Integral Equations

A differential equation can often be recast into an integral equation that may inspire an angle of approach that wasn't apparent in the original formulation.

Example

Here's an example of how to convert differential equation to integral equation

$$-u''(x) + V(x)u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1 \tag{1}$$

The Green's function for the differential operator $\hat{L} = \frac{d^2}{dx^2}$ with this BC is

$$G(x, y) = \begin{cases} x & x < y \\ 2x - y & x > y \end{cases} \tag{2}$$

thus we can rewrite (1) as

$$u''(x) = V(x)u(x) \tag{3}$$

and pretend the RHS is known to "solve" for u using Lagrange's identity[1]

$$u(x) = x + \int_0^x \{V(y)(x - y)\} u(y)dy \tag{4}$$

Voila! (1) has been converted into an integral equation (with BC encoded!)

Classification

The most general form of an integral equation is a Type II inhomogeneous Volterra equation ((4) is of this type!)

$$u(x) = f(x) + \int_{a(x)}^{b(x)} K(x, y)u(y)dy \tag{5}$$

where $K(x, y)$ is some known *integral kernel* and $f(x)$ is some known function. They are related to the boundary conditions and the differential operator in the differential equation from which (5) originates.

1. If the LHS is 0 not $u(x)$, (5) would be Type I instead of Type II
2. If $f(x) = 0$ (5) would be homogeneous rather than inhomogeneous
3. If $a(x), b(x)$ are independent of x, (5) would be Fredholm not Volterra

[1] This takes some work, refer to Green function with inhomogeneous boundary condition