Landau-Zener Transition

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We want to solve the Landau-Zener problem [1], which consists in solving the two-level problem when the frequency of the external field $\omega(t)$ varies across the resonant frequency ω_0. This is known as a *chirped pulse*

1 Analytical Solution

The two-level hamiltonian is

$$H = \hbar \begin{pmatrix} \omega_1 & \Omega^* \cos \omega t \\ \Omega \cos \omega t & \omega_2 \end{pmatrix} \tag{1}$$

Under the unitary transformation

$$\hat{U} \equiv \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} \tag{2}$$

and the rotating-wave approximation, the system becomes[1]

$$i\hbar \frac{d}{dt} \begin{pmatrix} c_g \\ c_e \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega^* e^{i\delta t} \\ \Omega e^{-i\delta t} & 0 \end{pmatrix} \begin{pmatrix} c_g \\ c_e \end{pmatrix} \tag{3}$$

where the parameter $\delta(t) \equiv \omega(t) - \omega_0$ is defined as the detuning from the resonant frequency.

We are interested in the the situation where the system is initially in the ground state

$$|c_g(t_i)|^2 = 1; \quad |c_e(t_i)|^2 = 0 \tag{4}$$

[1] The hamiltonian transforms as $\hat{H}' = \hat{U}^\dagger (\hat{H} - i\hbar\partial_t)\hat{U}$

THERE IS AN ERROR IN THE FOLLOWING EQUATION (I HAVE ERRONEOUSLY ASSUMED A CONSTANT DELTA IN THE DERIVATION). THE CORRECT RESULT HAS A SIMILAR FORM THOUGH

Solving the eq.(3) for c_e leads

$$\frac{d^2 c_e}{dt^2} + i\delta(t)\frac{dc_e}{dt} + \frac{|\Omega|^2}{4} c_e = 0 \tag{5}$$

with the initial conditions

$$c_e(t_i) = 0; \quad \frac{dc_e}{dt}(t_i) = -\frac{i\Omega}{2} e^{i\delta t_i} c_g(t_i) = \frac{\Omega}{2} \tag{6}$$

The eq.(5) can be easily solved numerically. However, there is an analytical solution when the frequency varies linearly in time. Let us consider $\delta(t) \equiv \alpha t$, and the change of variable $\tilde{c}_e \equiv e^{i\frac{\delta t}{4}} c_e$. The equation becomes

$$\frac{d^2 \tilde{c}_e}{dt^2} + \left(\frac{|\Omega|^2}{4} - \frac{i\alpha}{2} + \frac{\alpha^2 t^2}{4} \right) \tilde{c}_e = 0 \tag{7}$$

This equation can be written in a standard form via the change of variables $z \equiv \sqrt{\alpha} e^{-i\pi/4} t$ and $\nu \equiv \frac{i|\Omega|^2}{4\alpha}$ [1]

$$\frac{d^2 \tilde{c}_e}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4} \right) \tilde{c}_e = 0 \tag{8}$$

This is the Weber differential equation [2, 3], which has two independent solutions, $\tilde{c}_e = D_{-\nu-1}(-iz)$ and $\tilde{c}_e = D_\nu(z)$, where $D_\nu(z)$ is the parabolic cylinder function.

We conclude that the solution to eq.(5) is

$$\boxed{c_e(t) = \left(a D_{-\nu-1}(-i\sqrt{\alpha} e^{-i\pi/4} t) + b D_\nu(\sqrt{\alpha} e^{-i\pi/4} t) \right) e^{-i\delta t/4}} \tag{9}$$

where a and b are constants determined by the initial conditions.

When $t_i = -\infty$, there is an explicit solution for the asymptotic value of the excitation probability

$$\boxed{|c_e(t \to \infty)|^2 \to 1 - e^{-\frac{\pi|\Omega|^2}{2\alpha}}} \tag{10}$$

This result is called the *Landau-Zener formula* [1] (Recall that Ω is an *angular* frequency, and α is a rate of change in *angular* frequency)

2 Other forms

Experimentally, it is more convenient to work with linear frequencies. Let us define $\Delta f \equiv f(t) - f_0 \equiv \delta/2\pi$ and $f_R \equiv \Omega/2\pi$, then the eq.(5) becomes

$$\frac{d^2 c_e}{dt^2} + 2\pi \left(f(t) - f_0\right) \frac{dc_e}{dt} + \pi^2 f_R^2 c_e = 0 \tag{11}$$

$$c_e(t_i) = 0; \quad \frac{dc_e}{df}(t_i) = \pi f_R \tag{12}$$

Also, we could use frequency as the independent variable instead of time. Let us assume a linear dependency in time $f \equiv rt + f_i$, where r is the *linear* frequency rate of change. Then

$$\frac{d^2 c_e}{df^2} + 2\pi i \frac{f - f_0}{r} \frac{dc_e}{df} + \left(\frac{\pi f_R}{r}\right)^2 c_e = 0 \tag{13}$$

$$c_e(f_i) = 0; \quad \frac{dc_e}{df}(f_i) = \frac{\pi f_R}{r} \tag{14}$$

and the Landau-Zener formula becomes

$$|c_e(f \to \infty)|^2 \to 1 - e^{-\frac{\pi^2 f_R^2}{r}} \tag{15}$$

3 Numerical Calculation

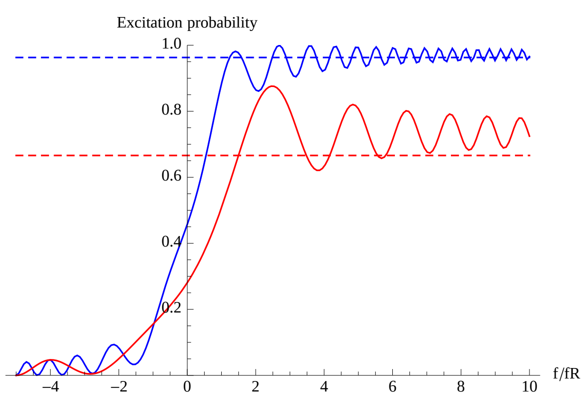

Figure 1: Excitation probability versus detuning for different sweep rates: $r = 3f_R^2$ (blue), $r = 9f_R^2$ (red). The initial detuning is $\Delta f_i = -5f_R$. The dashed line is the Landau-Zener limit ($\Delta f_i = -\infty$)

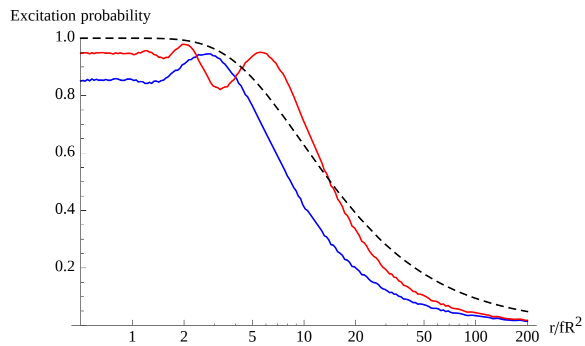

Figure 2: Asymptotic value of the excitation probability for different initial detunings: $\Delta f_i = -f_R$ (blue), $\Delta f_i = -2f_R$ (red) and the Landau-Zener limit $\Delta f_i = -\infty$ (dashed)

4 Chirped pulse vs fixed frequency pulse

See '20131016 Raman sweep vs pulse.nb'

5 Conclusions

- Most of the excitation occurs in the frequency range $\delta \in [-\Omega, \Omega]$
- The sweep speed controls the excitation probability. A rule of thumb would be 'a sweep rate α smaller than Ω^2 gives an excitation level higher than 70%'

6 Beyond Simple Models

Other works explore analytical [4–8] and numerical [9,10] solutions for different sweep functions. The reference [11, 12] explores Landau-Zener transitions in three-level systems.

References

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