come back to this later.

Independent constraints

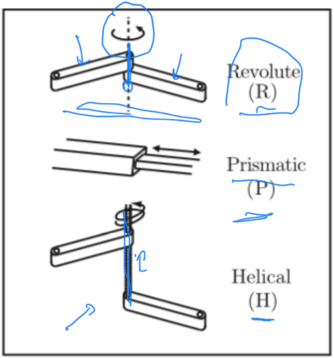
- How many DoF's?

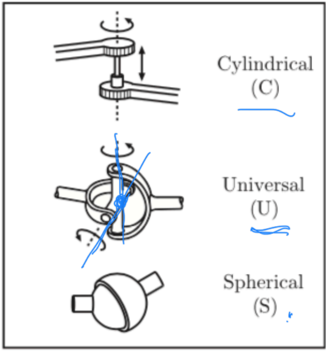


- We need a systematic way of computing the number of DoFs.

Degrees of Freedom of a robot

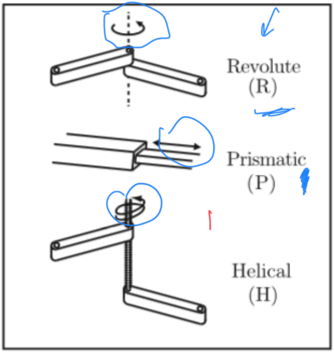
- A robot consists of rigid bodies attached with joints. Here are a few examples of joints:

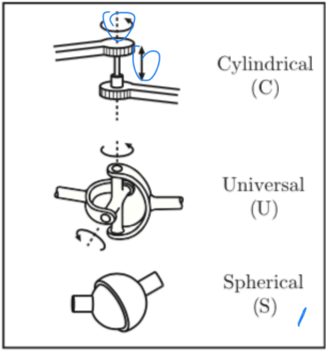




- Given 2 planar rigid bodies in 3d, attach them with joint (X), how many degrees of freedom does the resulting robot have? Similarly, how many constraints between the rigid bodies does the joint impose?

Degrees of Freedom of a robot





Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Grübler's formula

- Grübler's formula allows us to calculate in a systematic way the number of degrees of freedoms of a mechanism

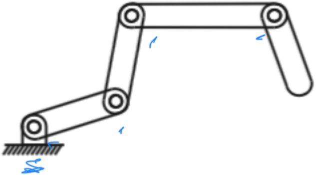
Proposition

Consider a mechanism consisting of N links, where ground is also regarded as a link. Let J be the number of joints, m be the number of degrees of freedom of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms), f_i be the number of dofs provided by joint i, and c_i be the number of constraints provided by joint i (thus $f_i + c_i = m$ for all i). Then

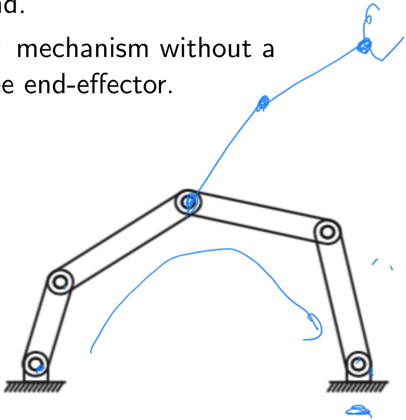
$$\begin{aligned} dof &= m(N-1) - \sum_{i=1}^{J} c_i \\ &= m(N-1) - \sum (m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i \end{aligned}$$

Open- and closed-chain mechanisms

- Closed-chain mechanism: mechanism with a closed-loop: e.g. both ends attached to the ground.
- Open-chain or serial mechanism: mechanism without a closed-loop: e.g. links with a free end-effector.

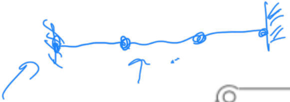


(a)



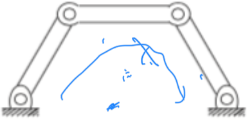
(b)

Open- and closed-chain mechanisms



$m = 3 \quad N = 3$

$J = 2$

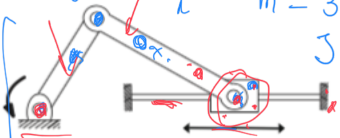


(a)

$N = 4$

$J = 4 \quad c_i = 2$

$m = 3$



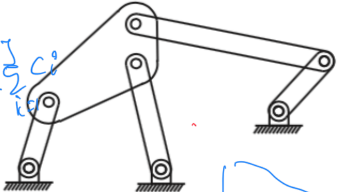
(b)

$N = 3 \quad J = 3$

$m = 3 \quad c_1 = c_2 = 2 \quad c_3 = 1$

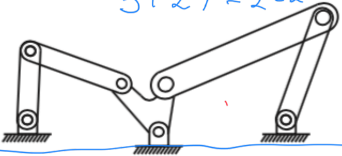
$3(2) - 2 - 2 - 1 = 1$

$\text{dof} = m(N-1) - \sum_{i=1}^{J} c_i$



(c)

$3(4-1) - 8 = 1$



(d)

$N = 4 \quad J = 4$

$c_1 = \dots = c_4 = 2$

$3 \cdot 3 - 4 \cdot 2 = 1$