Dirac Hamiltonian on 2D Square Lattice

Consider 1 atom 1 orbital per cell with spin.

$$H = -t\sum_{m,n}\left(\begin{array}{c} \frac{i}{2}c^\dagger_{m+1,n,\alpha}\sigma^x_\beta c_{m,n,\beta} + \frac{i}{2}c^\dagger_{m,n+1,\alpha}\sigma^y_{\alpha\beta}c_{m,n,\beta} \\ +\frac{i}{2}c^\dagger_{m+1,n,\alpha}\sigma^z_\beta c_{m,n,\beta} + \frac{i}{2}c^\dagger_{m,n+1,\alpha}\sigma^z_{\alpha\beta}c_{m,n,\beta} \\ +(2-M)\sum_{m,n}c^\dagger_{m,n}\sigma^z_{\alpha,\beta}c_{m,n,\beta} \end{array}\right) \tag{1}$$

Introduce Fourier transforms

$$c^\dagger_{m,n,\alpha} = \frac{1}{\sqrt{N}}\sum_q e^{-\vec{q}\cdot(m,n)a}c^\dagger_{q,\alpha} \tag{2}$$

and the Hamiltonian becomes

$$H_{\alpha\beta}(\vec{q}) = t\sin(q_x a)\sigma^x_{\alpha\beta} + t\sin(q_y a)\sigma^y_{\alpha\beta} + \left[(2-M) - t\cos(q_x a) - t\cos(q_y a)\right]\sigma^z \tag{3}$$

Expand around $\vec{q} = \vec{0}$

$$\begin{aligned} H_{\alpha\beta}(\vec{q}) =& q_x a\sigma^x + q_y a\sigma^y + \left(2 - M - (1 - \frac{q_x^2 a^2}{2}) - (1 - \frac{q_y^2 a^2}{2})\sigma^z\right) \\ =& q_x a\sigma^x + q_y a\sigma^y + (-M + \frac{q_x^2 + q_y^2}{a}^2 2)\sigma^z \end{aligned} \tag{4}$$

When $M = 0$, at $\vec{q} = \vec{0}$ there's a Dirac cone. The model has many discrte symmetries, inversion $P = \sigma^z$, charge conjugation $C = \sigma^x$, $C_2 = e^{i\pi\frac{\sigma^z}{2}}$, $C_4 = e^{\frac{i\pi}{2}\frac{\sigma^z}{2}}$. Looking at special points (Γ,M, etc.)

$$\begin{cases} H(0,0) = -M\sigma^z \\ H(\frac{\pi}{2},0) = (2-M)\sigma^z \\ H(0,\frac{\pi}{2}) = (2-M)\sigma^z \\ H(\frac{\pi}{a},\frac{\pi}{a}) = (4-M)\sigma^z \end{cases} \tag{5}$$

They all commute with $P = \sigma^z$. Further, there are transitions in the spin structure at $M = 0, 2, 4$ happening at $\Gamma, (X, Y), M$ points respectively. For each transition, the spin-up and spin-down bands touch and reseparates exchanging the spin of ground state.