Introduction to Robotics
Lecture 15: Statics of open chains and Kinematics of Closed Chains

Statics of open chains

- Considering the robot to be at static equilibrium, we can equate the power dissipated at the joints with the power at the end-effector:

$$\tau^\top \dot{\theta} = \mathcal{F}_b \mathcal{V}_b$$

- Since $\mathcal{V}_b = J_b(\theta)\dot{\theta}$, we get $\tau = J_b^\top(\theta)\mathcal{F}_b$ and $\tau = J_s^\top(\theta)\mathcal{F}_s$. We simply write

$$\tau = J^\top(\theta)\mathcal{F}$$

- We can use the above equation to derive the torques to apply at the joint to counteract the effect of an external wrench $-\mathcal{F}$ applied at the end-effector (e.g. by a load attached to it)
- Recall $J(\theta) \in \mathbb{R}^{6\times n}$. Depending on whether $n > 6$ or $n < 6$, the system is over- or under-determined.

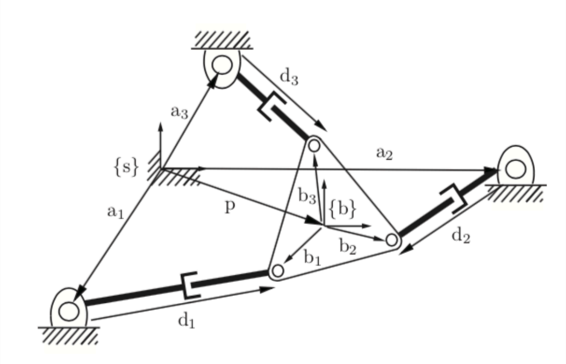
Statics of open chains

- If $n > 6$, the chain is called redundant. Immobilizing the end-effector does not fix the value of the torque joints.
- We need to consider dynamics to continue the analysis. We do not do it here.
- If $n < 6$ and $J^\top \in \mathbb{R}^{n\times 6}$, then we do not have enough joints to generate 6 DoFs of the end-effector twist. We cannot *actively* generate forces in the $6 - n$ directions in the nullspace of $J^\top(\theta)$

Kinematics of closed chains

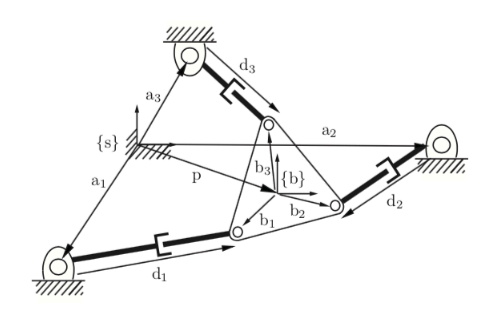
- A kinematic chain that contains one or more loop(s) (series of links joining the ground to the ground, i.e. without a free end-effector) is called a closed chain.
- Unlike open chains, closed-chain can have non-actuated, or passive, joints.
- Kinematics of closed-chains is more complex than the kinematics of open chains, since: 1) there are algebraic equations corresponding to the loops in the mechanism. These may be independent or dependent depending on mechanism. 2) Some joints are not actuated. 3) The chain is very often redundant.
- Approach: write algebraic equations corresponding to loops in the mechanism.

Kinematics of closed chains



- 3 DOFs planar $3\times$ RPR mechanism.
- The three prismatic joints are actuated, the 6 revolute joints are passive.
- Denote the lengths of the 3 legs by s_1, s_2, s_3 ($s_i = \|d_i\|$) and by R_{sb} the orientation of the body frame.
- Forward kinematics: $(s_1, s_2, s_3) \mapsto T_{sb}$
- Inverse kinematics: $T_{sb} \mapsto (s_1, s_2, s_3)$.

Kinematics of closed chains: 3RPR mechanism



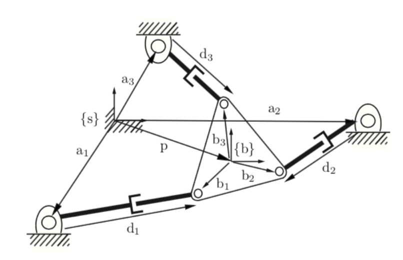
- Closed loops in the mechanism yield:

$$d_i = p + b_i - a_i.$$

We set $a_i = (a_{ix}, a_{iy})$ in s-frame coordinates, and similarly for p, d, and $b_i = (b_{ix}, b_{iy})$ in b-frame coordinates. If we denote by R_{sb} the rotation matrix of T_{sb}, we have

$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + R_{sb} \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix}$$

Kinematics of closed chains: 3RPR mechanism



- We have

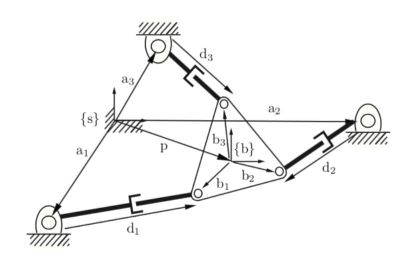
$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + R_{sb} \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix}$$

  and $s_i^2 = d_{ix}^2 + d_{iy}^2$, which yields

$$s_i^2 = (p_x + b_{ix}\cos\phi - b_{iy}\sin\phi - a_{ix})^2 + (p_y + b_{ix}\sin\phi + b_{iy}\cos\phi - a_{iy})^2,$$

  where R_{sb} is a rotation matrix by angle ϕ.
- The above equation solves the *inverse* kinematics problem: given T_{sb}, and thus ϕ, p, we can get the s_i.

Kinematics of closed chains: 3RPR mechanism

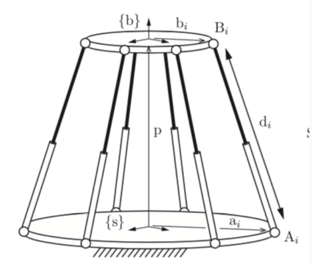


- Forward kinematics is harder. One needs to solve the equations above for p, ϕ, which is done numerically in general.
- Using the substitutions

$$t = \tan\frac{\phi}{2}, \sin\phi = \frac{2t}{1+t^2}, \cos\phi = \frac{1-t^2}{1+t^2},$$

we can reduce the three equations to a polynomial equation in t of degree 6.

Closed chains: 6SPS Steward-Gough platform



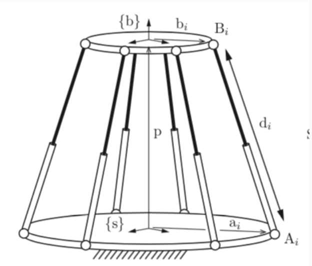
- The 12 spherical joints (at the ends of the 6 arms) are passive, the 6 prismatic joints (middle of the arms) are actuated.
- We make the definitions: p, a_i, d_i are expressed in s-frame, b_i in b-frame, and $R = R_{sb} \in SO(3)$.
- We have the 6 equations

$$d_i = p + Rb_i - a_i.$$

Denote by s_i the length of leg i. We have

$$s_i^2 = d_i^\top d_i = (p + Rb_i - a_i)^\top (p + Rb_i - a_i)$$

Closed chains: 6SPS Steward-Gough platform



- We have
$$s_i^2 = d_i^\top d_i = (p + Rb_i - a_i)^\top (p + Rb_i - a_i).$$
From this equation, the inverse kinematics is straightforward: given R, p, we can obtain s_i.
- The forward kinematics requires numerically solving the 6 equations above.

Velocity kinematics of closed chains

- Velocity kinematics for closed chains is generally difficult, and no nice systematic formulas (like PoE) can be used to obtain the Jacobian in general.
- The velocity kinematics is then obtained from first principles, by differentiating the forward or inverse kinematics map to obtain a Jacobian relating actuator velocities to the body twist.
- For example, the two previous examples produced analytic inverse kinematics maps: $s = g(R, p)$. We can differentiate these maps to obtain

$$\dot{s} = \frac{\partial g}{\partial(R, p)} \mathcal{V}_s,$$

  where $\frac{\partial g}{\partial(\omega, \theta, p)}$ is a matrix of partial derivatives and we expressed $R = Rot(\omega, \theta)$. This requires very lengthy computations.

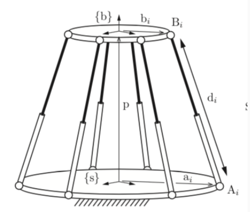
Velocity kinematics of closed chains

- We can however obtain relatively easily the inverse Jacobian from a static analysis.
- Recall that the Jacobian relates a wrench to the actuator forces according to

$$\mathcal{F}_s^\top J_s = \tau^\top \Rightarrow \mathcal{F}_s = J_s^{-\top} \tau$$

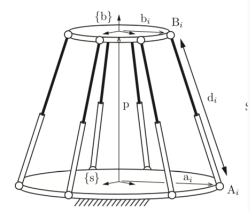
  If we have a 6DOFs mechanism and 6 actuators, the Jacobian is square and can be inverted at non-singular configurations.
- A static analysis of the mechanism thus can yield J^{-T} – see example in next slides.

Closed chains: 6SPS Steward-Gough platform



- If no external forces, the only forces on the platform are at the spherical joints. Since the spherical joints are passive, we can write these forces as $f_i = \tau_i \hat{n}_i$ in the s-frame.
- The moments the forces generate are $m_i = r_i \times f_i$, where r_i is the vector joining the origin of the s-frame to the spherical joint i, expressed in s-frame.
- Since the force is aligned with the (prismatic) joint axis, we can express the torque as $m_i = q_i \times f_i$, where q_i is the position of the lower spherical joint on the leg.

Closed chains: 6SPS Steward-Gough platform



- We thus have that the wrench on the moving platform is

$$\mathcal{F}_s = \sum_{i=1}^{6} \mathcal{F}_i = \sum_{i=1}^{6} \begin{bmatrix} r_\times \hat{n}_i \\ \hat{n}_i \end{bmatrix} \tau_i .$$

We can write it as

$$\mathcal{F}_s = \begin{bmatrix} -\hat{n}_1 \times q_i & \cdots & -\hat{n}_6 \times q_6 \\ \hat{n}_1 & \cdots & \hat{n}_6 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_6 \end{bmatrix}$$

from which we obtain J_s^{-T}.