1 Electromagnetism review

A point charge $q$ located at $r_q$ and moving at a velocity $v_q$ generates the magnetic field

$$B(r) = \frac{\mu_0}{4\pi} q v_q \times \frac{r - r_q}{|r - r_q|^3}. \quad (1)$$

In the case of a charge distribution, we replace $q v_q$ by $d^3r J$, where $J$ is the current flow. The magnetic field becomes

$$B(r) = \frac{\mu_0}{4\pi} \int d^3r' J(r') \times \frac{r - r'}{|r - r'|^3} \quad (2)$$

For a unidimensional conductor we can integrate the transverse direction. Equation (2) reduces to the Biot-Savart law,

$$B(r) = \frac{\mu_0 I}{4\pi} \int dr' \times \frac{r - r'}{|r - r'|^3}, \quad (3)$$

where $dr'$ is a line integration and $I$ is the current through the wire.

Far from the conductor, i.e. $|r| \ll |r'|$, the eq. (3) can be expanded via the series

$$\frac{1}{|r - r'|^3} = \frac{1}{|r'|^3} + \frac{3r \cdot r'}{|r'|^5} + \ldots$$

$$B(r) = \frac{\mu_0 I}{4\pi} \int dr' \times (r - r') \frac{1}{|r'|^3} + 3\frac{\mu_0 I}{4\pi} \int dr' \times (r - r') \frac{r \cdot r'}{|r'|^5}. \quad (4)$$
2 Anti-Helmholtz trap

This quadrupole trap is formed by two identical ring-shape wires shifted vertically and carrying current in opposite directions.

Let us first calculate the expression in (4) for a single ring of radius $R$ and shifted vertically in $+a$. The integration coordinate is $r' = R \cos \theta' \hat{x} + R \sin \theta' \hat{y} + a \hat{z}$ and its differential is $dr' = R \hat{\theta} d\theta$. Let us consider

$$\hat{\theta} \times (r - r') = (- \sin \theta', \cos \theta', 0) \times (x - R \cos \theta', y - R \sin \theta', -a)$$

$$= (-a \cos \theta', -a \sin \theta', R - y \sin \theta' - x \cos \theta')$$

(5)

and

$$r \cdot r' = xR \cos \theta' + yR \sin \theta' + za.$$ 

(6)

Given eqs. (5) and (6), the first and second integrals on the RHS of (4) are

$$\frac{\mu_0 I R^2}{2(R^2 + a^2)^{3/2}} \hat{z} + \frac{3 \mu_0 I a R^2}{2(R^2 + a^2)^{5/2}} \left( -\frac{x}{2}, -\frac{y}{2}, z \right).$$ 

(7)

To calculate the field produced by the second ring, we can simply do $I \to -I$ and $a \to -a$. This leads to

$$B_{-a}(r) = -\frac{\mu_0 I R^2}{2(R^2 + a^2)^{3/2}} \hat{z} + \frac{3 \mu_0 I a R^2}{2(R^2 + a^2)^{5/2}} \left( -\frac{x}{2}, -\frac{y}{2}, z \right).$$ 

(8)

Finally, the total field is\(^1\)

$$B(r) \equiv B_{+a}(r) + B_{-a}(r) = B_0 \left( \frac{x}{2}, \frac{y}{2}, -z \right),$$ 

(9)

where $B_0 \equiv -\frac{3 \mu_0 I a R^2}{2(R^2 + a^2)^{5/2}}$; and the potential is

$$U_{mag}(r) = g_F m_F \mu_B |B(r)|.$$ 

(10)

\(^1\)I think these results are calculated in Stan’s thesis via a multipole expansion.